# Derivation of Noise Correlations across Scales $E[< n_{sc}, n >]$

In this section we provide a detailed derivation of the expected noise correlations across scales, $E[< n_{sc}, n >]$, referred to in Section 3 of the paper. For simplicity, let us first derive the expression for the $1D$ case.

Let us denote a $1D$ noise patch, containing $M = 2m+1$ pixels and centered around a specific coordinate $l$ in the noisy image $N$ by: $n = (N_{l-m}, ..., N_{l+m})$. Let us denote by $l_{sc}$ the same relative coordinate at coarser scale $sc$ and the respective patch centered around $l_{sc}$ coordinate in $N_{sc}$ by: $n_{sc} = (N_{sc,l_{sc}-m}, ..., N_{sc,l_{sc}+m})$. The noise image $N_{sc}$ results from blurring and subsampling the original noise image $N$.

Fig. 1.a (see next page) displays the creation of a coarse scale (*e.g.,* half the original scale) from the fine scale via blur and subsample (the 3-tap blur kernel shown in the figure is for simplicity of illustration only). To illustrate the effect of blurring, each coarser pixel is colored according to the relative contribution (weight) of the original fine-scale pixels that were involved in the creation of this coarser pixel (*e.g.,* the middle coarse pixel = 0.25·red pixel +0.5·green pixel+0.25·blue pixel). The different weights are illustrated by the different sizes of colored areas within each coarse pixel.

Fig. 1.b (see next page) displays the pixel-wise correlation between a patch $n$ in the original scale (the red dashed rectangle) and the coarse patch $n_{sc}$ (the blue dashed rectangle). For simplicity, $l = 0$. Assuming that the noise pixels in the original scale are independent of each other (they are sampled from i.i.d. Gaussian noise), the correlation between a pixel $n(i)$ from the original patch and the coarse pixel $n_{sc}(i)$ is non-zero only if $n(i)$ took part in the creation of $n_{sc}(i)$. In this example, the fine-scale cyan pixel $n(-2)$ has zero correlation to the coarse-scale pixel $n_{sc}(-2)$, since it was not involved in its creation (no cyan color in $n_{sc}(-2)$). The same holds for the fine-scale gray pixel $n(2)$ versus the coarse-scale pixel $n_{sc}(2)$. On the other hand, the fine-scale green pixel $n(0)$ has high correlation to $n_{sc}(0)$, since it had a high weight in the creation of $n_{sc}(0)$ (and accordingly, the largest portion of $n_{sc}(0)$ is colored green). The fine-scale red pixel $n(-1)$ has a little correlation to $n_{sc}(-1)$ (and the same holds for the fine-scale blue pixel $n(1)$ versus $n_{sc}(1)$).

More formally: due to the blurring (prior to subsampling), any single pixel in a coarser scale is a linear combination of pixels from the original scale. Therefore, $N_{l_{sc}+i} = \Sigma_k \alpha_k^{l_{sc}+i} N_k$, where $\{N_k\}$ are pixels in the original scale ($k$ is a general spatial coordinate in the original scale), and $\{\alpha_k^{l_{sc}+i}\}$ are their respective weights in the blur process. Using the above relation and the linearity of the expectation operator:

$$E[N_{l_{sc}+i} N_{l+i}] = E[\Sigma_k \alpha_k^{l_{sc}+i} N_k N_{l+i}] = \Sigma_k \alpha_k^{l_{sc}+i} E[N_k N_{l+i}] \ .$$

Because $N_k$ are i.i.d. Gaussian noise samples, we have $E[N_k N_{l+i}] = \sigma^2 \delta_{k,l+i}$. This entails:

$$E[n_{l_{sc}+i} n_{l+i}] = \sigma^2 \alpha_{l+i}^{l_{sc}+i}$$

The above derivation shows that indeed the correlation between the original scale pixel and the coarse scale pixel depends only on the relative contribution of the pixel from the original scale to the pixel in the coarser scale during the blur process.

Finally, the expected correlation between $n$ and $n_{sc}$ is

$$E[< n_{sc}, n >] = E[\Sigma_i n_{sc}(i) n(i)] = \sigma^2 \Sigma_i \alpha_{l+i}^{l_{sc}+i} \ ,$$

This correlation will be relatively high (close to 1) for scales that are close to the original scale, and will diminish (towards 0) for very coarse scales.

The above derivation holds also for 2D patch correlation, only $l$ is now a 2D coordinate in the image $N$ (respectively $l_{sc}$ is a 2D coordinate in the image $N_{sc}$), and $\{\alpha_k\}$ are the respective weights in the blur process, which relate low scale 2D pixels to high scale 2D pixels.

**Empirical Validation of the Noise Correlations** While the noise correlations $E[< n_{sc}, n >]$ for the various scales $sc$ were derived analytically in the above paragraphs, these correlations can also be calculated *empirically*. Indeed, we empirically tested the multi-scale noise correlations by averaging over patches from 100 random noise images and their corresponding coarsened images. We further validated the correctness of our analytical expressions using these empirical calculations.

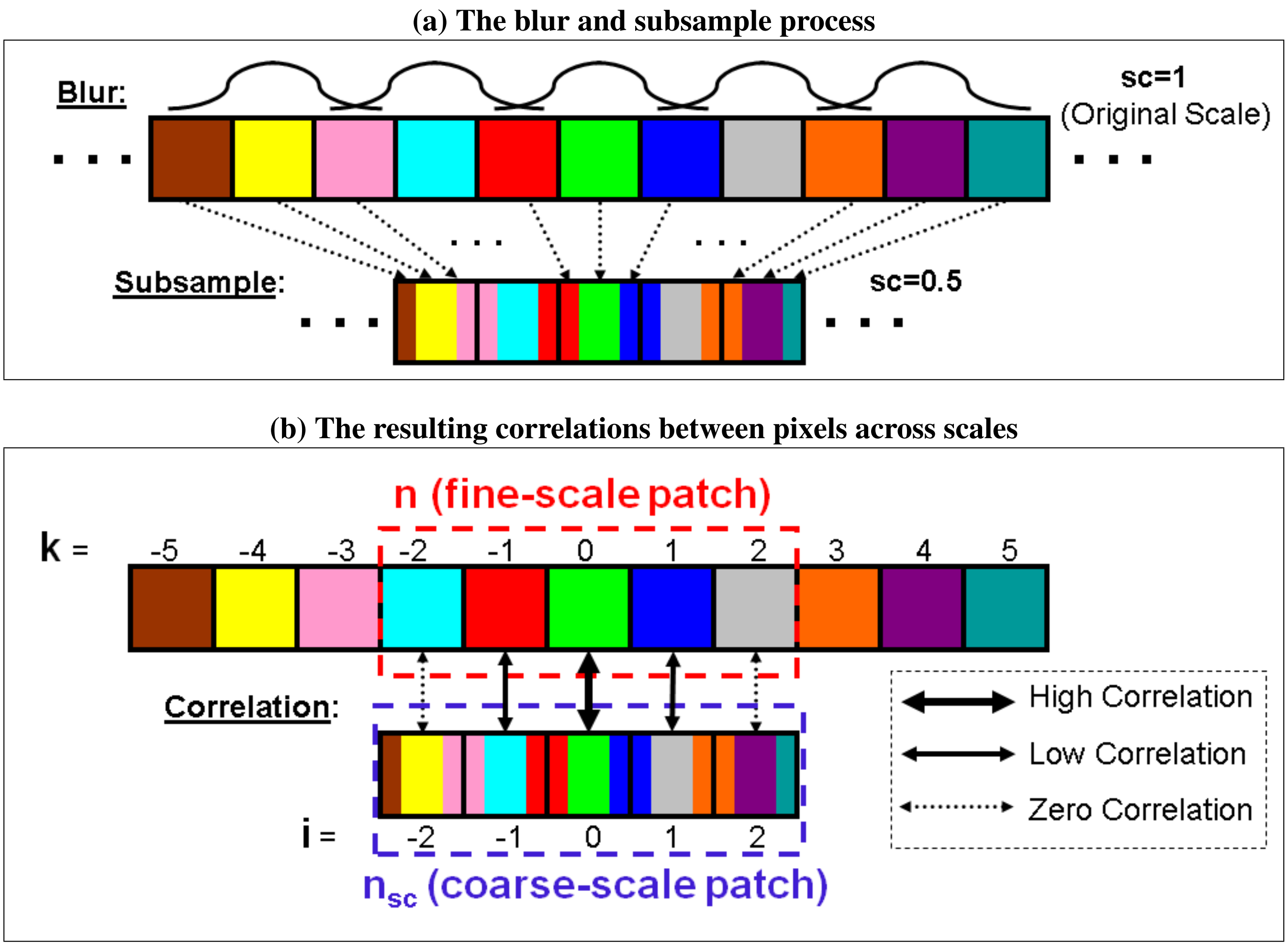


Figure 1. *Correlations across scales (in 1D).* See text for details