ALGEBRAIC AND ANALYTIC GEOMETRY OVER THE FIELD OF ONE ELEMENT

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Chapter I. Algebraic geometry

§1. $\mathbf{F}_1$-algebras

1.1. Definition and examples. First of all, a *semigroup* is a set provided with an associative binary operation. If a semigroup contains a neutral element, it is called a *monoid.* A homomorphism of semigroups (resp. monoids) is a map compatible with the operations on them (resp. and taking the neural element to the neutral element).

1.1.1. Definition. (i) An $\mathbf{F}_1$*-algebra* is a commutative multiplicative monoid A provided with elements $1 = 1_A$ and $0 = 0_A$ such that $1 \cdot f = f$ and $0 \cdot f = 0$ for all $f \in A$.

(ii) A *homomorphism of* $\mathbf{F}_1$*-algebras* $\varphi : A \to B$ is a map from A to B which is compatible with the operations on A and B and takes 0_A and 1_A to 0_B and 1_B, respectively.

The category of $\mathbf{F}_1$-algebras admits final and initial objects. Namely, the *trivial* $\mathbf{F}_1$-algebra, which consists of only one element (which is 0 as well as 1), is its final object, and the *field* $\mathbf{F}_1$, which consists of precisely two elements 0 and 1, is its initial object. Notice that the sets of homomorphisms of $\mathbf{F}_1$-algebras $\mathrm{Hom}(A, B)$ are provided with the canonical structure of a commutative semigroup.

1.1.2. Definition. (i) An element f of an $\mathbf{F}_1$-algebra A is said to be a *zero divisor* if it is nonzero and there exists a nonzero element $g \in A$ with $fg = 0$.

(ii) An $\mathbf{F}_1$-algebra A is said to be *integral* if the equality $fh = gh$ implies that either $f = g$ or $h = 0$.

(iii) An $\mathbf{F}_1$-algebra A is said to be $\mathbf{F}_1$*-field* if every nonzero element of A is invertible.

If A has no zero divisors, then the subset $\check{A} = A \backslash \{0\}$ is preserved under multiplication, i.e., it is a submonoid of A. The correspondence $A \mapsto \check{A}$ gives rise to an equivalence between the category of $\mathbf{F}_1$-algebras without zero divisors and that of commutative monoids without zero. Furthermore, the correspondence $A \mapsto \check{A}$ gives rise to an equivalence between the category of integral $\mathbf{F}_1$-algebras and that of commutative monoids with the cancellation property. Finally, an $\mathbf{F}_1$-algebra A is an $\mathbf{F}_1$-field if and only if $\check{A} = A^*$, and the correspondence $A \mapsto A^*$ gives rise to an equivalence between the category of $\mathbf{F}_1$-fields and that of abelian groups.

Let S be a sub-semigroup of an $\mathbf{F}_1$-algebra A. The *localization of A with respect to S* is a homomorphism of $\mathbf{F}_1$-algebras $A \to S^{-1}A$ such that the image of every element of S in $S^{-1}A$ is invertible and any homomorphism $A \to B$ to an $\mathbf{F}_1$-algebra B with the latter property goes through a unique homomorphism $S^{-1}A \to B$. The homomorphism $A \to S^{-1}A$ is unique up to a unique isomorphism, and $S^{-1}A$ can be constructed as the set of equivalences classes of pairs $(f, s) \in A \times S$ with respect to the following equivalence relation: $(f, s) \sim (f', s')$ if there is $t \in S$ with $fs't = f'st$. The equivalence class of a pair (f, s) is denoted by $\frac{f}{s}$. If S is generated by one element $f \in A$, $S^{-1}A$ is denoted by A_f. If A has no zero divisors, the localization of A with respect to $\check{A}$ is called the *fraction* $\mathbf{F}_1$*-field* of A and denoted by $\mathrm{Frac}(A)$.

1.1.3. Examples. (i) The multiplicative monoid $A^{\cdot}$ of any commutative ring A with unity (e.g., $\mathbf{Z}$) can be considered as an $\mathbf{F}_1$-algebra. If the ring is a field, the corresponding $\mathbf{F}_1$-algebra is an $\mathbf{F}_1$-field. For example, $\mathbf{F}_1$ corresponds to the field of two elements, i.e., $\mathbf{F}_1 = \mathbf{F}_2^{\cdot}$.

(ii) The sets of non-negative numbers $\mathbf{R}_+$ and of non-negative integers $\mathbf{Z}_+$ and the unit interval $[0, 1]$, provided with the usual multiplication, are $\mathbf{F}_1$-algebras. The $\mathbf{F}_1$-algebra $\mathbf{R}_+$ is an $\mathbf{F}_1$-field.

(iii) Given a set I, let $\mathbf{F}_1[T_i]_{i\in I}$ be the set consisting of 0 and expressions of the form $T_{i_1}^{\mu_1} \cdot \ldots \cdot T_{i_n}^{\mu_n}$ with $i_1, \ldots, i_n \in I$ and $\mu_1, \ldots, \mu_n \in \mathbf{Z}_+$ (the latter are called *monomials* in the variables $\{T_i\}_{i\in I}$). It is an integral $\mathbf{F}_1$-algebra with respect to the evident multiplication. More generally, for an $\mathbf{F}_1$-algebra A, let $A[T_i]_{i\in I}$ denote the set consisting of 0 and expressions of the form $aT_{i_1}^{\mu_1} \cdot \ldots \cdot T_{i_n}^{\mu_n}$ with $a \in \check{A}$, $i_1, \ldots, i_n \in I$ and $\mu_1, \ldots, \mu_n \in \mathbf{Z}_+$. It is also an $\mathbf{F}_1$-algebra with respect to the evident multiplication, and its elements are said to be *terms over A* (or *terms with coefficients in A*). Notice that there is an isomorphism of $\mathbf{F}_1$-algebras $\mathbf{F}_1[T_n]_{n\geq 1} \stackrel{\sim}{\to} \mathbf{Z}_+$ induced by the map that takes T_n to the n-th prime number, and it extends to an isomorphism $\mathbf{F}_3^{\cdot}[T_n]_{n\geq 1} \stackrel{\sim}{\to} \mathbf{Z}^{\cdot}$.

(iv) Let I be a poset (i.e., partially ordered set) which has unique maximal and minimal elements and in which every pair of elements $e, f \in I$ has supremum $\sup(e, f)$ (i.e., a unique minimal element that is greater or equal than each of them). Then I can be considered as an $\mathbf{F}_1$-algebra in which the supremum $\sup(e, f)$ is the product of e and f and the maximal and minimal elements are zero and one, respectively. In this $\mathbf{F}_1$-algebra every element e is idempotent, i.e.,

$e^2 = e$. Conversely, every *idempotent* $\mathbf{F}_1$*-algebra* I (i.e., an $\mathbf{F}_1$-algebra in which all elements are idempotents) can be considered as a poset with the above properties with respect to the following partial ordering: $e \le f$ if $ef = f$ (see §1.6).

1.2. Ideals and spectra.

1.2.1. Definition. An *ideal* of an $\mathbf{F}_1$-algebra A is an equivalence relation which is compatible with the operation on A, i.e., a subset $E \subset A \times A$ which is an equivalence relation and an $\mathbf{F}_1$-subalgebra. (The latter is what is usually called a congruence relation.)

Given an ideal $E \subset A \times A$, the set of equivalence classes A/E provided with the evident multiplication is an $\mathbf{F}_1$-algebra. For example, if Δ denotes the diagonal homomorphism $A \to A \times A$, then $\Delta(A)$ is an ideal which is contained in all other ideals of A (it is therefore called the *minimal ideal* of A). If G is a subgroup of A^*, then the set of pairs of the form (f, fg) with $f \in A$ and $g \in G$ is an ideal, and the corresponding quotient is the set A/G of orbits under the action of G on A. An ideal $E \subset A \times A$ is *generated by a subset* $S \subset A \times A$ if it is the minimal ideal that contains S. (Notice that the intersection of any family of ideals is again an ideal.) An ideal E is *nontrivial* if it does not coincide with $A \times A$, i.e., the quotient $\mathbf{F}_1$-algebra A/E is nontrivial.

1.2.2. Definition. A *Zariski ideal* is a subset $\mathbf{a} \subset A$ with the property that $fg \in \mathbf{a}$ whenever $f \in \mathbf{a}$ and $g \in A$.

A Zariski ideal $\mathbf{a}$ gives rise to the ideal $E_{\mathbf{a}} = \Delta(A) \cup (\mathbf{a} \times \mathbf{a})$. (For example, $E_{(0)} = \Delta(A)$.) The corresponding quotient $A/E_{\mathbf{a}}$ is denoted by $A/\mathbf{a}$. A Zariski ideal $\mathbf{a}$ is *nontrivial* if it does not coincide with A. Notice that the union of any family of nontrivial Zariski ideals is also a nontrivial Zariski ideal. In particular, there is a unique maximal Zariski ideal $\mathbf{m}_A$, it coincides with $A \backslash A^*$. For an ideal $E \subset A \times A$, the set $\mathbf{a}_E = \{f \in A \big| (f, 0) \in E\}$ is a Zariski ideal. For example, $\mathbf{a}_{E_{\mathbf{a}}} = \mathbf{a}$.

Let $\varphi : A \to B$ be a homomorphism of $\mathbf{F}_1$-algebras.

1.2.3. Definition. (i) The *kernel of* φ is the ideal $\mathrm{Ker}(\varphi) = \{(f, g) \in A \times A \big| \varphi(f) = \varphi(g)\}$.

(ii) The *Zariski kernel of* φ is the Zariski ideal $\mathrm{Zker}(\varphi) = \{f \in A \big| \varphi(f) = 0\}$.

(iii) The *preimage* of an ideal F of B is the ideal $\varphi^{-1}(F) = \{(f, g) \in A \times A \big| (\varphi(f), \varphi(g)) \in F\}$.

(iv) The preimage $\varphi^{-1}(\mathbf{b})$ of a Zariski ideal $\mathbf{b} \subset B$ is defined as the preimage of the associated ideal $E_{\mathbf{b}}$. The *Zariski preimage* of $\mathbf{b}$ is the Zariski ideal $z\varphi^{-1}(\mathbf{b}) = \{f \in A \big| \varphi(f) \in \mathbf{b}\}$.

Notice that the quotient $A/\mathrm{Ker}(\varphi)$ is canonically isomorphic to the image of φ, and that the ideal associated with $z\varphi^{-1}(\mathbf{b})$ does not necessarily coincide with $\varphi^{-1}(\mathbf{b})$.

1.2.4. Definition. (i) An ideal $\Pi \subset A \times A$ is *prime* if it is nontrivial and possesses the property that, if $(fh, gh) \in \Pi$, then either $(f, g) \in \Pi$ or $(h, 0) \in \Pi$, i.e., the quotient A/Π is a nontrivial integral $\mathbf{F}_1$-algebra. The set of prime ideals of A is called the *spectrum* of A and denoted by $\mathrm{Fspec}(A)$.

(ii) A Zariski ideal $\mathfrak{p} \subset A$ is *prime* if it is nontrivial and possesses the property that, if $fg \in \mathfrak{p}$, then either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$, i.e., the quotient $A/\mathfrak{p}$ is nontrivial and has no zero divisors. The set of Zariski prime ideals of A is called the *Zariski spectrum* of A and denoted by $\mathrm{Zspec}(A)$.

Notice that the union of any family of Zariski prime ideals is a Zariski prime ideal. The maximal Zariski prime ideal is the maximal Zariski ideal $\mathbf{m}_A$.

Given a Zariski prime ideal $\mathfrak{p} \subset A$, the fraction $\mathbf{F}_1$-field of $A/\mathfrak{p}$ is denoted by $\kappa(\mathfrak{p})$, and the localization of A with respect to the submonoid $A \backslash \mathfrak{p}$ is denoted by $A_\mathfrak{p}$. The maximal Zariski ideal of $A_\mathfrak{p}$ coincides with $\mathfrak{p}A_\mathfrak{p}$, and one has $A^*_\mathfrak{p} \xrightarrow{\sim} \kappa(\mathfrak{p})^*$. There is a canonical map $\mathrm{Fspec}(A) \to \mathrm{Zspec}(A)$: $\Pi \mapsto \mathfrak{p}_\Pi = \{f \in A \big| (f, 0) \in \Pi\}$. The prime ideals from the preimage of a Zariski prime ideal $\mathfrak{p} \subset A$ are said to be $\mathfrak{p}$-*prime*.

1.2.5. Proposition. *Given a Zariski prime ideal* $\mathfrak{p} \subset A$, *there is a canonical bijection between the set of* $\mathfrak{p}$*-prime ideals and the set of subgroups of the group* $\kappa(\mathfrak{p})^*$.

Proof. Given a $\mathfrak{p}$-prime ideal Π, the set G_Π of elements of $\kappa(\mathfrak{p})^*$ of the form $\frac{f}{g}$, where $f, g \notin \mathfrak{p}$ and $(f, g) \in \Pi$, is a subgroup of $\kappa(\mathfrak{p})^*$. Conversely, given a subgroup $G \subset \kappa(\mathfrak{p})^*$, the set Π_G of pairs (f, g) with either $f, g \in \mathfrak{p}$, or $f, g \notin \mathfrak{p}$ and $\frac{f}{g} \in G$ is a $\mathfrak{p}$-prime ideal of A. *We claim that the maps* $\Pi \mapsto G_\Pi$ *and* $G \mapsto \Pi_G$ *are inverse to each other.* (It is clear that the maps preserve the inclusion relation.)

The equality $G = G_{\Pi_G}$ and the inclusion $\Pi \subset \Pi_{G_\Pi}$ are trivial. Let $(f, g) \in \Pi_{G_\Pi}$. If $f, g \in \mathfrak{p}$, then $(f, g) \in \Pi$. Assume therefore that $f, g \notin \mathfrak{p}$. Then $\frac{f}{g} \in G_\Pi$, i.e., there exists an element $(u, v) \in \Pi$ with $u, v \notin \mathfrak{p}$ and $\frac{f}{g} = \frac{u}{v}$. The latter means that $fvh = guh$ for some $h \notin \mathfrak{p}$. Since the ideal Π is $\mathfrak{p}$-prime, it follows that $(fv, gu) \in \Pi$. But $(gu, gv) \in \Pi$ and, therefore, $(fv, gv) \in \Pi$. Again, since Π is prime, $\mathbf{a}_\Pi = \mathfrak{p}$ and $v \notin \mathfrak{p}$, it follows that $(f, g) \in \Pi$. ∎

1.2.6. Corollary. *Let* $A \to B$ *be an injective homomorphism of* $\mathbf{F}_1$*-algebras. Then for the induced commutative diagram*

$$\begin{array}{ccc} \mathrm{Zspec}(B) & \xrightarrow{\psi} & \mathrm{Zspec}(A) \\ \uparrow & & \uparrow \pi \\ \mathrm{Fspec}(B) & \xrightarrow{\varphi} & \mathrm{Fspec}(A) \end{array}$$

one has $\mathrm{Im}(\varphi) = \pi^{-1}(\mathrm{Im}(\psi))$. *In particular, surjectivity of* φ *is equivalent to that of* ψ.

Proof. The inclusion $\mathrm{Im}(\varphi) \subset \pi^{-1}(\mathrm{Im}(\psi))$ is trivial. Let $\mathfrak{p}$ be a Zariski prime ideal of A from the image of ψ. We have to show that all $\mathfrak{p}$-prime ideals of A lie in the image of φ. The assumption implies that $\mathfrak{p}B \cap A = \mathfrak{p}$ and, therefore, the canonical homomorphism $A/\mathfrak{p} \to B/\mathfrak{p}B$ is injective. We may therefore replace A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$ and assume that A has no zero divisors and $\mathfrak{p} = 0$. Furthermore, let $F = \kappa(\mathfrak{p})$ be the fraction $\mathbf{F}_1$-field of A. Then the canonical homomorphism from F to the localization of B with respect to $\check{A} = A\backslash\{0\}$ is injective. We may therefore replace A by F and B by that localization and assume that A is an $\mathbf{F}_1$-field. By Proposition 1.2.5, prime ideals correspond to subgroups of A^*. If G is such a subgroup, then the corresponding prime ideal is the intersection $\Pi \cap (A \times A)$, where Π is the prime ideal of B which is the union of $\mathbf{m}_B \times \mathbf{m}_B$ with the set of pairs $(f, g) \in B^* \times B^*$ with $\frac{f}{g} \in G$. ∎

The prime ideal that corresponds to the unit subgroup and the whole group $\kappa(\mathfrak{p})^*$ will be denoted by $\Pi_{\mathfrak{p}}$ and $\Pi_{(\mathfrak{p})}$, respectively. One has $\Pi_{\mathfrak{p}} = \{(f,g) \big|$ either $f, g \in \mathfrak{p}$, or $f, g \notin \mathfrak{p}$ and $fh = gh$ for some $h \notin \mathfrak{p}\}$, and $\Pi_{(\mathfrak{p})} = (\mathfrak{p} \times \mathfrak{p}) \cup (A\backslash\mathfrak{p} \times A\backslash\mathfrak{p})$. Notice that $\Pi_{\mathfrak{p}} = \mathrm{Ker}(A \to \kappa(\mathfrak{p}))$, and the set of ideals of the form $\Pi_{(\mathfrak{p})}$ coincides with the set of maximal ideals of A as well as with the set of ideals E such that $A/E = \mathbf{F}_1$.

In what follows, we will consider $\mathrm{Zspec}(A)$ as a partially ordered set (or, briefly, a poset) with respect to the partially ordering opposite to the inclusion relation (i.e., $\mathfrak{p} \le \mathfrak{q}$ if $\mathfrak{q} \subset \mathfrak{p}$). We notice that this partial ordering possesses the following property: every subset $S \subset \mathrm{Zspec}(A)$ has the infimum $\inf S$ (i.e., a unique maximal element x with the property that $x \le y$ for all $x \in S$). Namely the infimum corresponds to the union of the Zariski prime ideals from the subset. We call a poset X with the latter property an *inf-poset*. Thus, $\mathrm{Zspec}(A)$ is an inf-poset. Notice that, if a subset S of an inf-poset X admits an element $x \in X$ with $y \le x$ for all $y \in S$, then it has the supremum $\sup S$ (i.e., a unique minimal element with the latter property).

1.2.7. Lemma. *Let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ be Zariski prime ideals of A. Then*

(i) if a Zariski prime ideal $\mathfrak{p}$ contains $\mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n$, then $\mathfrak{p} \supset \mathfrak{p}_i$ for some $1 \le i \le n$;

(ii) if a prime ideal Π contains $\Pi_{\mathfrak{p}_1} \cap \dots \cap \Pi_{\mathfrak{p}_n}$, then there is a nonempty subset $J \subset \{1, \dots, n\}$ such that $\Pi \supset \Pi_{\mathfrak{q}}$, where $\mathfrak{q} = \bigcup_{i \in J} \mathfrak{p}_i$.

Proof. (i) Suppose that $\mathfrak{p} \not\supset \mathfrak{p}_i$ for all $1 \le i \le n$. Let $h_i \in \mathfrak{p}_i\backslash\mathfrak{p}$ and $h = \prod_{i=1}^n h_i$. then $h \in \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n$ and $h \notin \mathfrak{p}$, which contradicts the assumption.

(ii) If $\mathfrak{p} = \mathfrak{p}_\Pi$, then $\mathfrak{p} \supset \mathfrak{p}_1 \cap \dots \cap \mathfrak{p}_n$, and (i) implies that $\mathfrak{p} \supset \mathfrak{p}_i$ for some $1 \le i \le n$. Let J be the set of all $1 \le i \le n$ with $\mathfrak{p} \supset \mathfrak{p}_i$. *We claim that* $\Pi \supset \Pi_{\mathfrak{q}}$, *where* $\mathfrak{q} = \bigcup_{i \in J} \mathfrak{p}_i$. Indeed, let $(f, g) \in \Pi_{\mathfrak{q}}$. If $f, g \in \mathfrak{q} \subset \mathfrak{p}$, then $(f, g) \in \Pi$. Assume therefore that $f, g \notin \mathfrak{q}$. Then the inclusion

$(f,g) \in \Pi_{\mathfrak{q}}$ implies that $(f,g) \in \Pi_{\mathfrak{p}_i}$ for all $i \in J$. If $i \notin J$, take an element $h_i \in \mathfrak{p}_i \backslash \mathfrak{p}$ and set $h = \prod_{i \notin J} h_i$. Then $h \in \mathfrak{p}_i$ for all $i \notin J$. It follows that $(fh, gh) \in \Pi_{\mathfrak{p}_1} \cap \ldots \cap \Pi_{\mathfrak{p}_n} \subset \Pi$. Since Π is a prime ideal and $h \notin \mathfrak{p}$, we get $(f,g) \in \Pi$. ∎

1.3. Modules over an $\mathbf{F}_1$-algebra and K-vector subspaces.

1.3.1. Definition. (i) A *module over* an $\mathbf{F}_1$-algebra A (or an *A-module*) is a set M provided with an element $0 = 0_M$ and an action of A on M, i.e., a map $A \times M \to M : (f, m) \mapsto fm$, satisfying the following conditions: $(fg)m = f(gm)$, $1_A m = m$ and $0_A m = 0_M$ for all $f, g \in A$ and $m \in M$.

(ii) A homomorphism of A-modules is a map $M \to N$ compatible with the action of A.

Notice that such a homomorphism $M \to N$ takes 0_M to 0_N, and the set $\mathrm{Hom}_A(M, N)$ of homomorphisms of A-modules has a canonical structure of an A-module. The category of A-modules is denoted by A-Mod. An A-module is *trivial* if it has only one element 0. The trivial A-module is the initial and final object of the category A-Mod. An *A-algebra* is an $\mathbf{F}_1$-algebra B which is also an A-module. The structure of an A-algebra on B gives rise to a homomorphism of $\mathbf{F}_1$-algebras $A \to B$ and, conversely, the latter defines the former. If the canonical homomorphism $A \to B$ is injective, we will say that we are given an *extension of $\mathbf{F}_1$-algebras B/A*, and we will identify A with its image in B. An A-module M is said to be *integral* if the equality $am = an$ with $a \in A$ and $m, n \in M$ implies that either $a = 0$, or $m = n$, and the equality $am = bm$ with $a, b \in A$ and $m \in M$ implies that either $a = b$, or $m = 0$. For example, an $\mathbf{F}_1$-algebra A is integral as an A-module if it is an integral $\mathbf{F}_1$-algebra.

1.3.2. Definition. An *A-submodule* of an A-module M is an equivalence relation $E \subset M \times M$ such that $(fm, fn) \in E$ for every $f \in A$ and $(m, n) \in E$.

For example, A-submodules of A, considered as an A-module, are ideals of A. Given an A-submodule $E \subset M \times M$, the set of equivalence classes M/E provided with the evident action of A is an A-module. An A-submodule $E \subset M \times M$ is *generated by a subset* $S \subset M$ if it is the minimal A-submodule that contains S. (Notice that the intersection of any family of A-submodules is again an A-submodule.) An A-submodule E is *nontrivial* if it does not coincide with $M \times M$, i.e., the quotient M/E is nontrivial.

1.3.3. Definition. A *Zariski A-submodule* of an A-module M is a subset $N \subset M$ such that $fn \in N$ whenever $f \in A$ and $n \in N$.

A Zariski A-submodule N gives rise to an A-submodule E_N, which consists of the pairs (m,n) with either $m=n$ or $m,n \in N$. The corresponding quotient is denoted by M/N. The intersection and the union of any family of Zariski A-submodules over A is a Zariski A-submodule. For an A-submodule $E \subset M \times M$, the set $N_E = \{m \in M \big| (m,0) \in E\}$ is a Zariski A-submodule. For example, A itself is an A-module, and its A-submodules and Zariski A-submodules over A are precisely ideals and Zariski ideals, respectively.

Let $\varphi : M \to N$ be a homomorphism of A-modules.

1.3.4. Definition. (i) The *kernel of* φ is the A-submodule $\mathrm{Ker}(\varphi) = \{(m_1,m_2) \big| \varphi(m_1) = \varphi(m_2)\}$.

(ii) The *Zariski kernel of* φ is the Zariski A-submodule $\mathrm{Zker}(\varphi) = \{m \in M \big| \varphi(m) = 0\}$.

(iii) The *preimage* of an A-submodule F of N is the A-submodule $\varphi^{-1}(F) = \{(m,n) \in M \times N \big| (\varphi(m), \varphi(n)) \in F\}$.

(iv) The preimage $\varphi^{-1}(P)$ of a Zariski A-submodule $P \subset N$ is defined as the preimage of the associated A-submodule E_P. The *Zariski preimage* of P is the Zariski A-submodule $z\varphi^{-1}(P) = \{m \in M \big| \varphi(m) \in P\}$.

(v) The *image of* φ is the Zariski A-submodule of N defined by $\mathrm{Im}(\varphi) = \varphi(M)$.

Notice that φ gives rise to an isomorphism of A-modules, $M/\mathrm{Ker}(\varphi) \xrightarrow{\sim} \mathrm{Im}(\varphi)$.

The category of A-modules admits projective and inductive limits. The projective limits coincide with the corresponding set theoretic projective limits provided with the evident structure of an A-module. As for inductive limits, it suffices to construct coequalizers of two homomorphisms and direct sums. First of all, given two homomorphisms $\varphi, \psi : M \to N$ of A-modules, their coequalizer is the quotient of N by the A-submodule generated by the pairs $(\varphi(m), \psi(m))$ for $m \in M$. Furthermore, if $\{M_i\}_{i \in I}$ is a family of A-modules, their direct sum $\oplus_{i \in I} M_i$ is the union of M_i's in which their zeros are identified and which is provided with the evident action of A. An example of the latter is the direct sum $A^{(I)}$ of copies of A taken over a set I. A module over A isomorphic to $A^{(I)}$ for some I is called *free*. If I is a finite set of n elements, it is denoted by $A^{(n)}$. An A-module is said to be *finite* if there is a surjective homomorphism of A-modules $A^{(n)} \to M$. If $n=1$, M is said to be *cyclic*. An A-algebra is said to be *finite* if it is finite as an A-module.

1.3.5. Lemma. *The following properties of an A-module M are equivalent:*

(a) M is free;

(b) there exists an A-module N such that the A-module $M \oplus N$ is free;

(c) for any epimorphism $\pi : P \to Q$ and any homomorphism $\varphi : M \to Q$, there exists a

homomorphism $\psi : M \to P$ *with* $\pi\psi = \varphi$.

Proof. The implications (a)$\Longrightarrow$(b)$\Longrightarrow$(c) are trivial. Suppose that M possesses the property (c). We take an arbitrary epimorphism $\pi : F \to M$ from a free A-module $F = A^{(I)}$. By the property (c) applied to the identity homomorphism $M \to M$, the epimorphism π has a section $\sigma : M \to F$. For $i \in I$, let A_i be the corresponding free A-module of rank one, and we set $J = \{i \in I\big|$ there exists $m \in M\backslash\{0\}$ with $\sigma(m) \in A_i\}$. *We claim that* π *induces an isomorphism* $A^{(J)} \stackrel{\sim}{\to} M$. Indeed, surjectivity of the latter homomorphism follows from that of π. For $i \in J$, let e_i be the canonical generator of A_i and set $m_i = \pi(e_i)$. Let also m be a nonzero element of M with $\sigma(m) \in A_i$, i.e., $\sigma(m) = ae_i$ for some $a \in A$. It follows that $m = am_i$ and, therefore, $ae_i = a\sigma(m_i)$. The latter implies that $\sigma(m_i) \in A_i$ and, in fact, $\sigma(m_i) = e_i$. The claim follows. ∎

The category of A-modules is a symmetric strict monoidal category with respect to the tensor product which is defined as follows. Given A-modules M, N and P, a map $\varphi : M \times N \to P$ is called *A-bilinear* if $\varphi(fm, n) = \varphi(m, fn) = f\varphi(m, n)$ for all $f \in A$ and $(m, n) \in M \times N$. The *tensor product* of M and N over A is an A-module $M \otimes_A N$ provided with a bilinear homomorphism $M \times N \to M \otimes_A N$ such that, for any A-bilinear homomorphism $\varphi : M \times N \to P$, there exists a unique homomorphism of A-modules $M \otimes_A N \to P$ which is compatible with φ. The tensor product is unique up to a unique isomorphism, and is constructed as follows. It is the quotient of $M \times N$ by the A-submodule generated by the relations $(fm, n) \sim (m, fn)$ for $f \in A$, $m \in M$ and $n \in N$. If A' is an A-algebra, then $M' = M \otimes_A A'$ is an A'-module. Notice that, for an A-submodule E of M, there is a canonical isomorphism of A'-modules $M/E \otimes_A A' \stackrel{\sim}{\to} M'/E'$, where E' is the A'-submodule of M' generated by the image of E. Furthermore, if B and C are A-algebras, then so is $B \otimes_A C$.

Modules over an $\mathbf{F}_1$-field K are said to be *K-vector spaces*. Every K-vector space M has a canonical decomposition into a direct sum of cyclic K-vector spaces. Indeed, if $I = \check{M}/K^*$ is the set of orbits of the multiplicative group K^* acting on the set $\check{M} = M\backslash\{0\}$ and $\{m_i\}_{i\in I}$ is a set of representatives, then $M = \oplus_{i\in I} Km_i$. Such a set of representatives, called a *basis* of M, defines a surjective homomorphism of K-vector spaces $K^{(I)} \to M$ which is bijective if and only if M is a free K-vector space, or if and only if the action of K^* on $\check{M}$ is free. If M is a cyclic K-vector space, the stabilizers of any two nonzero elements of M in K^* coincide (and are called the *stabilizer* of M), and the isomorphism class of M is determined by this stabilizer. If, in addition, K' is an $\mathbf{F}_1$-field that contains K, then the K'-vector space $M \otimes_K K'$ is also cyclic, and its stabilizer coincide with the stabilizer of M. Thus, if M is an arbitrary K-vector space and $M = \oplus_{i\in I} Km_i$ is its canonical decomposition into a direct sum of cyclic K-vector spaces, then $M \otimes_K K' = \oplus_{i\in I} K'm_i$, then the isomorphism class of M is determined by the family $\{G_i\}_{i\in I}$ of stabilizers of the cyclic K-vector spaces Km_i, and the stabilizer of each $K'm_i$ in K'^* coincides with that in K^* (i.e., G_i). We are now going to describe the above canonical decomposition for finite modules over finitely generated K-algebras.

Let A be the $\mathbf{F}_1$-algebra $\mathbf{F}_1[T_1, \dots, T_n]$. We consider the free A-module $A^{(m)}$ for $m \geq 1$. Its basis elements will be denoted by $e_1, \dots, e_m$, and its nonzero elements will be called *monomials*. Let us fix a monomial order $\leq$ on the set of monomials, i.e., a total order that possesses the property that, if $f < g$ and $T^\mu \neq 1$, then $f < T^\mu f < T^\mu g$ (see [Eis, §15]). We extend this order to a total order on the whole $A^{(m)}$ by $0 < f$ for all $f \neq 0$. The simple but important fact is that every nonempty subset of $A^{(m)}$ has a unique minimal element (see [Eis, 15.2]).

Let now K be an $\mathbf{F}_1$-field, and $B = K[T_1, \dots, T_n]$. Nonzero elements of $B^{(m)}$ will be called *terms*. For a nonzero term $f = aT^\mu e_i \in B^{(m)}$, we set $\mathrm{in}(f) = T^\mu e_i \in A$, and we set $\mathrm{in}(0) = 0$. For a B-submodule E of $B^{(m)}$, let $\mathrm{in}(E)$ denote the Zariski A-submodule of $A^{(m)}$ whose nonzero elements are of the form $\max\{\mathrm{in}(f), \mathrm{in}(g)\}$ for $(f, g) \in E$ with $\mathrm{in}(f) \neq \mathrm{in}(g)$. For example, $\mathrm{in}(\Delta(B^{(m)})) = 0$ and, for a Zariski B-submodule $N \subset B^{(m)}$, $\mathrm{in}(N) = \mathrm{in}(E_N)$ consists of elements of the form $\mathrm{in}(f)$ with $f \in N$. Notice that $\mathrm{in}(N_E) \subset \mathrm{in}(E)$. The following is a version of a theorem of Macaulay ([Eis, 15.3]).

1.3.6. Lemma. *For any B-submodule E of $B^{(m)}$, the images of monomials from $A^{(m)} \backslash \mathrm{in}(E)$ in the quotient $B^{(m)}/E$ form a basis of its canonical decomposition into a direct sum of cyclic K-vector spaces.*

Proof. First of all, the inclusion $(f, ag) \in E$ for $a \in K^*$ and two distinct elements $f, g \in A^{(m)} \backslash \mathrm{in}(E)$ is impossible since $\max\{\mathrm{in}(f), \mathrm{in}(ag)\}$ is f or g, but both of them are outside $\mathrm{in}(E)$. It remains to show that for every element $f \in B^{(m)} \backslash N_E$ there exist elements $g \in A^{(m)} \backslash \mathrm{in}(E)$ and $a \in K^*$ with $(f, ag) \in E$. Multiplying f by an element of K^*, we may assume that $f \in A^{(m)}$. If $f \in A^{(m)} \backslash \mathrm{in}(E)$, there is nothing to prove, and so assume that $f \in \mathrm{in}(E)$. Then there exists an element $(f, ag) \in E$ with $a \in K^*$, $g \in A^{(m)}$ and $f > g$. We may assume that for such a pair g is minimal, and in this case *we claim that* $g \in A^{(m)} \backslash \mathrm{in}(E)$. Indeed, if this is not true, then $g \in \mathrm{in}(E)$ and, therefore, there exists an element $(g, bh) \in E$ with $b \in K^*$, $h \in A$ and $g > h$. It follows that $(f, abh) \in E$, and this contradicts the minimality of g. ∎

For a B-submodule E of $B^{(m)}$, let $\overline{\mathrm{in}}(E)$ denote the Zariski A-submodule of $A^{(m)}$ whose nonzero elements are of the form $\max\{\mathrm{in}(f), \mathrm{in}(g)\}$ for $(f, g) \in E$ with $f \neq g$. One has $\mathrm{in}(E) \subset \overline{\mathrm{in}}(E)$. If

$K = \mathbf{F}_1$, then $\overline{\mathrm{in}}(E)$ always coincides with $\mathrm{in}(E)$.

1.3.7. Corollary. *The following properties of a B-submodule E of $B^{(m)}$ are equivalent:*

(a) $\overline{\mathrm{in}}(E) = \mathrm{in}(E)$;

(b) the quotient $B^{(m)}/E$ is a free K-vector space.

Proof. (a)$\Longrightarrow$(b). Suppose (b) is not true. By Lemma 1.3.6, we can find a minimal element $f \in A^{(m)} \backslash \mathrm{in}(E)$ with $(f, af) \in E$ for some $a \in K^* \backslash \{1\}$. By the assumption, there exists an element $(f, bg) \in E$ with $g \in A^{(m)}$, $b \in K^*$ and $g < f$. It follows that $(g, ag) \in E$, which contradicts the minimality of f.

(b)$\Longrightarrow$(a). Suppose there exists an element $f \in \overline{\mathrm{in}}(E) \backslash \mathrm{in}(E)$. This implies that $(f, af) \in E$ for some $a \in K^* \backslash \{1\}$, i.e., a stabilizes the image of f in $B^{(m)}/E$, which is a contradiction. ∎

1.4. The $\mathbf{F}_1$-algebra of terms.

1.4.1. Proposition. *Let A be an $\mathbf{F}_1$-algebra, B the A-algebra of terms $A[T_i]_{i \in I}$, and $\mathcal{P}(I)$ the set of all subsets of I. Then there is an isomorphism of partially ordered sets* $\mathrm{Zspec}(A) \times \mathcal{P}(I) \stackrel{\sim}{\to} \mathrm{Zspec}(B) : (\mathfrak{p}, J) \mapsto \mathfrak{p}_J$.

Proof. For a Zariski primes ideal $\mathfrak{p} \subset A$ and a subset $J \subset I$, let $\mathfrak{p}_J$ be the Zariski ideal generated by $\mathfrak{p}$ and the elements T_i for $i \notin J$. It is a Zariski prime ideal, and $B/\mathfrak{p}_J \stackrel{\sim}{\to} A/\mathfrak{p}[T_i]_{i \in J}$. (For example, $(\mathbf{m}_A)_\emptyset = \mathbf{m}_B$, and $(0)_I = (0)$.) The map $(\mathfrak{p}, J) \mapsto \mathfrak{p}_J$ is evidently injective and preserves the partial orderings of both sets. Let now $\mathfrak{q}$ be a nonzero Zariski prime ideal of B, and let $\mathfrak{p} = \mathfrak{q} \cap A$ and $J = \{i \big| T_i \notin \mathfrak{q}\}$. *We claim that* $\mathfrak{q} = \mathfrak{p}_J$. Indeed, it is clear that $\mathfrak{p}_J \subset \mathfrak{q}$. Assume that $fT^\nu = fT_{i_1}^{\nu_1} \cdot \ldots \cdot T_{i_n}^{\nu_n} \in \mathfrak{q}$, where $f \in A \backslash \mathfrak{p}$ and $\nu_j \geq 0$. Then there exists $1 \leq k \leq n$ with $\nu_k \geq 1$ and $T_{i_k} \in \mathfrak{q}$. By the definition of J, one has $i_k \notin J$ and, therefore, $fT^\nu \in \mathfrak{p}_J$, i.e., $\mathfrak{q} \subset \mathfrak{p}_J$. ∎

1.4.2. Corollary. *If A is an $\mathbf{F}_1$-algebra whose Zariski spectrum is finite, then the Zariski spectrum of any finitely generated A-algebra is finite.* ∎

For example, let A be an $\mathbf{F}_1$-algebra finitely generated over an $\mathbf{F}_1$-field K. Every surjective homomorphism $\varphi : K[T_1, \ldots, T_n] \to A$ gives rise to an isomorphism between the partially ordered set $\mathrm{Zspec}(A)$ and a subset $\mathcal{I}(A) \subset \mathcal{P}(\{1, \ldots, n\})$ which is preserved under intersections. Namely, for a Zariski prime ideal $\mathfrak{p} \subset A$ let $I_\mathfrak{p}$ be the subset of $\{1, \ldots, n\}$ for which $z\varphi^{-1}(\mathfrak{p})$ is generated by T_i with $i \notin I_\mathfrak{p}$. One evidently has $\mathfrak{p} \subset \mathfrak{q}$ if and only if $I_\mathfrak{q} \subset I_\mathfrak{p}$ and $I_{\mathfrak{p} \cup \mathfrak{q}} = I_\mathfrak{p} \cap I_\mathfrak{q}$, and so $\mathrm{Zspec}(A)$ is identified with the set $\mathcal{I}(A) = \{I_\mathfrak{p} \big| \mathfrak{p} \in \mathrm{Zspec}(A)\}$. We are now going to deduce from the above observation an additional property of the Zariski spectra.

We say that a poset X is *sup-complete* if it possesses the following property: if every finite subset of a set $S \subset X$ has the supremum, then S itself has the supremum.

1.4.3. Proposition. *The Zariski spectrum* $\mathrm{Zspec}(A)$ *of any* $\mathbf{F}_1$*-algebra* A *is a sup-complete inf-poset.*

First of all, we remark the following fact which easily follows from the definition of the spectra.

1.4.4. Lemma. *Suppose that* A *is a filtered inductive limit* $\varinjlim A_i$ *of a family of* $\mathbf{F}_1$*-algebras* $\{A_i\}_{i\in I}$*. Then there are canonical bijections* $\mathrm{Zspec}(A) \stackrel{\sim}{\to} \varprojlim \mathrm{Zspec}(A_i)$ *and* $\mathrm{Fspec}(A) \stackrel{\sim}{\to} \varprojlim \mathrm{Fspec}(A_i)$. ∎

Proof of Proposition 1.4.3. That $\mathrm{Zspec}(A)$ is an inf-poset was already noticed in §1.2. This implies that, to prove the statement, it suffices to show if, a subset $S \subset \mathrm{Zspec}(A)$ possesses the property that the intersection of every finite family of elements from S contains a Zariski prime ideal, then the intersection of all elements from S contains a Zariski prime ideal. Let $\{A_i\}_{i\in I}$ be the filtered family of $\mathbf{F}_1$-subalgebras of A which are finitely generated over $\mathbf{F}_1$. By Corollary 1.4.2, the Zariski spectra $\mathrm{Zspec}(A)$ are finite sets. The assumption implies that the image of S in $\mathrm{Zspec}(A_i)$ has the supremum $\mathfrak{p}_i \in \mathrm{Zspec}(A_i)$. If $j \geq i$, then the preimage $\mathfrak{p}_{ji}$ of $\mathfrak{p}_j$ in A_i lies in $\mathfrak{p}_i$, i.e., $\mathfrak{p}_i \leq \mathfrak{p}_{ji}$. Using again the finiteness of $\mathrm{Zspec}(A)$, we can find such $j \geq i$ that, for every $k \geq j$, one has $\mathfrak{p}_{ki} = \mathfrak{p}_{ji}$. We denote the latter Zariski prime ideal of A_i by $\mathfrak{q}_i$. Then for every pair $j \geq i$ the preimage of $\mathfrak{q}_j$ in A_i coincides with $\mathfrak{q}_i$. By Lemma 1.4.4, the tuple $\{\mathfrak{q}_i\}_{i\in I}$ defines a Zariski prime ideal $\mathfrak{q} \subset A$ which has the property that $\mathfrak{p} \leq \mathfrak{q}$ for all $\mathfrak{p} \in S$. It follows that the set S has the supremum. ∎

Let now A be the ring of integers of a finite extension of $\mathbf{Q}$. Every nonzero prime ideal $\mathfrak{p} \subset A$ is a Zariski prime ideal of the $\mathbf{F}_1$-algebra $A^{\cdot}$ with $\kappa(\mathfrak{p}) = U_{\mathfrak{p}} \cup \{0\}$, where $U_{\mathfrak{p}}$ is the group of units of $A_{\mathfrak{p}}$, the localization of A with respect to the complement of $\mathfrak{p}$. The image of $\mathfrak{p}$ under the canonical map $\mathrm{Spec}(A) \to \mathrm{Fspec}(A^{\cdot})$ corresponds to the subgroup $U^1_{\mathfrak{p}} = \{a \in U_{\mathfrak{p}} \big| a \equiv 1 (\mathrm{mod}\ \mathfrak{p})\}$. Furthermore, the union $\mathfrak{p}_S$ of any set S of nonzero prime ideals of A is a Zariski prime ideal of $A^{\cdot}$, and one has $\kappa(\mathfrak{p}) = U_S \cup \{0\}$, where U_S is the group of units of the localization of A with respect to the complement of $\mathfrak{p}_S$.

1.4.5. Proposition. *In the above situation, each nonzero Zariski prime ideal of* $A^{\cdot}$ *is of the form* $\mathfrak{p}_S$ *for some set* S *of prime ideals of* A*.*

1.4.6. Lemma. *Let* A *be an* $\mathbf{F}_1$*-algebra, and let* B *be an* $\mathbf{F}_1$*-algebra that contains* A *and such that, for every element* $g \in B$*, there exists* $n \geq 1$ *with* $g^n \in A$*. Then* $\mathrm{Zspec}(B) \stackrel{\sim}{\to} \mathrm{Zspec}(A)$*.*

Proof. Let $\mathfrak{p}$ is a Zariski prime ideal of A. Then $\mathfrak{q} = \{g \in B \big| g^n \in \mathfrak{p}$ for some $n \geq 1\}$ is a Zariski prime ideal of B. Indeed, if $gh \in \mathfrak{q}$, then there exists $n \geq 1$ with $g^n, h^n \in A$ and $g^n h^n \in \mathfrak{p}$. It follows that either $g^n \in \mathfrak{p}$, i.e., $g \in \mathfrak{q}$, or $h^n \in \mathfrak{p}$, i.e., $h \in \mathfrak{q}$. Thus, the map considered is surjective. Let now $\mathfrak{q}'$ be another Zariski prime ideal of B over $\mathfrak{p}$. If $g \in \mathfrak{q}$, then $g^n \in \mathfrak{p} \subset \mathfrak{q}'$ for some $n \geq 1$ and, therefore, $\mathfrak{q} \subset \mathfrak{q}'$. On the other hand, if $g \in \mathfrak{q}'$, then $g^n \in \mathfrak{q}' \cap A = \mathfrak{p}$ for some $n \geq 1$ and, therefore, $g \in \mathfrak{p}$, i.e., $\mathfrak{q}' \subset \mathfrak{q}$. ∎

Proof of Proposition 1.4.5. Since the class number of A is finite, there exists $n \geq 1$ such that n-th power $\mathfrak{p}^n$ of every prime ideal of A is a principal ideal. We fix its generator $f_\mathfrak{p}$. Let K be the $\mathbf{F}_1$-field $A^* \cup \{0\}$. Then there is an injective homomorphism of $\mathbf{F}_1$-algebras $B = K[T_\mathfrak{p}]_\mathfrak{p} \to A^\cdot$ that takes $T_\mathfrak{p}$ to the element $f_\mathfrak{p}$. Since A is a Dedekind ring and its class number is finite, it follows that $g^n \in B$ for all elements $g \in A$. Lemma 1.4.6 implies that $\mathrm{Zspec}(A^\cdot) \overset{\sim}{\to} \mathrm{Zspec}(B)$, and the required fact follows from Proposition 1.4.1. ∎

1.5. Finitely generated integral K-algebras. Let K be an $\mathbf{F}_1$-field, and let A be a finitely generated integral K-algebra. It can be considered as a K-subalgebra of its fraction $\mathbf{F}_1$-field L. To relate $\mathrm{Zspec}(A)$ to a familiar object, consider the finitely generated abelian group L^*/K^* as an additive group N. The cone C of $N_\mathbf{R} = N \otimes_\mathbf{Z} \mathbf{R}$ generated by the image of $\check{A}$ is a rational convex polyhedral cone. Let $\mathrm{face}(C)$ denote the set of faces of C. It is a partially ordered set with respect to the inclusion relation which admits the infimum of every pair of elements.

1.5.1. Proposition. *For every Zariski prime ideal $\mathfrak{p} \subset A$, the cone $F_\mathfrak{p}$ of $N_\mathbf{R}$ generated by the image of $A \backslash \mathfrak{p}$ is a face of C, and the correspondence $\mathfrak{p} \mapsto F_\mathfrak{p}$ gives rise to an isomorphism of partially ordered sets* $\mathrm{Zspec}(A) \overset{\sim}{\to} \mathrm{face}(C) : \mathfrak{p} \mapsto F_\mathfrak{p}$.

Proof. Since $\mathrm{Zspec}(A/K^*) \overset{\sim}{\to} \mathrm{Zspec}(A)$ and L/K^* is the fraction $\mathbf{F}_1$-field of the integral $\mathbf{F}_1$-algebra A/K^*, we can replace A by A/K^* and assume that $K = \mathbf{F}_1$. In this case the monoid $\check{A}$ is finitely generated and, therefore, the same is true for the monoid $\check{B} = \{\lambda \in L^* \big| \lambda^n \in \check{A}$ for some $n \geq 1\}$ (the saturation of $\check{A}$). It follows that $B = \check{B} \cup \{0\}$ is also a finitely generated integral $\mathbf{F}_1$-algebra and, by Lemma 1.4.6, $\mathrm{Zspec}(B) \overset{\sim}{\to} \mathrm{Zspec}(A)$. We can therefore replace A by B and assume that the monoid $\check{A}$ is saturated. Furthermore, since $\mathrm{Zspec}(A/A^*_{\mathrm{tors}}) \overset{\sim}{\to} \mathrm{Zspec}(A)$, we can replace A by A/A^*_{tors}, and so we may also assume that the group L^* has no torsion. In particular, we can identify the monoid $\check{A}$ with the monoid of integral points $C \cap N$ in the cone C. If now $\mathfrak{p}$ is a Zariski prime ideal of A, then the cone $F_\mathfrak{p}$ generated by the monoid $A \backslash \mathfrak{p}$ is a face of C. Since $A \backslash \mathfrak{p} = F_\mathfrak{p} \cap N$, it follows that the map $\mathfrak{p} \mapsto F_\mathfrak{p}$ is injective. On the other hand, if F is a face of

C, then $\mathfrak{p} = A \backslash F$ is a Zariski prime ideal of A with $F_\mathfrak{p} = F$, i.e., the above map is bijective. Since $\mathfrak{p} \subset \mathfrak{q}$ if and only if $F_\mathfrak{q} \subset F_\mathfrak{p}$, the required fact follows. ∎

The *Zariski-Krull dimension* of an $\mathbf{F}_1$-algebra A is the maximal length n of a chain of Zariski prime ideals $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ (with $\mathfrak{p}_i \neq \mathfrak{p}_{i+1}$ for $0 \leq i \leq n-1$). The *height* $\mathrm{ht}(\mathfrak{p})$ (resp. *depth* $\mathrm{dt}(\mathfrak{p})$) of a Zariski prime ideal $\mathfrak{p}$ is the maximal length n of a chain of Zariski prime ideals $\mathfrak{p}_0 = \mathfrak{p} \supset \mathfrak{p}_1 \supset \ldots \supset \mathfrak{p}_n$ (resp. $\mathfrak{p}_0 = \mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$). One evidently has $\mathrm{ht}(\mathfrak{p}) = \dim(A_\mathfrak{p})$ and $\mathrm{dt}(\mathfrak{p}) = \dim(A/\mathfrak{p})$.

1.5.2. Corollary. *In the situation of Proposition 1.5.1, the following is true:*

(i) for every Zariski prime ideal $\mathfrak{p} \subset A$, one has $\dim(F_\mathfrak{p}) = \mathrm{dt}(\mathfrak{p}) + \mathrm{rk}(A^*/K^*)$;

(ii) if $n \geq 1$ (i.e., A is not an $\mathbf{F}_1$-field) and $f_i(A)$ denotes the number of Zariski prime ideals of depth i, then $\sum_{i=0}^{n} (-1)^i f_i(A) = 0$.

Here $\dim(F_\mathfrak{p})$ is the (topological) dimension of the face $F_\mathfrak{p}$, and $\mathrm{rk}(A^*/K^*)$ is the (rational) rank of the abelian group A^*/K^*.

Proof. The statement (i) is a direct consequence of Proposition 1.5.1, and (ii) is a consequence of the Euler relation for polytopes. ∎

1.6. Idempotent $\mathbf{F}_1$-algebras. Let I be an idempotent $\mathbf{F}_1$-algebra. By Example 1.1.3(iv), I can be considered as a poset. The restriction of the partial ordering to the subset of nonzero elements $\check{I}$ gives a poset with the following two properties: $\check{I}$ has a unique minimal element and every pair of elements $x, y \in \check{I}$, for which there exists $z \in S$ with $x, y \leq z$, has supremum $\sup(x, y)$. Conversely, any poset with the latter two properties can be considered as the subset of nonzero elements of an idempotent $\mathbf{F}_1$-algebra.

Let I be an idempotent $\mathbf{F}_1$-algebra. Then $\kappa(\mathfrak{p}) = \mathbf{F}_1$ for all Zariski prime ideals $\mathfrak{p}$ of I and, therefore, the canonical map $\mathrm{Fspec}(I) \to \mathrm{Zspec}(I)$ is a bijection. Let M be an I-module (e.g., $M = I$). For a Zariski prime ideal $\mathfrak{p} \subset I$, let $F_\mathfrak{p}$ denote the I-submodule of M generated by the prime ideal $\Pi_\mathfrak{p} = \{(e, f) \big|$ either $e, f \in \mathfrak{p}$, or $e, f \notin \mathfrak{p}\}$, i.e., $F_\mathfrak{p}$ is generated by pairs of the form (em, m) and $(fm, 0)$ with $m \in M$, $e \notin \mathfrak{p}$ and $f \in \mathfrak{p}$.

1.6.1. Lemma. *(i) $F_\mathfrak{p} = \{(m, n) \big|$ there exists $e \notin \mathfrak{p}$ with either $em = en$, or $em, en \in \mathfrak{p}M\}$;*

(ii) $\bigcap_{\mathfrak{p} \in \mathrm{Zspec}(I)} F_\mathfrak{p} = \Delta(M)$.

Proof. (i) The set on the right hand side is an I-submodule of M that contains the above generators of $F_\mathfrak{p}$ and, therefore, it contains $F_\mathfrak{p}$. If (m, n) is a pair with $em = en$ (resp. $em, en \in \mathfrak{p}M$)

for some $e \notin \mathfrak{p}$, then the inclusions $(m, em), (n, en) \in F_\mathfrak{p}$ imply that $(m, n) \in F_\mathfrak{p}$, i.e., the set on the right hand side is contained in $F_\mathfrak{p}$.

(ii) Let (m, n) be a pair from the intersection, and suppose $m \neq 0$. Let $\mathfrak{p}$ be the maximal Zariski ideal of I with $m \notin \mathfrak{p}M$. *We claim that $\mathfrak{p}$ is prime.* Indeed, assume that $fg \in \mathfrak{p}$ for some $f, g \notin \mathfrak{p}$. By the maximality of $\mathfrak{p}$, we have $m = fu = gv$ for some $u, v \in M$. It follows that $m = fm = gm$ and, therefore, $m = fgm \in \mathfrak{p}M$, which is a contradiction. Since $(m, n) \in F_\mathfrak{p}$, there exists $e \notin \mathfrak{p}$ with either $em = en$, or $em, en \in \mathfrak{p}M$. By the maximality of $\mathfrak{p}$ again, we have $m = eu$ for some $u \in M$ and, therefore, $em = eu = m \notin \mathfrak{p}M$. It follows that $en \notin \mathfrak{p}M$ and $m = em = en$. If $\mathfrak{q}$ is the similar Zariski prime ideal of I that corresponds to the element n, then $\mathfrak{q} \supset \mathfrak{p}$ and there exists an element $f \notin \mathfrak{q}M$ with $n = fn = fm$. It follows that $m = efm = efn = n$. ∎

1.6.2. Corollary. *Let A be an $\mathbf{F}_1$-algebra that contains I (and so $F_\mathfrak{p}$ is an ideal of A), E an ideal of I, and F the ideal of A generated by E. Then*

(i) $F_\mathfrak{p} \cap (I \times I) = \Pi_\mathfrak{p}$ and, in particular, the ideal $F_\mathfrak{p}$ is nontrivial;

(ii) $F = \bigcap_{E \subset \Pi_\mathfrak{p}} F_\mathfrak{p}$;

(iii) $E = F \cap (I \times I)$.

Proof. (i) Let $(f, g) \in F_\mathfrak{p} \cap (I \times I)$. By Lemma 1.6.1(i), one has there exists $e \notin \mathfrak{p}$ with either $fe = ge$, or $fe, ge \in \mathfrak{p}A$. In the latter case, one has $fe = ua$ and $ge = vb$ for some $u, v \in \mathfrak{p}$ and $a, b \in A$ and, therefore, $fe = feu \in \mathfrak{p}$ and $ge = gev \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime and does not contain e, it follows that $f, g \in \mathfrak{p}$, i.e., $(f, g) \in \Pi_\mathfrak{p}$. Assume therefore that $fe, ge \notin \mathfrak{p}A$. Then $fe = ge \notin \mathfrak{p}$ and, therefore, $f, g \notin \mathfrak{p}$, i.e., $(f, g) \in \Pi_\mathfrak{p}$.

(ii) The statement follows from Lemma 1.6.1(ii) applied to the I-module A/F.

(iii) By (ii), one has $E = \bigcap_{E \subset \Pi_\mathfrak{p}} \Pi_\mathfrak{p}$ and $F = \bigcap_{E \subset \Pi_\mathfrak{p}} F_\mathfrak{p}$ and, therefore, the required fact follows from (i). ∎

Furthermore, every element $e \in \check{I}$ defines the Zariski prime ideal $\mathfrak{p}_e = \{f \in \check{I} \big| f \not\leq e\}$ (it is the maximal Zariski ideal that does not contain the element e), and so we get an injective map $\check{I} \to \mathrm{Zspec}(I) : e \mapsto \mathfrak{p}_e$, which preserves the partial orderings of both sets. The prime ideal that corresponds to $\mathfrak{p}_e$ is denoted by Π_e (instead of $\Pi_{\mathfrak{p}_e}$), and one has $\Pi_e = \{(f, g) \in I \times I \big|$ either $f, g \leq e$, or $f, g \not\leq e\}$.

1.6.3. Lemma. *In the above situation, if $\check{I}$ is an inf-poset, then the map $\check{I} \to \mathrm{Zspec}(I)$ is an inf-map.*

A map of inf-posets is said to be an *inf-map* if it takes the infimum of a family of elements to the infimum of their images.

Proof. Let J be a subset of $\check{I}$, and set $f = \inf(J)$ and $\mathfrak{p} = \bigcup_{e \in J} \mathfrak{p}_e$. The inclusion $\mathfrak{p}_f \supset \mathfrak{p}$ is trivial. Suppose they do not coincide, i.e., there exists an element $g \in \mathfrak{p}_f \backslash \mathfrak{p}$. Then $\mathfrak{p}_g \supset \mathfrak{p}_e$ for all $e \in J$. This implies that $g \leq e$ for all $e \in J$ and, therefore, $g \leq f$. But the latter is impossible since $g \in \mathfrak{p}_f$. ∎

Every element $e \in I$ defines a map $\mathrm{Zspec}(I) \to \mathbf{F}_1 = \{0, 1\}$. If we consider $\{0, 1\}$ as a poset in which $0 < 1$, then the above map belongs to $\mathrm{Hom}_{\inf}(\mathrm{Zspec}(I), \{0, 1\})$ where, for inf-posets P and Q, $\mathrm{Hom}_{\inf}(P, Q)$ denotes the set of maps $P \to Q$ that commute with the partial orderings and take the infimum of any family of elements of P to the infimum of their images in Q. In this way we get an injective homomorphism of idempotent $\mathbf{F}_1$-algebras $I \hookrightarrow \mathrm{Hom}_{\inf}(\mathrm{Zspec}(I), \{0, 1\})$.

The latter idempotent $\mathbf{F}_1$-algebra can be described as follows. Let $\Sigma(I)$ denote the set $\mathrm{Zspec}(I) \cup \{0_\Sigma\}$ in which the image of $\mathfrak{p} \in \mathrm{Zspec}(I)$ is denoted by $\mathfrak{p}_\Sigma$. We provide $\Sigma(I)$ with multiplication as follows: if the intersection $\mathfrak{p} \cap \mathfrak{q}$ does not contain a Zariski prime ideal of I, then $\mathfrak{p}_\Sigma \cdot \mathfrak{q}_\Sigma = 0_\Sigma$ and, otherwise, $\mathfrak{p}_\Sigma \cdot \mathfrak{q}_\Sigma = \mathfrak{r}_\Sigma$, where $\mathfrak{r}$ is the maximal Zariski prime ideal lying in $\mathfrak{p} \cap \mathfrak{q}$. An element $\mathfrak{p}_\sigma \in \Sigma(I)$ defines a map $\varphi_\mathfrak{p} : \mathrm{Zspec}(I) \to \{0, 1\}$ that takes $\mathfrak{q} \in \mathrm{Zspec}(I)$ to 1, if $\mathfrak{p} \leq \mathfrak{q}$ (i.e., $\mathfrak{q} \subset \mathfrak{p}$) and to 0, otherwise. The correspondence $\mathfrak{p}_\Sigma \mapsto \varphi_\mathfrak{p}$ gives rise to a homomorphism of idempotent $\mathbf{F}_1$-algebras and if, for $\varphi \in \mathrm{Hom}_{\inf}(\mathrm{Zspec}(I), \{0, 1\})$, $\mathfrak{p}$ is the minimal element of $\mathrm{Zspec}(I)$ with $\varphi(\mathfrak{p}) = 1$, then $\varphi = \varphi_\mathfrak{p}$. Thus, there is a canonical isomorphism of idempotent $\mathbf{F}_1$-algebras $\Sigma(I) \xrightarrow{\sim} \mathrm{Hom}_{\inf}(\mathrm{Zspec}(I), \{0, 1\})$. The composition of the latter with the map $I \to \Sigma(I) : e \mapsto \mathfrak{p}_e$ gives rise to an injective homomorphism $I \hookrightarrow \mathrm{Hom}_{\inf}(\mathrm{Zspec}(I), \{0, 1\})$.

1.6.4. Lemma. *The following properties of an idempotent $\mathbf{F}_1$-algebra I are equivalent:*

(a) the poset $\check{I}$ is noetherian;

(b) $\check{I} \xrightarrow{\sim} \mathrm{Zspec}(I)$;

(c) $I \xrightarrow{\sim} \mathrm{Hom}_{\inf}(\mathrm{Zspec}(I), \{0, 1\})$.

A poset is called noetherian if any ascending sequence of elements stabilizes.

Proof. (a)$\Longrightarrow$(b). The assumption implies that any subset of $\check{I}$ has a maximal element. In particular, given a Zariski prime ideal $\mathfrak{p} \subset I$, there exists a maximal element e in the subset $I \backslash \mathfrak{p}$. It follows that e is a unique maximal element outside $\mathfrak{p}$ and that $\mathfrak{p}$ is a maximal Zariski ideal of I that does not contain e, i.e., $\mathfrak{p} = \mathfrak{p}_e$.

(b)$\Longrightarrow$(c). Let φ be a map of partially ordered sets $\mathrm{Zspec}(I) \to \{0, 1\}$ that commutes with the infimum operation, and assume that it is not identically zero. Since the partially ordered set $\mathrm{Zspec}(I)$ admits the infimum of any set of elements, there is a unique minimal element $\mathfrak{p} \in \mathrm{Zspec}(I)$

with $\varphi(\mathfrak{p}) = 1$. By the assumption, one has $\mathfrak{p} = \mathfrak{p}_e$ for some element $e \in \check{I}$, and it is easy to see that φ is precisely the map associated to the element e.

(c)$\Longrightarrow$(a). Let $e_1 \leq e_2 \leq \ldots$ be an ascending sequence of elements of $\check{I}$, and let $\mathfrak{p}$ be the maximal Zariski ideal of I with $e_i \notin \mathfrak{p}$ for all $i \geq 1$. Then the map $\varphi : \mathrm{Zspec}(I) \to \{0,1\}$, defined by $\varphi(\mathfrak{q}) = 1$, $\mathfrak{p} \leq \mathfrak{q}$ (i.e., $\mathfrak{q} \subset \mathfrak{p}$), and $\varphi(\mathfrak{q}) = 0$, otherwise, belongs to $\mathrm{Hom}_{\mathrm{inf}}(\mathrm{Zspec}(I), \{0,1\})$ and, therefore, it corresponds to an element $e \in \check{I}$, i.e., $\varphi(\mathfrak{q}) = 1$, if $e \notin \mathfrak{q}$, and $\varphi(\mathfrak{q}) = 0$, otherwise. It follows that $\mathfrak{p} = \mathfrak{p}_e$, i.e., e is a unique maximal element outside $\mathfrak{p}$. In particular, $e_i \leq e$ for all $i \geq 1$. If $e \neq e_i$ for all $i \geq 1$, then $e_i \notin \mathfrak{p} \cup Ie$ for all $i \geq 1$, which contradicts maximality of $\mathfrak{p}$. Thus, $e = e_i$ for some $i \geq 1$ and, therefore, the sequence stabilizes. ∎

Of course, if I is finite, the poset $\check{I}$ is noetherian. But the converse is not true in general. For example, this is not true for the idempotent $\mathbf{F}_1$-algebra $I = \{0, 1, e_1, e_2, \ldots\}$ with $e_i e_j = 0$ for $i \neq j$. An idempotent $\mathbf{F}_1$-algebra possessing the equivalent properties of Lemma 1.6.4 will be said to be *almost finite*.

Notice that a noetherian poset is an inf-poset if and only if it has a unique minimal element.

1.6.5. Corollary. *The correspondence $I \mapsto \check{I}$ gives rise to an anti-equivalence between the category of almost finite idempotent $\mathbf{F}_1$-algebras and the category of noetherian inf-posets (with inf-maps as morphisms).*

Proof. A homomorphism of almost finite idempotent $\mathbf{F}_1$-algebras $\varphi : I \to I'$ induces a map between their spectra $\mathrm{Zspec}(I') \to \mathrm{Zspec}(I)$, and Lemma 1.6.4 implies that the correspondence considered is a contravariant functor. The map $\alpha_\varphi : \check{I}' \to \check{I}$, induced by the latter map, takes an element $e' \in \check{I}'$ to the maximal element $e \in \check{I}$ with $\varphi(e) \leq e'$. Conversely, any inf-map $\alpha : \check{I}' \to \check{I}$ is induced by the homomorphism $\varphi_\alpha : I \to I'$ that takes an element $e \in \check{I}$ to zero, if there is no an element $e' \in \check{I}'$ with $e \leq \psi(e')$, and to the infimum of all $e' \in \check{I}'$ with $e \leq \psi(e')$, otherwise. It is easy to see that one has $\varphi_{\alpha_\varphi} = \varphi$ and $\alpha_{\varphi_\alpha} = \alpha$ and, in particular, the functor considered is fully faithful. Let now $\check{I}$ be a noetherian inf-poset. We introduce the structure of an idempotent $\mathbf{F}_1$-algebra on the set $I = \{0\} \cup \{e_i\}_{i \in \check{I}}$ as follows: $e_i e_j = 0$, if $\sup(i,j)$ does not exist in $\check{I}$, and $e_i e_j = e_{\sup(i,j)}$, otherwise. The multiplication operation defined in this way is associative and commutative, and the unit for it is the idempotent e_i for the minimal element i of $\check{I}$. The poset associated with this idempotent $\mathbf{F}_1$-algebra is the poset $\check{I}$. It follows that I is almost finite, and we get the required statement. ∎

§2. Commutative algebra of $\mathbf{F}_1$-algebras

2.1 Noetherian $\mathbf{F}_1$-algebras.

2.1.1. Definition. A module M over an $\mathbf{F}_1$-algebra A is said to be *noetherian* (resp. *Zariski noetherian*) if any increasing sequence of A-submodules (resp. Zariski A-submodules) of M stabilizes. If $M = A$, A is said to be *noetherian* (resp. *Zariski noetherian*).

An A-module M is noetherian (resp. Zariski noetherian) if and only if all of its A-submodules (resp. Zariski A-submodules) are finitely generated. If M is noetherian, then it is Zariski noetherian, but the converse is not true in general. For example, any $\mathbf{F}_1$-field K is Zariski noetherian but, if the group K^* is not finitely generated, K is not noetherian. The proof of the following analog of Hilbert Basis Theorem for Zariski ideals imitates the proof of the latter.

2.1.2. Proposition. *If an $\mathbf{F}_1$-algebra A is Zariski noetherian, then any finite module over a finitely generated A-algebra is Zariski noetherian.*

Proof. It suffices to prove the statement for the A-algebra $B = A[T]$. Every nonzero element $g \in B$ has the form fT^n with $f \in A$ and $n \geq 0$. The integer n is the *degree* of g, and the element f is the *initial coefficient* of g. Let $\mathbf{b}$ be an ideal of B. We construct as follows a sequence of elements $g_0, g_1, \dots$ of $\mathbf{b}$. First of all, g_0 is an element of $\mathbf{b}$ of minimal degree. Assuming that elements $g_0, \dots, g_n$ are already constructed and the ideal generated by them does not coincide with $\mathbf{b}$, we choose an element $g_{n+1} \in \mathbf{b} \backslash \cup_{i=0}^n g_i A$ of minimal degree. Consider the ideal $\mathbf{a}$ of A generated by the initial coefficients f_i of g_i. By the assumption, $\mathbf{a} = \cup_{i=1}^n f_i A$ for some $n \geq 0$, and *we claim that* $\mathbf{b} = \cup_{i=1}^n g_n B$. Indeed, if the latter is not true, then for the element $g_{n+1} \in \mathbf{b} \backslash \cup_{i=0}^n g_i B$ one has $f_{n+1} = f_i h$ for some $1 \leq i \leq n$ and $h \in A$. Since the degree of g_i is at most the degree of g_{n+1}, the latter implies that $g_{n+1} \in g_i B \subset \mathbf{b}$, which is a contradiction. ∎

2.1.3. Corollary. *Let A be a Zariski noetherian $\mathbf{F}_1$-algebra. Then for every Zariski ideal $\mathbf{a} \subset A$ and every finite A-module M one has*

$$\bigcap_{i=1}^{\infty} \mathbf{a}^i M = \{m \in M \big| m = am \ \text{ for some } \ a \in \mathbf{a}\} \ .$$

Proof. Let $a_1, \dots, a_n$ be generators of the ideal $\mathbf{a}$. The homomorphism of A-algebras $\varphi : A[T_1, \dots, T_n] \to A$ that takes T_i to a_i makes A an $A[T_1, \dots, T_n]$-algebra, and the homomorphism of A-modules $\varphi : M[T_1, \dots, T_n] \to M : T_i m \mapsto a_i m$ is in fact a homomorphism of finite $A[T_1, \dots, T_n]$-modules. If $m \in \bigcap_{i=1}^{\infty} \mathbf{a}^i M$, for every $i \geq 1$ there exists $F_i \in M[T_1, \dots, T_n]$ of degree i with $\varphi(F_i) = m$. By Proposition 2.1.2, the Zariski submodule of $M[T_1, \dots, T_n]$ generated by the F_i's is generated by $F_1, \dots, F_k$ for some $k \geq 1$. It follows that $F_{k+1} = GF_i$ for some $1 \leq i \leq k$ and $G \in A[T_1, \dots, T_n]$, and we get $m = \varphi(G)m$ with $\varphi(G) \in \mathbf{a}$. ∎

The proof of the following analog of Hilbert Basis Theorem for ideals (Proposition 2.1.6) is based on Proposition 2.1.2 and the idea of Gröbner bases (see [Eis, §15]), which was already used in the previous subsection. Let K be an $\mathbf{F}_1$-field.

2.1.4. Definition. A *K-vector subspace* of a K-vector space M is a K-submodule $E \subset M \times M$ possessing the following property: if $(f, \lambda f) \in E$ for $\lambda \in K$, then either $f = \lambda f$ or $f \in N_E$ (i.e., $(f, 0) \in E$).

In other words, a K-vector subspace is a K-submodule E with the property that for every element $f \notin N_E$ the stabilizer of f in K^* coincides with that of its image in M/E. For example, if M is a free K-vector space, the latter condition on E means that the quotient M/E is also a free K-vector space. Furthermore, the K-submodule E_N associated with any Zariski K-submodule $N \subset M$ is a K-vector subspace. The intersection of any family of K-vector subspaces is a K-vector subspace. If A is a K-algebra, its ideals which are K-vector subspaces are said to be *K-ideals*. For example, the ideals of $K[T_1, \ldots, T_n]$ that possess the equivalent properties of Corollary 1.3.7 are precisely K-ideals.

2.1.5. Lemma. *The following properties of a module M over a K-algebra A are equivalent:*

(a) every increasing sequence of A-submodules, which are K-vector subspaces, stabilizes;

(b) every A-submodule, which is a K-vector subspace, is finite.

Proof. First of all, we notice that the union of any increasing sequence of K-vector subspaces of a K-vector space is a K-vector subspace. This immediately gives the implication (b)$\Longrightarrow$(a). Assume that (a) is true. Since the A-submodules of M associated with Zariski A-submodules are K-vector subspaces, it follows that any such A-submodule is finite. Let now E be an arbitrary A-submodule of M, which is a K-vector subspace. By the above remark, N_E is is finite. If E is not finitely generated, we can find an increasing sequence of finitely generated A-submodules $E_1 \subset E_2 \subset \ldots$ which does not stabilize. Since N_E is finite, we can increase the A-submodules E_i and assume that $N_{E_i} = N_E$ for all $i \geq 1$. Then all E_i's are K-vector subspaces of M, and we get a contradiction. ∎

A module M over a K-algebra A possessing the equivalent properties of Lemma 2.1.5 is said to be *K-noetherian*. The K-algebra itself is said to be *K-noetherian* if it is K-noetherian as an A-module. If $K = \mathbf{F}_1$, this definition coincides with that introduced at the beginning of this subsection.

2.1.6. Proposition. *Any finite module over a finitely generated K-algebra is K-noetherian.*

In particular, any finitely generated K-algebra is K-noetherian.

Let us set $B = K[T_1, \dots, T_n]$ and $A = \mathbf{F}_1[T_1, \dots, T_n]$, and fix a monomial order on $A^{(m)}$ as in §1.3. A *Gröbner basis* of an B-submodule E of $B^{(m)}$ is a system of elements $(f_1, g_1), \dots, (f_k, g_k) \in E$ with $\mathrm{in}(f_i) \neq \mathrm{in}(g_i)$ for all $1 \leq i \leq k$ and such that the monomials $\max\{\mathrm{in}(f_1), \mathrm{in}(g_1)\}$, ..., $\max\{\mathrm{in}(f_k), \mathrm{in}(g_k)\}$ generate the Zariski A-submodule $\mathrm{in}(E)$ of $A^{(m)}$. By Proposition 2.1.2, all Zariski A-submodules of $A^{(m)}$ are finite and, therefore, every B-submodule of $B^{(m)}$ admits a Gröbner basis.

2.1.7. Lemma. *Let E be a B-submodule of $B^{(m)}$ which is a K-vector subspace. Then any Gröbner basis of E generates E.*

Proof. Let $(f_1, g_1), \dots, (f_k, g_k)$ be a Gröbner basis of a B-submodule E with $\mathrm{in}(f_i) > \mathrm{in}(g_i)$ for all $1 \leq i \leq m$, and let E' be the ideal generated by them. Multiplying each pair (f_i, g_i) by an elements of K, we may assume that $f_i = T^{\nu_i} e_{\sigma(i)} \in A$ for all $1 \leq i \leq m$ and some $1 \leq \sigma(i) \leq m$. If $E' \neq E$, we can find a pair $(f, g) \in E \backslash E'$ with $\mathrm{in}(f) \geq \mathrm{in}(g)$ and minimal $\mathrm{in}(f)$. Then $f \neq g$, and since E is a K-vector subspace, it follows that $\mathrm{in}(f) > \mathrm{in}(g)$. Multiplying (f, g) by an element of K, we may assume that $f = T^\mu e_j \in A$. By the assumption, the monomial $T^\mu e_j$ is divisible by $T^{\nu_i} e_{\sigma(i)}$ for some $1 \leq i \leq m$. (In particular, $e_j = e_{\sigma(i)}$.) It follows that $(f, g_i T^{\mu - \nu_i}) = (f_i T^{\mu - \nu_i}, g_i T^{\mu - \nu_i}) \in E'$ and, therefore, $(g, g_i T^{\mu - \nu_i}) \in E \backslash E'$. This contradicts the minimality of $\mathrm{in}(f)$. Indeed, we have $f > \mathrm{in}(g)$, and since $f_i = T^{\nu_i} e_j > \mathrm{in}(g_i)$, then $f = f_i T^{\mu - \nu_i} > \mathrm{in}(g_i) T^{\mu - \nu_i}$. ∎

Proof of Proposition 2.1.6. Lemma 2.1.7 and Proposition 2.1.2 imply the required fact in the case $K = \mathbf{F}_1$. We reduce the general case to this one as follows.

For a K-vector space M, let $\overline{M}$ denote the quotient M/K^* which is an $\mathbf{F}_1$-vector space, and, for an element $m \in M$, let $\overline{m}$ denote its image in $\overline{M}$. Furthermore, for a K-vector subspace E of M, let $\overline{E}$ denote the $\mathbf{F}_1$-vector subspace of $\overline{M}$ that consists of the pairs $(\overline{m}, \overline{n})$ with $(m, n) \in E$. *We claim that, if $E' \subset E''$ are K-vector subspaces of M such that $\overline{E}' = \overline{E}''$, then $E' = E''$.* Indeed, since $N_{E'}$ and $N_{E''}$ are the preimages of $N_{\overline{E}'}$ and $N_{\overline{E}''}$, respectively, it follows that $N_{E'} = N_{E''}$. Furthermore, let $(m, n) \in E'' \backslash (N_{E''} \times N_{E''})$. By the assumption, there exists an element $\lambda \in K^*$ with $(m, \lambda n) \in E'$. Since $E' \subset E''$, it follows that $(n, \lambda n) \in E''$. Since E'' is a K-vector subspace, we get $n = \lambda n$ and, therefore, $(m, n) \in E'$, i.e, the claim is true.

Let M be a finite module over a finitely generated K-algebra A, and assume we are given an increasing sequence $E_1 \subset E_2 \subset \dots$ of A-submodules which are K-vector subspaces of M. By Step 1, the finitely generated $\mathbf{F}_1$-algebra $\overline{A} = A/K^*$ and the finite $\overline{A}$-module $\overline{M} = M/K^*$ are noetherian. It follows that the sequence $\overline{E}_1 \subset \overline{E}_2 \subset \dots$ of $\overline{A}$-submodules of $\overline{M}$ stabilizes, i.e., there is $n \geq 1$ with $\overline{E}_n = \overline{E}_{n+1} = \ldots$. Step 2 implies that $E_n = E_{n+1} = \ldots$, i.e., M is K-noetherian. ∎

2.2. Radicals of $\mathbf{F}_1$-algebras and modules. Let A be an $\mathbf{F}_1$-algebra, and M an A-module (e.g., $M = A$).

2.2.1. Definition. (i) The the *Zariski annihilator* of a subset $N \subset M$ is the Zariski ideal $\mathrm{zann}(N) = \{f \in A \big| fm = 0 \text{ for all } m \in N\}$.

(ii) The *Zariski nilradical* of M is the Zariski ideal $\mathrm{z}\mathbf{n}(M) = \{f \in A \big| f^n \in \mathrm{zann}(M) \text{ for some } n \geq 1\}$.

(iii) The *Zariski radical* $\mathrm{z}\mathbf{r}(N) = \mathrm{z}\mathbf{r}_M(N)$ of a Zariski A-submodules $N \subset M$ is the Zariski ideal $\mathrm{z}\mathbf{n}(M/N)$. The *Zariski radical* $\mathrm{z}\mathbf{r}(E)$ of an A-submodule E is the Zariski ideal $\mathrm{z}\mathbf{n}(M/E)$.

For example, $\mathrm{zann}(A) = 0$, and $\mathrm{z}\mathbf{n}(A)$ is the set of nilpotent elements of A. An A-module is said to be *Zariski reduced* if $\mathrm{z}\mathbf{n}(M) = \mathrm{zann}(M)$. Furthermore, $\mathrm{z}\mathbf{r}_M(0) = \mathrm{z}\mathbf{n}(M)$.

2.2.2. Proposition. *One has* $\mathrm{z}\mathbf{n}(M) = \bigcap_{\mathfrak{p}} \mathfrak{p}$, *where* $\mathfrak{p}$ *runs through Zariski prime ideals that contain* $\mathrm{zann}(M)$*.*

Proof. Let f be an element of A which is not nilpotent at M. By Zorn's Lemma, there exists a Zariski ideal $\mathbf{a}$ maximal among those which do not intersect with the set $\{f^k\}_{k \geq 1}$. *We claim that* $\mathbf{a}$ *is prime.* For this we assume that $gh \in \mathbf{a}$ and that both g and h are not in $\mathbf{a}$. Then the Zariski ideals of A generated by $\mathbf{a}$ and g and h, respectively, contain some powers of f, i.e., $f^k = ag$ and $f^l = bh$ for some $k, l \geq 1$ and $a, b \in A$. It follows that $f^{k+l} = abgh \in \mathbf{a}$ which is a contradiction. It remains to show that every element $g \in \mathrm{zann}(M)$ lies in $\mathbf{a}$. If $g \notin \mathbf{a}$, then the maximality of $\mathbf{a}$ implies that $f^k = ga$ for some $k \geq 1$ and $a \in A$. It follows that f^k annihilates M, which is a contradiction. ∎

2.2.3. Corollary. *For a Zariski A-submodule $N \subset M$, one has* $\mathrm{z}\mathbf{r}(N) = \bigcap_{\mathfrak{p}} \mathfrak{p}$, *where* $\mathfrak{p}$ *runs through Zariski prime ideals that contain* $\mathrm{zann}(M/N)$*.* ∎

2.2.4. Definition. (i) The *annihilator* of a subset $N \subset M$ is the ideal

$$\mathrm{ann}(N) = \{(f, g) \in M \times M \big| fm = gm \text{ for all } m \in N\}.$$

(ii) The *nilradical* of M is the ideal

$$\mathbf{n}(M) = \{(f, g) \in M \times M \big| \text{ there exists } k \geq 1 \text{ with } f^i m = g^i m \text{ for all } i \geq k \text{ and } m \in M\} \ .$$

(iii) The *radical* $\mathbf{r}(E)$ of an A-submodule E of M is the preimage of $\mathbf{n}(M/E)$ in $A \times A$.

Notice that it is enough to require that the actions of f^i and g^i on M coincide for two successive values of i. Indeed, assume that $f^k m = g^k m$ and $f^{k+1} = g^{k+1} m$ for some $k \geq 0$ and all $m \in M$. Then $f^k gm = g^{k+1} m = f^{k+1} m$ and, therefore, $f^i gm = f^{i+1} m$ for all $i \geq k$. By induction on j, we get $f^i g^j m = f^{i+j} m$ for all $i \geq k$ and $j \geq 0$ and, by symmetry, we have $f^i g^j m = g^{i+j} m$ for all $i \geq 0$ and $j \geq k$. It follows that $f^i m = g^i m$ for all $i \geq 2k$. Notice also that $\mathbf{a}_{\mathrm{ann}(M)} = \mathrm{zann}(M)$ and $\mathbf{a}_{\mathbf{n}(M)} = \mathrm{z}\mathbf{n}(M)$.

An A-module M is said to be *reduced* if $\mathbf{n}(M) = \mathrm{ann}(M)$. For example, any integral $\mathbf{F}_1$-algebra A is reduced. Indeed, if for non-nilpotent elements f and g one has $f^i = g^i$ for all $i \geq n$, then $f \cdot f^i = f^{i+1} = g^{i+1} = g \cdot f^i$ and, therefore, $f = g$. Notice that for any Zariski prime ideal $\mathfrak{p}$ of an $\mathbf{F}_1$-algebra A the canonical map $\mathbf{n}(A) \to \mathbf{n}(A/\mathfrak{p})$ is surjective. In particular, if A is reduced, then the quotient $A/\mathfrak{p}$ is reduced for any Zariski prime ideal $\mathfrak{p} \subset A$.

2.2.4. Proposition. *One has* $\mathbf{n}(M) = \bigcap \Pi$, *where* Π *runs through prime ideals that contain* $\mathrm{ann}(M)$*; in particular, one has* $\mathbf{n}(A) = \bigcap \Pi_{\mathfrak{p}}$, *where* $\mathfrak{p}$ *run through Zariski prime ideals of* A.

Proof. We can replace A by $A/\mathrm{ann}(M)$ and assume that $\mathrm{ann}(M) = \Delta(A)$. It follows that $\mathbf{n}(M) = \mathbf{n}(A)$ and, therefore, it suffices to show that $\mathbf{n}(A) = \bigcap \Pi_{\mathfrak{p}}$, where $\mathfrak{p}$ run through Zariski prime ideals of A. That $\mathbf{n}(A)$ is contained in the intersection is trivial. Let (f, g) be an element from the intersection. If both f and g are contained in all Zariski prime ideals of A, then, by Proposition 2.2.2, they are nilpotent and, in particular, $(f, g) \in \mathbf{n}(A)$. Assume therefore that it is not the case. It is easy to see that the image of (f, g) in $A_f \times A_f$ is contained in the similar intersection for A_f. Since the image of f in A_f is invertible, it follows that the image of g in A_f is also invertible. Since $(f, g) \in \Pi_{\mathbf{m}_{A_f}}$, there exists an element $h \in (A_f)^*$ with $fh = gh$ in A_f and, therefore, the images of f and g in A_f coincide, i.e., there exists $m \geq 0$ with $f^m g = f^{m+1}$. By symmetry, there exists $n \geq 0$ with $fg^n = g^{n+1}$. It follows that $f^i = g^i$ for all $i \geq m + n$, i.e., $(f, g) \in \mathbf{n}(A)$. ∎

Proposition 2.2.4 implies that the radical of an ideal coincides with the intersection of all prime ideals that contain it. An ideal E of A is said to be *radical* if $E = \mathbf{r}(E)$.

For an $\mathbf{F}_1$-algebra A, let I_A denote the set of all idempotents in A. It is an idempotent $\mathbf{F}_1$-subalgebra of A (the *idempotent* $\mathbf{F}_1$*-subalgebra* of A).

2.2.5. Proposition. *Let* A *be an* $\mathbf{F}_1$*-algebra, and let* $B = A/\mathbf{n}(A)$. *Then* $I_A \xrightarrow{\sim} I_B$.

Proof. The injectivity of the map considered is trivial. Suppose that an element $f \in A$ represent an idempotent in B. This means that $(f^2, f) \in \mathbf{n}(A)$, i.e., there exists $n \geq 1$ with $f^{2i} = f^i$ for all $i \geq n$. *We claim that* $f^{n+i} = f^{n+1}$ *for all* $i \geq 1$. Indeed, if $i = 1$, there is nothing to prove, and so assume that $i \geq 2$ and that the claim is true for all smaller values of i. We have $f^{n+i+1} = f^{n+i-1} \cdot f^2 = f^{2(n+i-1)} \cdot f^2 = f^{2(n+i)} = f^{n+i}$, and so the claim is true. It follows that for the idempotent $e = f^{n+1}$, one has $e^i = f^i$ for all $i \geq i+1$, i.e., $(e, f) \in \mathbf{n}(A)$. ∎

2.3. Primary decomposition for Zariski ideals and Zariski modules. Let A be an $\mathbf{F}_1$-algebra, and M an A-module (e.g., $M = A$).

2.3.1. Definition. A Zariski A-submodule $N \subset M$ is said to be *primary* if it is nontrivial and possesses the property that, if $fm \in N$, then either $m \in N$ or $f \in \mathrm{z}\mathbf{r}(N)$.

If N is primary, the Zariski radical $\mathrm{z}\mathbf{r}(N)$ is a Zariski prime ideal $\mathfrak{p}$ and N is said to be $\mathfrak{p}$-*primary*. For example, any Zariski A-submodule $N \subset M$ with $\mathrm{z}\mathbf{r}(N) = \mathbf{m}_A$ is primary. Notice that the intersection of two Zariski $\mathfrak{p}$-primary A-submodules is a Zariski $\mathfrak{p}$-primary A-submodule.

A Zariski A-submodule $N \subset M$ is said to be *decomposable* if it admits a *primary decomposition*, i.e., a representation in the form $\bigcap_{i=1}^k N_i$, where N_i are primary Zariski A-submodules. A primary decomposition $N = \bigcap_{i=1}^k N_i$ is said to be *minimal* if all of the Zariski prime ideals $\mathfrak{p}_i = \mathrm{z}\mathbf{r}(N_i)$ are pairwise distinct and, for every $1 \leq i \leq k$, $\bigcap_{j \neq i} N_j \not\subset N_i$. Notice that every decomposable Zariski A-submodule admits a minimal primary decomposition.

For a Zariski A-submodule $P \subset M$ and a subset $Q \subset M$, one denotes by $(P : Q)$ the Zariski ideal $\{f \in A \big| fQ \subset P\}$. For example, $(0 : Q) = \mathrm{zann}(Q)$.

2.3.2. Proposition *(The first uniqueness theorem). Let N be a decomposable Zariski A-submodule provided with a minimal primary decomposition $\bigcap_{i=1}^n N_i$, and let $\mathfrak{p}_i = \mathrm{z}\mathbf{r}(N_i)$. Then $\{\mathfrak{p}_i\}_{1 \leq i \leq n}$ coincides with the set of Zariski prime ideals of the form $\mathrm{z}\mathbf{r}(N : m)$ with $m \in M$. In particular, the set $\{\mathfrak{p}_i\}_{1 \leq i \leq n}$ does not depend on the choice of the minimal primary decomposition.*

2.3.3. Lemma. *(i) $(P : Q) = A$ if and only if $Q \subset P$;*

(ii) $(\bigcap_{i=1}^n P_i : Q) = \bigcap_{i=1}^n (P_i : Q)$;

(iii) if P is $\mathfrak{p}$-primary and $Q \not\subset P$, then $(P : Q)$ is a Zariski $\mathfrak{p}$-primary ideal of A.

Proof. (i) One has $(P : Q) = A$ if and only if $1 \in (P : Q)$, i.e., $Q \subset P$.

(ii) One has $f \in (\bigcap_{i=1}^n P_i : Q)$ if and only if $fQ \subset \bigcap_{i=1}^n P_i$. The latter is obviously equivalent to the inclusion $f \in \bigcap_{i=1}^n (P_i : Q)$.

(iii) *We claim that* $\mathrm{z}\mathbf{r}(P : Q) = \mathfrak{p}$. Indeed, the inclusion $\mathfrak{p} \subset \mathrm{z}\mathbf{r}(P : Q)$ is trivial. Suppose that $f \in \mathrm{z}\mathbf{r}(P : Q)$, i.e., there is $k \geq 1$ with $f^k m \in P$ for all $m \in Q$. If $m \in Q \backslash P$, the latter inclusion implies that $(f^k)^l \in \mathrm{z}\mathbf{r}(P) = \mathfrak{p}$ for some $l \geq 1$, and the claim follows. Assume now that

$fg \in (P : Q)$, i.e., $fgm \in P$ for all $m \in Q$. If $g \notin \mathfrak{p}$, then $fm \in P$ for all $m \in Q$ and, therefore, $f \in (P : Q)$. ∎

Proof of Proposition 2.3.2. By Lemma 2.3.3, one has $\mathrm{z}\mathbf{r}(N : m) = \cap_{m \notin N_i} \mathfrak{p}_i$ for every element $m \in M$. If the intersection is a Zariski prime ideal, Lemma 1.2.7(i) implies that it coincides with some $\mathfrak{p}_i$. Conversely, if $m \in (\bigcap_{j \neq i} N_j) \backslash N_i$, then $\mathrm{z}\mathbf{r}(N : m) = \mathfrak{p}_i$, and the required statement follows. ∎

In the situation of Proposition 2.3.2, $\mathfrak{p}_i$ are said to be the Zariski prime ideals *associated to* N, the minimal (resp. non-minimal) elements of $\{\mathfrak{p}_1, \dots, \mathfrak{p}_n\}$ are said to be the *isolated* (resp. *embedded*) Zariski prime ideals associated to N.

2.3.4. Corollary. *Suppose that the zero ideal of A is decomposable, and let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ (resp. $\mathfrak{p}_{m+1}, \dots, \mathfrak{p}_n$) be the associated isolated (resp. embedded) Zariski prime ideals. Then*

(i) $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ are precisely the minimal Zariski prime ideals of A;

(ii) the set of zero divisors in A coincides with $\bigcup_{i=1}^n \mathfrak{p}_i$;

(iii) if $f \notin \bigcup_{i=1}^m \mathfrak{p}_i$ and $fg = 0$, then $g \in \mathrm{z}\mathbf{n}(A)$.

Proof. (i) Let $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ be the isolated ideals. Then $\mathrm{z}\mathbf{n}(A) = \bigcap_{i=1}^m \mathfrak{p}_i$. If $\mathfrak{p}$ is a Zariski prime ideal of A, then it contains the latter intersection and, therefore, it contains $\mathfrak{p}_i$ for some $1 \leq i \leq m$.

(ii) The set of zero divisors of A is a Zariski ideal which coincides with the union $\bigcup_{f \neq 0} (0 : f)$. On the other hand, it coincides with its own radical, i.e., with the union $\bigcup_{f \neq 0} \mathrm{z}\mathbf{r}(0 : f)$. By Proposition 2.3.2, the Zariski ideals $\mathfrak{p}_i$ are among sets in the union. Since $\mathrm{z}\mathbf{r}(0 : f) = \bigcap_{f \notin \mathfrak{p}_i} \mathfrak{p}_i$, the required fact follows.

(iii) If $g \notin \mathrm{z}\mathbf{n}(A)$, (i) implies that $g \notin \mathfrak{p}_i$ for some $1 \leq i \leq m$. Since $f \notin \mathfrak{p}_i$, it follows that $fg \notin \mathfrak{p}_i$ which contradicts the assumption $fg = 0$. ∎

2.3.5. Corollary. *Let $\varphi : A \to B$ be a homomorphism of $\mathbf{F}_1$-algebras, and suppose that* $\mathrm{Zspec}(A)$ *is finite. Then a Zariski prime ideal $\mathfrak{p} \subset A$ lies in the image of* $\mathrm{Zspec}(B)$ *if and only if* $\varphi^{-1}(\mathfrak{p}B) = \mathfrak{p}$.

Proof. The direct implication is trivial. To prove the converse implication, we can replace A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$, and so we may assume that A has no zero divisors, $\mathfrak{p} = 0$ and the homomorphism φ is Zariski injective, i.e., $\mathrm{Zker}(\varphi) = 0$. Consider first the case when B is finitely generated over A. By Corollary 1.4.2, the Zariski spectrum of B is finite and, in particular, the Zariski nilradical $\mathrm{z}\mathbf{n}(B)$ is decomposable, i.e., $\mathrm{z}\mathbf{n}(B) = \bigcap_{i=1}^n \mathfrak{q}_i$ for some Zariski prime ideals of B. Since φ is Zariski injective, it follows that $\bigcap_{i=1}^n \varphi^{-1}(\mathfrak{q}_i) = 0$. Corollary 2.3.4 implies that at least one of the Zariski prime ideals $\varphi^{-1}(\mathfrak{q}_i)$ should coincide with 0. In the general case, let

$\{B_i\}_{i\in I}$ be the filtered system of finitely generated A-subalgebras of B. By the previous case, for every $i \in I$ the finite set of Zariski prime ideals $\mathfrak{q} \subset B_i$ with $\varphi^{-1}(\mathfrak{q}) = 0$ is nonempty. Since $\mathrm{Zspec}(B) \xrightarrow{\sim} \varprojlim \mathrm{Zspec}(B_i)$, it follows that there is a Zariski prime ideal $\mathfrak{q} \subset B$ with $\varphi^{-1}(\mathfrak{q}) = 0$. ■

2.3.5. Proposition *(The second uniqueness theorem). In the situation of Proposition 2.3.2, let $\mathfrak{p}$ be a Zapiski prime ideal of the form $\mathfrak{p}_{i_1} \cup \ldots \cup \mathfrak{p}_{i_k}$, and set $N^{(\mathfrak{p})} = \bigcap_{\mathfrak{p}_i \subset \mathfrak{p}} N_i$. Then $N^{(\mathfrak{p})} = \{m \in M \big| fm \in N$ for some $f \in A \backslash \mathfrak{p}\}$. In particular, the Zariski A-submodules N_i that correspond to the isolated Zariski prime ideals associated to N are determined by N.*

Proof. Assume first that $m \in N^{(\mathfrak{p})}$. We pick up an element $f_j \in \mathfrak{p}_j \backslash \mathfrak{p}$ for every j with $\mathfrak{p}_j \not\subset \mathfrak{p}$, and denote by f their product. Then we can replace f by its power so that $fm \in N_j$ for all j with $\mathfrak{p}_j \not\subset \mathfrak{p}$. It follows that $fm \in N$. Conversely, assume that $fm \in N$ for some $f \in A \backslash \mathfrak{p}$. Since for every i with $\mathfrak{p}_i \subset \mathfrak{p}$ one has $fm \in N_i$ and $f \notin \mathfrak{p}_i$, it follows that $m \in N_i$, i.e., $m \in N^{(\mathfrak{p})}$. ■

2.3.6. Definition. An A-module M is said to be *Zariski decomposable* if the zero Zariski A-submodule of M is decomposable. In this case, the corresponding associated Zariski prime ideals will be said to be *Zariski associated to* M, and their set is denoted by $\mathrm{Zass}(M) = \mathrm{Zass}_A(M)$.

2.3.7. Lemma. *If M is a Zariski decomposable A-module, then for any sub-semigroup $S \subset A$ the $S^{-1}A$-module $S^{-1}M$ is also Zariski decomposable and* $\mathrm{Zass}(S^{-1}M) = \mathrm{Zass}(M) \cap \mathrm{Zspec}(S^{-1}A)$.

Proof. Let $\mathfrak{p}$ be a Zariski prime ideal of A with $\mathfrak{p} \cap S = \emptyset$. *We claim that, if N is a $\mathfrak{p}$-primary Zariski A-submodule of M, then $S^{-1}N$ is a $S^{-1}\mathfrak{p}$-primary Zariski $S^{-1}A$-submodule of $S^{-1}M$.* Indeed, suppose that $\frac{f}{s} \cdot \frac{m}{t} \in S^{-1}N$, where $s, t \in S$. It follows that $\alpha fm \in N$ for some $\alpha \in S$ and, therefore, one has either $m \in N$ or $\alpha f \in \mathfrak{p}$. Since $\mathfrak{p} \cap S = \emptyset$, the latter inclusion implies that $f \in \mathfrak{p}$, and the claim follows. Let $0 = \bigcap_{i=1}^k N_i$ be a minimal primary decomposition with $\mathfrak{p}_i = \mathrm{z}\mathbf{r}(N_i)$, and suppose that $\mathfrak{p}_i \cap S = \emptyset$ for only $1 \le i \le l$. It follows that $0 = \bigcap_{i=1}^l S^{-1}N_i$ and, by the above claim, each $S^{-1}N_i$ is $S^{-1}\mathfrak{p}_i$-primary. In particular, the Zariski $S^{-1}A$-module $S^{-1}M$ is decomposable and $\mathrm{Zass}(S^{-1}M) \subset \mathrm{Zass}(M) \cap \mathrm{Zspec}(S^{-1}A)$. Finally, by Proposition 2.3.1, each $\mathfrak{p}_i$ is of the form $\mathrm{z}\mathbf{r}(0:m)$ for some $m \in M$. This easily implies that $\mathrm{z}\mathbf{r}_{S^{-1}A}(0:m) = S^{-1}\mathfrak{p}_i$, and the converse inclusion follows. ■

2.3.8. Proposition. *Let A be a Zariski noetherian $\mathbf{F}_1$-algebra, and M a Zariski noetherian A-module. Then*

(i) M is Zariski decomposable;

(ii) $\mathrm{Zass}(M)$ *coincides with the set of Zariski prime ideals of the form* $\mathrm{zann}(m)$ *with $m \in M$.*

Proof. (i) A Zariski A-submodule $N \subset M$ is said to be *irreducible*, if it possesses the property that if $N = P \cap Q$ then either $N = P$ or $N = Q$. To prove (i), it suffices to verify the following two facts:

(1) every Zariski A-submodule is the intersection of a finite number of irreducible Zariski A-submodules;

(2) every irreducible Zariski A-submodule is primary.

(1) If the statement is not true, then the set of all Zariski A-submodules for which it does not hold is nonempty and, therefore, contains a maximal element N. Since N is not irreducible, there exist Zariski A-submodules P and Q different of N and such that $N = P \cap Q$. Then P and Q can be represented as intersections of finite sets of irreducible Zariski A-submodules, and so the same is true for N, which is a contradiction.

(2) If N is an irreducible Zariski A-submodule, then replacing M by M/N, we may assume that $N = 0$. Assume that $fm = 0$ and $m \neq 0$. For $k \geq 1$, let P_k denote the Zariski A-submodule of M that consists of the elements n with $f^k n = 0$. The chain of Zariski A-submodules $P_1 \subset P_2 \subset \dots$ stabilizes and, therefore, $P_k = P_{k+1}$ for some $k \geq 1$. *We claim that* $f^k M \cap Am = 0$. Indeed, if $f^k n = gm$ for some $n \in M$ and $g \in A$, then $f^{k+1} n = gfm = 0$, i.e., $n \in P_{k+1}$. We get $n \in P_k$, i.e., $f^k n = 0$, and the claim follows. Since the zero ideal is irreducible, it follows that $f^k M = 0$.

(ii) First of all, *we claim that every Zariski ideal maximal among nontrivial Zariski ideals of the form* $\mathrm{zann}(m) = (0 : m)$ *with* $m \in M$ *is Zariski prime.* Indeed, let $\mathrm{zann}(m)$ be such a Zariski ideal, and suppose that $fg \in \mathrm{zann}(m)$ and $g \notin \mathrm{zann}(m)$, i.e., $fgm = 0$ and $gm \neq 0$. Then $fgm = 0$ and, therefore, $Af \cup \mathfrak{p} \subset \mathrm{zann}(gm)$. The maximality of $\mathrm{zann}(m)$ implies that $f \in \mathrm{zann}(m)$, and the claim follows. Let now $\mathfrak{p} \in \mathrm{Zass}(M)$. To prove the required property of $\mathfrak{p}$, consider first the case when $\mathfrak{p}$ is the maximal Zariski ideal of A, i.e., $A \backslash \mathfrak{p} = A^*$. By Proposition 2.3.2, one has $\mathfrak{p} = \mathrm{z}\mathbf{r}(0 : m)$ for some $m \in M$. Since A is Zariski noetherian, the above claim implies that there is a Zariski prime ideal $\mathfrak{q}$ of the form $\mathrm{zann}(n)$ with $n \in M$ that contains $\mathrm{zann}(m)$. For the same reason, there is $k \geq 1$ with $\mathfrak{p}^k \subset \mathrm{zann}(m)$ and, since $\mathfrak{p}$ is the maximal Zariski ideal of A, it follows that $\mathfrak{p} = \mathfrak{q}$. In the general case, we can use Lemma 2.3.7 and apply the previous case to the localizations $A_\mathfrak{p}$ and $M_\mathfrak{p}$. It follows that there is an element $\frac{m}{s} \in M_\mathfrak{p}$ such that $\mathfrak{p}A_\mathfrak{p} = \mathrm{zann}(\frac{m}{s})$. This easily implies that $\mathfrak{p} = \mathrm{zann}(m)$. ∎

2.4. Primary decomposition for ideals and submodules. Let A be an $\mathbf{F}_1$-algebra and M an A-module (e.g., $M = A$).

2.4.1. Definition. (i) An A-submodule E of M is said to be *primary* if it is nontrivial and possesses the property that, if $(fm, fn) \in E$, then either $(m, n) \in E$ or $f \in \mathrm{z}\mathbf{r}(E) = \mathrm{z}\mathbf{n}(M/E)$.

(ii) An A-module M is said to be *quasi-integral* if the minimal A-submodule $\Delta(M)$ is primary.

An A-submodule E of an A-module M is primary if and only if the A-module M/E is quasi-integral. If E is a primary A-submodule of M, then $\mathrm{ann}(M/E)$ is a primary ideal of A. In this case, the radical and the Zariski radical of the latter, which coincide with $\mathbf{r}(E)$ and $\mathrm{z}\mathbf{r}(E)$, are a prime ideal Π and a Zariski prime ideal $\mathfrak{p}$, respectively, and E is said to be Π-*primary* or $\mathfrak{p}$-*primary.* If E is a $\mathfrak{p}$-primary A-submodule, then $\mathrm{ann}(M/E)$ is a $\mathfrak{p}$-primary ideal. Notice that, given a finite system of $\mathfrak{p}$-primary A-submodules $\{E_i\}_{i\in I}$ with $\Pi_i = \mathbf{r}(E_i)$, the intersection $\bigcap_{i\in I} E_i$ is a Π-primary A-submodule with $\Pi = \bigcap_{i\in I} \Pi_i$. Notice also that an A-module M is primary if and only if $\mathrm{z}\mathbf{n}(M)$ is a Zariski prime ideal $\mathfrak{p}$ of A and the canonical homomorphism $M \to M_{\mathfrak{p}}$ is injective.

2.4.2. Lemma. *Let M be a nonzero A-module, $\mathfrak{p}$ a Zariski prime ideal of A, N a nontrivial Zariski $A_{\mathfrak{p}}$-submodule of $M_{\mathfrak{p}}$ with $\mathrm{z}\mathbf{r}(N) = \mathfrak{p}A_{\mathfrak{p}}$, and $E = \mathrm{Ker}(M \to M_{\mathfrak{p}}/N)$. Then*

(i) E is a $\mathfrak{p}$-primary A-submodule of M;

(ii) if $M = A$, then E is $\Pi_{\mathfrak{p}}$-primary;

(iii) the Zariski ideal $\mathbf{a} = \mathrm{zann}_A(M_{\mathfrak{p}}/N)$ is $\mathfrak{p}$-primary, and $F = \mathrm{Ker}(M \to M_{\mathfrak{p}}/\mathbf{a}M_{\mathfrak{p}})$ is unique minimal among primary A-submodules of M with $\mathrm{zann}(M/F) = \mathbf{a}$.

Proof. (i) The assumption $\mathrm{z}\mathbf{r}(N) = \mathfrak{p}A_{\mathfrak{p}}$ implies that $\mathrm{z}\mathbf{r}(E) = \mathfrak{p}$. Assume now that $(fm, fn) \in E$. If $f \notin \mathrm{z}\mathbf{r}(E) = \mathfrak{p}$, then the image of f in $A_{\mathfrak{p}}$ is invertible and, therefore, the images of m and n in $M_{\mathfrak{p}}/N$ coincide, i.e., $(m, n) \in E$. Thus, the A-submodule E is $\mathfrak{p}$-primary.

(ii) Let $(f, g) \in \mathbf{r}(E)$, i.e., there exists $k \geq 1$ such that $(f^i, g^i) \in E$ for all $i \geq k$. It follows that, if $fg \in \mathfrak{p}$, then $f, g \in \mathfrak{p}$ and, therefore, $(f, g) \in \Pi_{\mathfrak{p}}$. Assume therefore that $f, g \notin \mathfrak{p}$. Then their images in $A_{\mathfrak{p}}$ are invertible. Since the images of f^i and g^i in $A_{\mathfrak{p}}/N$ are equal for all $i \geq k$, it follows that the same is true for the images of f^i and g^i in $A_{\mathfrak{p}}$, and this implies that the images of f and g in $A_{\mathfrak{p}}$ coincide, i.e., $(f, g) \in \Pi_{\mathfrak{p}}$.

(iii) Since $\mathbf{a} = \mathrm{zann}(M/E)$, (i) implies that the Zariski ideal $\mathbf{a}$ is $\mathfrak{p}$-primary and the ideal F is $\mathfrak{p}$-primary. Furthermore, if an element $f \in A$ annihilates $M_{\mathfrak{p}}/\mathbf{a}M_{\mathfrak{p}}$, it annihilates $M_{\mathfrak{p}}/N$ and, therefore, $f \in \mathbf{a}$, i.e., $\mathrm{zann}(M/F) = \mathbf{a}$. That the $\mathfrak{p}$-primary ideal F is minimal with the latter property is trivial. ∎

2.4.3. Definition. (i) An A-submodule E of M is said to be *decomposable* if it admits a *primary decomposition*, i.e., a representation in the form $\bigcap_{i=1}^k E_i$, where E_i are primary A-submodules. It is said to be *weakly decomposable* if its radical $\mathbf{r}(E)$ is decomposable.

(iii) A primary decomposition $E = \bigcap_{i=1}^k E_i$ is said to be *minimal* if all of the Zariski prime ideals $\mathrm{z}\mathbf{r}(E_i)$ are pairwise distinct and, for every $1 \leq i \leq k$, $\bigcap_{j\neq i} E_j \not\subset E_i$.

For example, if the Zariski spectrum of A is finite then, by Corollary 2.2.5, all ideals of A

are weakly decomposable. Notice also that every decomposable A-submodule admits a minimal primary decomposition. Indeed, let $E = \bigcap_{i=1}^{k} E_i$ be a primary decomposition of an A-submodule E of M, and let $\Pi_i = \mathbf{r}(E_i)$ and $\mathfrak{p}_i = \mathrm{z}\mathbf{r}(E_i)$. If $\mathfrak{p}_i = \mathfrak{p}_j$, we replace the pair of ideals E_i, E_j by their intersection $E_i \cap E_j$, which is a $\mathfrak{p}_i$-primary ideal. Furthermore, withdrawing from the decomposition all of the ideals E_i that contain $\bigcap_{j\neq i} E_j$, we get a minimal primary decomposition.

For an A-submodule $E \subset M \times M$ and a subset $F \subset M \times M$, we denote by $(E : F)$ the Zariski ideal $\{f \in A \big| (fm, fn) \in E$ for all $(m,n) \in F\}$. For example, $(\Delta(M) : F) = \mathrm{ann}(F)$.

2.4.4. Proposition *(The first uniqueness theorem). Let E be a decomposable A-submodule of M provided with a minimal primary decomposition $E = \bigcap_{i=1}^{k} E_i$, and let $\mathfrak{p}_i = \mathrm{z}\mathbf{r}(E_i)$. Then $\{\mathfrak{p}_i\}_{1\leq i\leq k}$ coincides with the set of Zariski prime ideals of the form $\mathrm{z}\mathbf{r}(E : (m,n))$ with $m,n \in M$. In particular, this set does not depend on the choice of the minimal primary decomposition.*

2.4.5. Lemma. *(i) $(\bigcap_{i=1}^{k} E_i : F) = \bigcap_{i=1}^{k}(E_i : F)$;*

(ii) $(E : F) = A$ if and only if $F \subset E$;

(iii) if $F \not\subset E$ and E is $\mathfrak{p}$-primary, then $(E : F)$ is a Zariski $\mathfrak{p}$-primary ideal.

Proof. (i) One has $f \in (\bigcap_{i=1}^{k} E_i : F)$ if and only if $(fm, fn) \in \bigcap_{i=1}^{k} E_i$ for all $(m,n) \in F$. The latter is obviously equivalent to the inclusion $f \in \bigcap_{i=1}^{k}(E_i, F)$.

(ii) One has $(E : F) = A$ if and only if $1 \in (E : F)$, i.e., $(m,n) \in E$ for all $(m,n) \in F$.

(iii) First of all, we show that $\mathrm{z}\mathbf{r}(E : F) = \mathfrak{p}$. Since $\mathrm{zann}(M/E) \subset (E : F)$, it follows that $\mathfrak{p} = \mathrm{z}\mathbf{r}(E) \subset \mathrm{z}\mathbf{r}(E : F)$. Conversely, assume that $f \in \mathrm{z}\mathbf{r}(E : F)$. Then $f^k \in (E : F)$ for some $k \geq 1$, i.e., $(f^k m, f^k n) \in E$ for all $(m,n) \in F$. Since there exists $(m,n) \in F\backslash E$, it follows that $f^k \in \mathfrak{p}$ and, therefore, $f \in \mathfrak{p}$. Thus, $\mathrm{z}\mathbf{r}(E : F) = \mathfrak{p}$. Assume now that $fg \in (E : F)$, i.e., $(fgm, fgn) \in E$ for all $(m,n) \in F$. If $g \notin \mathfrak{p}$, it follows that $(fm, fn) \in E$ for all $(m,n) \in F$ and, therefore, $f \in (E : F)$, i.e., $(E : F)$ is a Zariski $\mathfrak{p}$-primary ideal. ∎

Proof of Proposition 2.4.4. By Lemma 2.4.5, one has

$$\mathrm{z}\mathbf{r}(E : (m,n)) = \bigcap_{i=1}^{k} \mathrm{z}\mathbf{r}(E_i : (m,n)) = \bigcap_{(m,n)\notin E_i} \mathfrak{p}_i \ .$$

If the latter is a Zariski prime ideal, it coincides with some $\mathfrak{p}_i$. Conversely, if $(m,n) \in (\bigcap_{j\neq i} E_j)\backslash E_i$, then $\mathrm{z}\mathbf{r}(E : (m,n)) = \mathfrak{p}_i$, and the required statement follows. ∎

In the situation of Proposition 2.4.4, $\mathfrak{p}_i$ are said to be the Zariski prime ideals *associated to E*.

2.4.6. Proposition. *(The second uniqueness theorem). In the situation of Proposition 2.4.4, let $\mathfrak{p}$ be a Zariski prime ideal of the form $\mathfrak{p}_{i_1} \cup \ldots \cup \mathfrak{p}_{i_l}$, and set $E^{(\mathfrak{p})} = \bigcap_{\mathfrak{p}_i\subset\mathfrak{p}} E_i$. Then*

$E^{(\mathfrak{p})} = \{(m,n) \big| (fm, fn) \in E$ *for some* $f \notin \mathfrak{p}\}$. *In particular, the ideals* E_i *that correspond to the Zariski prime ideals* $\mathfrak{p}_i$ *minimal among* $\{\mathfrak{p}_i\}_{1\le i\le k}$ *are determined by* E.

Proof. Assume first that $(m,n) \in E^{(\mathfrak{p})}$. For every $\mathfrak{p}_j \not\subset \mathfrak{p}$, take an element $f_j \in \mathfrak{p}_j \backslash \mathfrak{p}$. Replacing f_j by its power, we may assume that $f_j \in \mathrm{zann}(M/E_j)$. Then $(f_j m, f_j m) \in E_j$. If f is the product of the above elements f_j, we get $(fm, fn) \in E$ and $f \notin \mathfrak{p}$. Conversely, assume that $(fm, fn) \in E$ for some $f \notin \mathfrak{p}$. It follows that, for every $1 \le i \le k$, either $(m,n) \in E_i$ or $f^l \in \mathfrak{p}_i$ for some $l \ge 1$. If $\mathfrak{p}_i \subset \mathfrak{p}$, the second inclusion is impossible and, therefore, $(m,n) \in E^{(\mathfrak{p})}$. ∎

2.4.7. Definition. (i) An A-module M is said to be *decomposable* (resp. *weakly decomposable*) if the minimal A-submodule $\Delta(M)$ is decomposable (resp. weakly decomposable).

(ii) If M is decomposable, the Zariski prime ideals associated to $\Delta(M)$ are said to be *associated to* M, and their set is denoted by $\mathrm{Ass}(M) = \mathrm{Ass}_A(M)$.

Notice that, if M is decomposable, it is also Zariski decomposable and $\mathrm{Zass}(M) \subset \mathrm{Ass}(M)$.

2.4.8. Lemma. *If an* A*-module* M *is decomposable, then for any sub-semigroup* $S \subset A$ *the* $S^{-1}A$*-module* $S^{-1}M$ *is also decomposable, and one has* $\mathrm{Ass}(S^{-1}M) = \mathrm{Ass}(M) \cap \mathrm{Zspec}(S^{-1}A)$.

Proof. For an A-submodule E, let $S^{-1}E$ be the preimage of E in $S^{-1}A$. Suppose that E is Π-primary with a $\mathfrak{p}$-prime ideal Π. *We claim that*

(1) if $\mathfrak{p} \cap S \neq \emptyset$*, then the* $S^{-1}A$*-submodule* $S^{-1}E$ *is trivial;*

(2) if $\mathfrak{p} \cap S = \emptyset$*, then the* $S^{-1}A$*-submodule* $S^{-1}E$ *is* $S^{-1}\Pi$*-primary.*

(1) Let $f \in \mathfrak{p} \cap S$. Then there is $k \ge 1$ with $(f^k m, 0) \in E$ for all $m \in M$. This implies that $(\frac{m}{s}, 0) \in S^{-1}E$ for all $m \in M$ and $s \in S$, i.e., $S^{-1}E$ is a trivial $S^{-1}A$-submodule.

(2) Suppose that $(\frac{f}{\alpha} \cdot \frac{m}{s}, \frac{f}{\alpha} \cdot \frac{n}{t}) \in S^{-1}E$. Then $(f\alpha\beta tm, f\alpha\beta sn) \in E$ for some $\beta \in S$ and, therefore, one has either $(tm, sn) \in E$, i.e., $(\frac{m}{s}, \frac{n}{t}) \in S^{-1}E$, or $f\alpha\beta \in \mathfrak{p}$. Since $\mathfrak{p} \cap S = \emptyset$, the latter inclusion implies that $f \in \mathfrak{p}$, and the claim follows.

Let $\Delta(M) = \bigcap_{i=1}^k E_i$ be a minimal primary decomposition with $\mathfrak{p}_i = \mathrm{z}\mathbf{r}(E_i)$, and suppose that $\mathfrak{p}_i \cap S = \emptyset$ precisely for $1 \le i \le l$. By the above claim, $\Delta(S^{-1}M) = \bigcap_{i=1}^l S^{-1}E_i$ is a primary decomposition and, in particular, $S^{-1}M$ is decomposable and $\mathrm{Ass}(S^{-1}M) \subset \mathrm{Ass}(M) \cap \mathrm{Fspec}(S^{-1}A)$. By Proposition 2.4.4, each $\mathfrak{p}_i$ is of the form $\mathrm{z}\mathbf{r}(\Delta(M) : (f,g))$ for some $(f,g) \in M \times M$. This easily implies that $\mathrm{z}\mathbf{r}_{S^{-1}A}(\Delta(S^{-1}(M) : (f,g)) = S^{-1}\mathfrak{p}_i$, and the converse inclusion follows. ∎

The following proposition summarizes properties of minimal primary decompositions for decomposable $\mathbf{F}_1$-algebras.

2.4.9. Proposition. *Suppose that A is decomposable, and let $\Delta(A) = \bigcap_{i=1}^n E_i$ be a minimal primary decomposition, and $\mathfrak{p}_i = \mathrm{z}\mathbf{r}(E_i)$. Then*

(i) the set $\{\mathfrak{p}_i\}_{1\le i\le n}$ contains all minimal Zariski prime ideals of A;

(ii) for every $1 \le i \le n$, E_i is $\Pi_{\mathfrak{p}_i}$-primary and the $\mathfrak{p}_i$-primary ideal $E'_i = \mathrm{Ker}(A \to A_{\mathfrak{p}_i}/\mathbf{a}_i A_{\mathfrak{p}_i})$ with $\mathbf{a}_i = \{f \in A \big| (f, 0) \in E_i\}$ lies in E_i; in particular, $\Delta(A) = \bigcap_{i=1}^n E'_i$ is also a minimal primary decomposition;

(iii) if A is reduced, then $\Delta(A) = \bigcap_{i=1}^n \Pi_{\mathfrak{p}_i}$, and so it is also a minimal primary decomposition, and all of the ideals $\Pi_{\mathfrak{p}_1}, \dots, \Pi_{\mathfrak{p}_n}$ lie in the set of minimal prime ideals of A.

Proof. (i) follows from Corollary 2.3.4(i).

(ii) By Lemma 2.4.2, $E'_i = \mathrm{Ker}(A \to A_{\mathfrak{p}_i}/\mathbf{a}_i A_{\mathfrak{p}_i})$ is a unique minimal among primary ideals F with $\mathbf{a}_i = \{f \in A \big| (f, 0) \in F\}$ and, therefore, $E'_i \subset E_i$. Since E'_i is $\Pi_{\mathfrak{p}_i}$-primary, then so is E_i.

(iii) Suppose $(a, b) \in \bigcap_{i=1}^n \Pi_{\mathfrak{p}_i}$. Then there exists $m \ge 1$ such that, for every $1 \le i \le n$, the images of a^m and b^m in $A_{\mathfrak{p}_i}$ lie in $\mathbf{a}_i A_{\mathfrak{p}_i}$. It follows that $(a^j, b^j) \in \bigcap_{i=1}^n E'_i$ for all $j \ge m$. Since the latter is $\Delta(A)$ and A is reduced, we get $a = b$. ∎

2.4.10. Example. Let A be a decomposable $\mathbf{F}_1$-algebra isomorphic to the quotient $B/\mathbf{b}$ of an integral domain B by a Zariski ideal $\mathbf{b}$. Then $\mathrm{Ass}(A) = \mathrm{Zass}(A)$. Indeed, let $\mathfrak{p} \in \mathrm{Ass}(A)$. By Proposition 2.4.4, there exist elements $f, g \in B$ such that the preimage $\mathfrak{q}$ of $\mathfrak{p}$ in B coincides with the set $\{b \in B \big|$ either $b^n f = b^n g$, or $b^n f, b^n g \in \mathbf{b}$ for some $n \ge 1\}$. Since B is an integral domain, it follows that $\mathfrak{q} = \{b \in B \big| b^n f \in \mathbf{b}$ for some $n \ge 1\}$, i.e., $\mathfrak{r} \in \mathrm{Zass}(A)$, by Proposition 2.3.2.

2.4.10. Corollary. *In the situation of Proposition 2.4.4, if the ideal E is radical, then all of the prime ideals associated with E lie in the set of minimal prime ideals that contain E.* ∎

For an A-module M, let $\mathrm{ass}(M)$ denote the set of all prime ideals of A of the form $\mathrm{ann}(m)$ with $m \in M$. Notice that, if such an ideal $\Pi = \mathrm{ann}(m)$ is $\mathfrak{p}$-prime, then $\mathfrak{p} = \mathrm{zann}(m)$ and, by Proposition 2.3.8, the image of $\mathrm{ass}(M)$ under the canonical map $\mathrm{Fspec}(A) \to \mathrm{Zspec}(A)$ lies in $\mathrm{Zass}(M)$.

2.4.11. Proposition. *Let A be a noetherian $\mathbf{F}_1$-algebra, and M a noetherian A-module. Then*

(i) M is decomposable;

(ii) the set $\mathrm{ass}(M)$ is finite, and the map $\mathrm{ass}(M) \to \mathrm{Zass}(M)$ is surjective;

(iii) there is a chain of Zariski A-submodules $N_0 = 0 \subset N_1 \subset \dots \subset N_k = M$ such that each quotient N_i/N_{i-1} is isomorphic to an A-module of the form A/Π, where Π is a prime ideal of A.

The proof of (i) uses only the assumption that M is noetherian. In §2.7, the conclusions (i)-(iii) are extended to a more general case.

Proof. (i) An A-submodule $E \subset M \times M$ is said to be *irreducible* if it possesses the property that, if $E = E' \cap E''$, then either $E = E'$ or $E = E''$. To prove (i), it suffices to verify the following two facts:

(1) every A-submodule is the intersection of a finite number of irreducible A-submodules;

(2) every irreducible A-submodule is primary.

(1) is verified in the same way as the corresponding fact from the proof of Proposition 2.3.8(i).

(2) If E is an irreducible A-submodule, then replacing M by M/E, we may assume that $E = \Delta(M)$. Assume that $fm = fn$ and $m \neq n$. For $k \geq 1$, let E_k be an A-submodule of M consisting of the pairs (p, q) with $f^k p = f^k q$. One has $(m, n) \in E_1 \subset E_2 \subset \dots$ and, therefore, there exists $k \geq 1$ with $E_k = E_i$ for all $i \geq k$. *We claim that* $\Delta(M) = E_k \cap F$*, where* F *is the* A*-submodule* $\Delta(M) \cup (f^k M \times f^k M)$. Indeed, let (p, q) be an element from the intersection with $p \neq q$. Then $p = f^k p'$ and $q = f^n q'$ for some $p', q' \in M$. Since $f^k p = f^k q$, we get $f^{2k} p' = f^{2k} q'$ and, therefore, $(p' q') \in E_{2k} = E_k$, i.e., $p = f^k p' = f^n q' = q$, which is a contradiction. By the assumption, one has either $E_k = \Delta(M)$, or $F = \Delta(M)$. The former case is impossible since $m \neq n$, and in the latter case we have $f^k p = 0$ for all $p \in M$, i.e., $f \in \mathrm{z}\mathbf{n}(M)$.

(ii) First of all, *we claim that every ideal maximal among nontrivial ideals of the form* $\mathrm{ann}(m)$ *with* $m \in M$ *is prime.* Indeed, let $\mathrm{ann}(m)$ be such an ideal, and suppose that $(fh, gh) \in \mathrm{ann}(m)$ and $h \notin \mathrm{ann}(m)$, i.e., $fhm = ghm$ and $hm \neq 0$. Then $(f, g) \in \mathrm{ann}(hm)$ and, therefore, the ideal generated by (f, g) and $\mathrm{ann}(m)$ is contained in $\mathrm{ann}(hm)$. The maximality of $\mathrm{ann}(m)$ implies that $(f, g) \in \mathrm{ann}(m)$, and the claim follows.

Let $\mathfrak{p} \in \mathrm{Zass}(M)$, and suppose first that $\mathfrak{p}$ is the maximal Zariski ideal of A, i.e., $A \backslash \mathfrak{p} = A^*$. By Proposition 2.3.2, one has $\mathfrak{p} = \mathrm{z}\mathbf{r}(\mathrm{zann}(m))$ for some $m \in M$. Then there is $k \geq 1$ with $\mathfrak{p}^k \subset \mathrm{zann}(m)$ and, therefore, the ideal $E = \Delta(A) \cup (\mathfrak{p}^k \times \mathfrak{p}^k)$ associated to $\mathfrak{p}^k$ is contained in $\mathrm{ann}(m)$. Since A is noetherian, the above claim implies that there exists a prime ideal Π of the form $\mathrm{ann}(n)$ with $n \in M$ that contains E. The maximality of $\mathfrak{p}$ implies that Π is an $\mathfrak{p}$-ideal. In the general case, we apply Lemma 2.3.7 to the localizations $A_\mathfrak{p}$ and $M_\mathfrak{p}$. It follows that there exists an $\mathfrak{p}A_\mathfrak{p}$-prime ideal $\Pi' \subset A_\mathfrak{p} \times A_\mathfrak{p}$ of the form $\mathrm{ann}_{A_\mathfrak{p}}(m)$ with $m \in M$. *We claim that the* $\mathfrak{p}$*-prime ideal* $\Pi \subset A \times A$*, which is the preimage of* Π'*, has the form* $\mathrm{ann}(tm)$ *for some* $t \in A \backslash \mathfrak{p}$. Indeed, since A is noetherian, Π is generated by a finite set of pairs $(f_1, g_1), \dots, (f_k, g_k)$. For every $1 \leq i \leq k$, the inclusion $(\frac{f_i}{1}, \frac{g_i}{1}) \in \mathrm{ann}_{A_\mathfrak{p}}(m)$ implies that there is $t_i \in A \backslash \mathfrak{p}$ with $f_i t_i m = g_i t_i m$. If t is the product of all t_i's, then $t \in A \backslash \mathfrak{p}$ and, in particular, $tm \neq 0$, and $f_i tm = g_i tm$ for all $1 \leq i \leq k$. The required claim follows.

For $\mathfrak{p} \in \mathrm{ann}(M)$, let $M^{(\mathfrak{p})}$ denote the subset of M consisting of zero and elements m with $\mathrm{ann}(m) = \mathfrak{p}$. Since the quotient of A by a prime ideal is an integral domain, it follows that $M^{(\mathfrak{p})}$ is a Zariski A-submodule of M. It follows also that, for distinct $\mathfrak{p}, \mathfrak{q} \in \mathrm{ass}(M)$, one has $M^{(\mathfrak{p})} \cap M^{(\mathfrak{q})} = 0$. Since M is Zariski noetherian, this implies that the set $\mathrm{ass}(M)$ is finite.

(iii) If M is nonzero, we can find a prime ideal $\Pi \in \mathrm{ass}(M)$, i.e., $\Pi = \mathrm{ann}(m)$ for some $m \in M$. Then $A/\Pi \xrightarrow{\sim} N_1$, where N_1 is the Zariski A-submodule of M which is the image of the homomorphism $A \to M$ that takes 1 to m. If $N_1 \neq M$, we can apply the same construction to the quotient A-module M/N_1 and so on. Since M is Zariski noetherian, this procedure stabilizes, and we get the required fact. ∎

2.5. Artinian $\mathbf{F}_1$-algebras.

2.5.1. Definition. (i) An $\mathbf{F}_1$-algebra A is said to be *Zariski artinian* if it satisfies the descending chain condition for Zariski ideals.

(ii) A Zariski artinian $\mathbf{F}_1$-algebra A is said to be *local* if $\mathbf{m} = \mathbf{m}_A$ is its only Zariski prime ideal (and, in particular, $\mathbf{m} = \mathrm{z}\mathbf{n}(A)$).

Notice that, in comparison with rings, a Zariski artinian $\mathbf{F}_1$-algebra is not necessarily Zariski noetherian or satisfies the descending chain condition for ideals. Indeed, let I be the idempotent $\mathbf{F}_1$-algebra $\{0, 1, e_1, e_2, \dots\}$ with $e_i \cdot e_j = e_{\min\{i,j\}}$ for all $i, j \geq 1$. If $\mathbf{a}$ is a Zariski ideal that does not contain an element e_n, then $\mathbf{a}$ lies in the Zariski ideal $\mathbf{a}_{n-1} = \{0, e_1, \dots, e_{n-1}\}$ and, therefore, A is Zariski artinian. On the other hand, the ascending chain of Zariski ideals $\mathbf{a}_1 \subset \mathbf{a}_2 \subset \dots$ does not stabilize, i.e., I is not Zariski noetherian. Furthermore, for every $n \geq 1$ the union of $\Delta(I)$ and the set of pairs (e_i, e_j) with $i, j \geq n$ is an ideal E_n, and the descending chain of ideals $E_1 \supset E_2 \supset \dots$ does not stabilize. Notice also that if an idempotent $\mathbf{F}_1$-algebra I is Zariski artinian, it is almost finite. Indeed, an ascending chain of nonzero idempotents $e_1 \leq e_2 \leq \dots$ gives rise to the descending chain of Zariski prime ideals $\mathfrak{p}_{e_1} \supset \mathfrak{p}_{e_2} \supset \dots$ and, therefore, it stabilizes.

2.5.2. Proposition. *Let A be a Zariski artinian $\mathbf{F}_1$-algebra, and I_A the idempotent $\mathbf{F}_1$-subalgebra of A.. Then*

(i) the Zariski nilradical $\mathrm{z}\mathbf{n}(A)$ is nilpotent;

(ii) I_A is Zariski artinian and, in particular, it is almost finite;

(iii) the canonical map $\mathrm{Zspec}(A) \to \mathrm{Zspec}(I_A)$ is a bijection.

Proof. (i) Let $\mathbf{n} = \mathrm{z}\mathbf{n}(A)$. By the assumption, the descending chain $\mathbf{n} \supset \mathbf{n}^2 \supset \dots$ stabilizes, i.e., there is $m \geq 1$ with $\mathbf{n}^m = \mathbf{n}^{m+1} = \dots$. *We claim that the ideal* $\mathbf{a} = \mathbf{n}^m$ *is zero.* Indeed, assume $\mathbf{a} \neq 0$, and let S be the set of nonzero Zariski ideals $\mathbf{b}$ with $\mathbf{a} \cdot \mathbf{b} \neq 0$. Since A is an element in S and it is Zariski artinian, there exists minimal $\mathbf{b}$ with the above property. Let $f \in \mathbf{b}$ be such that $f\mathbf{a} \neq 0$. Since $fA \subset \mathbf{b}$ and $\mathbf{b}$ is minimal, it follows that $\mathbf{b} = fA$. Furthermore, one has $(f\mathbf{a})\mathbf{a} = f\mathbf{a}^2 = f\mathbf{a} \neq 0$. Since $f\mathbf{a} \subset \mathbf{b}$, it follows that $f\mathbf{a} = \mathbf{b} = fA$ and, in particular, $f = fg$ for some $g \in \mathbf{a}$. It follows that $f = fg = fg^2 = \dots$. Since g is nilpotent, we get $f = 0$, which is a contradiction.

(ii) Corolary 1.6.2(iii) implies that, for every Zariski ideal $\mathbf{a} \subset I_A$, one has $\mathbf{b} \cap I = \mathbf{a}$, where $\mathbf{b} = \mathbf{a}A$. It follows that a descending chain $\mathbf{a}_1 \supset \mathbf{a}_2 \supset \dots$ of Zariski ideals of I_A gives rise to a descending chain $\mathbf{b}_1 \supset \mathbf{b}_2 \supset \dots$ of Zariski ideals of A and, therefore, it stabilizes.

(iii) That the map considered is a surjection follows from Corollary 1.6.2. Let $\mathfrak{p}$ be a Zariski prime ideal of A.

Step 1. *There exists an element* $h \notin \mathfrak{p}$ *with* $A_\mathfrak{p} = A_h$. Indeed, for every pair $(f, g) \in \mathrm{Ker}(A \to A_\mathfrak{p})$, there is an element $h = h_{(f,g)} \notin \mathfrak{p}$ with $fh = gh$. For a finite subset $J \subset \mathrm{Ker}(A \to A_\mathfrak{p})$, let $\mathbf{a}_J$ be the Zariski ideal generated by the element $h_J = \prod h_{(f,g)}$, where the product is taken over elements of J. Since A is Zariski artinian, the family of Zariski ideals $\{\mathbf{a}_J\}_J$ has a minimal element. Let h be the element h_J that corresponds to such a minimal ideal $\mathbf{a}_J$, and let $(f, g) \in \mathrm{Ker}(A \to A_\mathfrak{p})$. If $(f, g) \in J$, then of course $fh = gh$. Assume therefore that $(f, g) \notin J$. Since $\mathbf{a}_J = \mathbf{a}_{J \cup \{(f,g)\}}$, it follows that $h = hh_{(f,g)}u$ with $u \in A$, and, therefore, $fh = gh$. It follows that $A_\mathfrak{p} = A_h$.

Step 2. *One can find an idempotent* $e_\mathfrak{p}$ *among the elements* h *with the property of Step 1.* Indeed, since A is Zariski artinian, the descending chain of Zariski ideals $hA \supset h^2 \supset h^3A \supset \dots$ stabilizes. It follows that $h^n = h^{2n}v$ for some $n \geq 1$ and $v \in A \backslash \mathfrak{p}$ and, therefore, the element $e_\mathfrak{p} = h^n v$, which also possesses the property of Step 1, is an idempotent.

Step 3. *The element* $e = e_\mathfrak{p}$ *is a unique maximal idempotent that does not lie in* $\mathfrak{p}$, *and one has* $\mathfrak{p} = \mathfrak{q}_e$. Indeed, let e' be an idempotent outside $\mathfrak{p}$. Then the images of e and $f = e'e$ in $A_\mathfrak{p}$ are equal to 1 and, therefore, $f = fe = e$. In particular, $e' \leq e$. Let now $\mathfrak{p}'$ be a Zariski prime ideal that contains $\mathfrak{p}$. Then $e_{\mathfrak{p}'} \leq e$. If $e \notin \mathfrak{p}'$, then $e \leq e_{\mathfrak{p}'}$ and, therefore, $e_{\mathfrak{p}'} = e$. It follows that $A_{\mathfrak{p}'} \stackrel{\sim}{\to} A_\mathfrak{p}$, and this implies that $\mathfrak{p}' = \mathfrak{p}$, i.e., $\mathfrak{p} = \mathfrak{q}_e$. In particular, the map $\mathrm{Zspec}(A) \to \check{I}_A : \mathfrak{p} \mapsto e_\mathfrak{p}$ is injective.

Since $\check{I}_A \stackrel{\sim}{\to} \mathrm{Zspec}(I_A)$, it follows that $\mathrm{Zspec}(A) \stackrel{\sim}{\to} \mathrm{Zspec}(I_A)$. ∎

2.5.3. Corollary. *A Zariski artinian* $\mathbf{F}_1$*-algebra* A *is local if and only if* $I_A = \{0, 1\}$. *Furthermore, in this case* A *has finite dimension over the* $\mathbf{F}_1$*-field* $K = A^* \cup \{0\}$, *and, in particular, it* *is Zariski noetherian.*

Proof. The first statement follows Proposition 2.5.2(iv). Suppose that A is local. By (i), $\mathbf{m}^n = 0$ for some $n \geq 1$, and so to prove the second statement it is enough to show that, for every $1 \leq i \leq n$, $\mathbf{m}^i/\mathbf{m}^{i+1}$ is a finitely dimensional K-vector space. But this is clear because every Zariski K-submodule of the latter corresponds to a Zariski ideal of A. ∎

2.5.4. Proposition. *The following properties of a Zariski artinian* $\mathbf{F}_1$*-algebra* A *are equivalent:*

(a) A *is Zariski noetherian;*

(b) the Zariski spectrum $\mathrm{Zspec}(A)$ *is finite;*

(c) the idempotent algebra I_A *is finite.*

2.5.5. Definition. An $\mathbf{F}_1$-algebra A is said to be *artinian* if it is Zariski artinian and possesses the equivalent properties of Proposition 2.5.4.

Notice that every quotient of an artinian $\mathbf{F}_1$-algebra is artinian, and every local Zariski artinian $\mathbf{F}_1$-algebra is artinian.

Proof of Proposition 2.5.4. The equivalence (b)$\Leftrightarrow$(c) follows from Proposition 2.5.2.

(a)$\Longrightarrow$(c). Suppose that I_A is not finite. *We claim that* I_A *is not Zariski noetherian.* Indeed, suppose that I_A is Zariski noetherian, and let $e_1, e_2, \dots$ be a sequence of pairwise distinct nonzero idempotents. If $\mathbf{a}_n$ denotes the Zariski ideal generated by $e_1, \dots, e_n$, then the ascending chain $\mathbf{a}_1 \subset \mathbf{a}_2 \subset \dots$ stabilizes, i.e., $\mathbf{a}_n = \mathbf{a}_{n+1} = \dots$ for some $n \geq 1$. This implies that $e_n < e_{n+1} < \dots$. If we apply the same reasoning to the sequence $e_{n+1}, e_n, e_{n+3}, e_{n+2}, \dots$, we get a contradiction, and the claim follows. Let $\mathbf{a}_1 \subset \mathbf{a}_2 \subset \dots$ be a sequence of Zariski ideals of I_A that does not stabilize. If $\mathbf{b}_i$ is the Zariski ideal of A generated by $\mathbf{a}_i$ then, by Corollary 1.6.2(ii), $\mathbf{a}_i = \mathbf{b}_i \cap I_A$, and so the sequence of Zariski ideals $\mathbf{b}_1 \subset \mathbf{b}_2 \subset \dots$ does not stabilize, which contradicts the assumption (a).

(b)$\Longrightarrow$(a). Let n be the number of Zariski prime ideals of A. If $n = 1$, A is local and the property (a) follows from Proposition 2.5.2(ii). Suppose $n \geq 2$, and the property (a) holds for Zariski artinian $\mathbf{F}_1$-algebras with strictly smaller number of Zariski prime ideals. Let $\mathbf{a}_1 \subset \mathbf{a}_n \subset \dots$ be an increasing sequence of Zariski ideals, and let $\mathfrak{p}$ be a minimal Zariski prime ideal of A. The sequence of Zariski ideals $\mathbf{a}_1 \cap \mathfrak{p} \subset \mathbf{a}_2 \cap \mathfrak{p} \subset \dots$ stabilizes because this is true for the local Zariski artinian algebra $A_{\mathfrak{p}}$. We may therefore assume that $\mathbf{a}_i \cap \mathfrak{p} = \mathbf{a}_j \cap \mathfrak{p}$ for all $i, j \geq 1$. By the induction hypothesis applied for the quotient $A/\mathfrak{p}$, the sequence $\mathbf{a}_1 \cup \mathfrak{p} \subset \mathbf{a}_2 \cup \mathfrak{p} \subset \dots$ stabilizes and, therefore, the previous fact implies that the same is true for the sequence $\mathbf{a}_1 \subset \mathbf{a}_2 \subset \dots$. ∎

2.5.6. Proposition. *Every ideal of an artinian $\mathbf{F}_1$-algebra A is decomposable.*

Proof. It suffices to verify the statements for the minimal ideal $\Delta(A)$. For $e \in \check{I}_A$, let F_e be the ideal of A generated by Π_e (see §1.6). One has $A/F_e \xrightarrow{\sim} A_e/\mathbf{p}_e A_e$. It follows that the only idempotents in A/F_e are 0 and 1. Corollary 2.5.3 then implies that A/F_e is a local artinian $\mathbf{F}_1$-algebra and, in particular, it is quasi-integral. It follows that F_e is a primary ideal. By Corollary 1.6.2(i), one has $\Delta(A) = \bigcap_{e \in \check{I}_A} F_e$ and, therefore, A is decomposable. ∎

2.5.7. Corollary. *The following properties of an artinian $\mathbf{F}_1$-algebra A are equivalent:*

(a) A is reduced;

(b) for any $e \in \check{I}_A$, $\mathfrak{p}_e A_e$ is the maximal Zariski ideal of A_e.

Proof. The implication (b)$\Longrightarrow$(a) follows from fact that A is embedded in the direct product $\prod_{e \in \check{I}_A} A/F_e$ and, by (b), all of the multipliers $A/F_e = A_e/\mathfrak{p}_e A_e$ are $\mathbf{F}_1$-fields. Suppose now that A is reduced, and let a be an element of A whose image in A_e lies in $\mathbf{m}_{A_e} \backslash \mathfrak{p}_e A_e$ for some $e \in \check{I}_A$. Since A_e is also reduced, we can replace A by A_e and assume that $e = 1$. We have $a \notin \mathfrak{p}_1 A$ and $a^n \in \mathfrak{p}_1 A$ for some $n \geq 2$. the latter means that $a^n = bf$ for some idempotent $f \neq 1$. It follows that $a^i = (af)^i$ for all $i \geq n$. Since A is reduced, we get $a = af$ which contradicts the assumption $a \notin \mathfrak{p}_1 A$. ∎

2.6. Integral extensions of $\mathbf{F}_1$-algebras. Let A be an $\mathbf{F}_1$-subalgebra of an $\mathbf{F}_1$-algebra B.

2.6.1. Definition. An element $f \in B$ is said to be *integral over A* if $f^m = af^n$ for some element $a \in A$ and integers $m > n \geq 0$.

For elements $f_1, \ldots, f_n \in B$, let $A[f_1, \ldots, f_n]$ denote the A-subalgebra B generated by those elements.

2.6.2. Proposition. *The following properties of an element $f \in B$ are equivalent:*

(a) f is integral over A;

(b) the $\mathbf{F}_1$-algebra $A[f]$ is a finite A-module;

(c) the element f is contained in an $\mathbf{F}_1$-subalgebra $C \subset B$ which is a finite A-module.

Proof. The only non-evident implication is (c)$\Longrightarrow$(a). To prove it, we need the following simple fact.

2.6.3. Lemma. *Let M be a finite A-module, $x_1, \ldots, x_d$ generators of M, and $\mathbf{a}$ a Zariski ideal of A. Then for any endomorphism $\varphi : M \to M$ with $\varphi(M) \subset \mathbf{a}M$ there exist elements $a_1, \ldots, a_d \in \mathbf{a}$ and integers $m > n \geq 0$ such that $\varphi^m(x_i) = a_i \varphi^n(x_i)$ for all $1 \leq i \leq d$.*

Proof. There exist elements $a_1, \dots, a_d \in \mathbf{a}$ and a map $\sigma : \{1, \dots, d\} \to \{1, \dots, d\}$ such that $\varphi(x_i) = a_i x_{\sigma(i)}$ for all $1 \le i \le d$. For some $k \ge 1$ the images of the maps σ^k and σ^{k+1} coincide. Replacing therefore φ by φ^k, we may assume that for the image I of σ one has $I = \sigma(I)$, i.e., there is a permutation τ of the set I such that $\sigma(i) = \tau(i)$ for all $i \in I$. It follows that $\sigma^k(i) = \tau^k(i)$ for all $k \ge 1$ and $i \in I$. Furthermore, we can find $k \ge 1$ such that τ^k is the identity map on I. Thus, replacing φ by φ^k, we may assume that σ induces the identity map on the set I. We get $\varphi^2(x_i) = \varphi(a_i x_{\sigma(i)}) = a_i^2 x_{\sigma(i)} = a_i \varphi(x_i)$ for all $1 \le i \le d$. ∎

Let $x_1 = 1, x_2, \dots, x_d$ be generators of the finitely generated A-module C. We apply Lemma 2.6.3 to the multiplication by the element f on C and the trivial ideal $\mathbf{a} = A$. It follows that there exist elements $a_1, \dots, a_d \in A$ and integers $m > n \ge 0$ such that $f^m x_i = a_i f^n x_i$ for all $1 \le i \le d$. Since $x_1 = 1$, we get $f^m = a_1 f^n$. ∎

2.6.4. Corollary. *(i) The set C of elements of B integral over A is an A-subalgebra of B;*

(ii) if $f_1, \dots, f_n \in C$, the A-subalgebra $A[f_1, \dots, f_n]$ is a finite A-module.

Proof. (i) If $f, g \in A$, the A-subalgebras $A[f]$ and $A[g]$ are finite A-modules. It follows that the $\mathbf{F}_1$-subalgebra $A[f] \cdot A[g]$ is a finite A-module. Since the element fg is contained in the latter, it follows that it is integral over A.

(ii) The statement is obtained by induction using the simple fact that, given $\mathbf{F}_1$-algebras $A \subset A' \subset A''$ such that A' is a finite A-module and A'' is a finite A'-module, then A'' is a finite A-module. ∎

The A-subalgebra C from Corollary 2.6.4 is called the *integral closure of A in B*. If $C = B$, B is said to be *integral over A* (or that $A \subset B$ *is an integral extension* of $\mathbf{F}_1$-algebras). If $C = A$, A is said to be *integrally closed in B*. An integral $\mathbf{F}_1$-algebra is said to be *normal* if it is integrally closed in its fraction $\mathbf{F}_1$-field.

2.6.5. Proposition. *Let $A \subset B$ be an integral extension of $\mathbf{F}_1$-algebras. Then*

(i) the canonical map $\mathrm{Zspec}(B) \to \mathrm{Zspec}(A)$ is surjective, and it takes $\mathbf{m}_B$ to $\mathbf{m}_A$;

(ii) the canonical map $\mathrm{Fspec}(B) \to \mathrm{Fspec}(A)$ is surjective;

(iii) (Lifting Theorem) given chains of Zariski prime ideals $\mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_n$ in A and $\mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_m$ in B with $m < n$ and $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $1 \le i \le m$, the second chain can be extended to a chain $\mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_n$ such that $\mathfrak{q}_i \cap A = \mathfrak{p}_i$ for all $1 \le i \le n$.

2.6.6. Lemma. *In the situation of Proposition 2.6.5, if B is an integral $\mathbf{F}_1$-algebra, then A is an $\mathbf{F}_1$-field if and only if B is an $\mathbf{F}_1$-field.*

Proof. Since B is an integral $\mathbf{F}_1$-algebra, it follows that, for every element $g \in B$, there exists $n \geq 1$ with $g^n \in A$, and the required fact follows from Lemma 1.4.4. ∎

Proof of Proposition 2.6.5. (i) Let $\mathfrak{p}$ be a Zariski prime ideal of A. Then the localization $B_{\mathfrak{p}}$ of B with respect to $A \backslash \mathfrak{p}$ is integral over $A_{\mathfrak{p}}$. We can therefore replace A and B by $A_{\mathfrak{p}}$ and $B_{\mathfrak{p}}$, respectively, and we have only to verify the last statement. The intersection $\mathfrak{p} = \mathbf{m}_B \cap A$ is a Zariski prime ideal of A, and $B/\mathbf{m}_B$ is integral over $A/\mathfrak{p}$. By Lemma 2.6.6, $A/\mathfrak{p}$ is an $\mathbf{F}_1$-field, i.e., $\mathfrak{p} = \mathbf{m}_A$.

(ii) follows from (i) and Corollary 1.2.6.

(iii) The situation is easily reduced to the case when $n = 2$ and $m = 1$. Replacing A and B by $A/\mathfrak{p}_1$ and $B/\mathfrak{q}_1$, we may assume that $\mathfrak{p}_1 = 0$ and $\mathfrak{q}_1 = 0$. In this case, the required fact follows from (i). ∎

Let K be an $\mathbf{F}_1$-field. Lemma 2.6.6 implies that any integral domain L, which contains K and is integral over it, is also an $\mathbf{F}_1$-field. Such an $\mathbf{F}_1$-field L is said to be an *algebraic extension* of K. An element $x \in L$ integral over K satisfies an equation $x^n = a$ with $n \geq 1$ and $a \in K$ and, if n is minimal with this property, this equation is said to be *minimal.* (If $x^m = b$ for $m \geq 1$ and $b \in K$, then $m = qn$ and $b = a^q$ for some $q \geq 1$.) An $\mathbf{F}_1$-field L is said to be a *finite extension* of K if it contains K and is a finitely generated free K-module. Of course, it is then an algebraic extension of K.

Furthermore, let A be an an integral domain, and K the fraction $\mathbf{F}_1$-field of A. Then the integral closure of A in an $\mathbf{F}_1$-field L that contains K corresponds to the saturation of $\check{A} = A \backslash \{0\}$ in L^* (i.e., the set of elements $x \in K^*$ with $x^n \in \check{A}$ for some $n \geq 1$).

2.6.7. Lemma. *Let K be an $\mathbf{F}_1$-field, and let A be an integral finitely generated K-algebra which is an integral domain. Then the integral closure of A in any finite extension of its fraction $\mathbf{F}_1$-field is a finite A-algebra.*

Proof. If L is the fraction $\mathbf{F}_1$-field of A, then L/K^* is the fraction $\mathbf{F}_1$-field of the integral domain A/K^*. Thus, replacing A by A/K^*, the situation is reduced to the case when A is a finitely generated $\mathbf{F}_1$-algebra. Furthermore, by Corollary 2.6.4(ii), it suffices to verify that the integral closure considered is finitely generated over $\mathbf{F}_1$. The required statement therefore follows from the well known fact that the saturation of a finitely generated submonoid in a finitely generated group is a finitely generated monoid. ∎

An $\mathbf{F}_1$-field K is said to be *algebraically closed* if the group K^* is divisible (i.e., any equation $T^n = a$ with $n \geq 1$ and $a \in K$ has a solution in K). This is equivalent to the property that, for any algebraic extension L of K, there exists a K-homomorphism $L \to K$. Every $\mathbf{F}_1$-field K has an *algebraic closure*, i.e., an algebraically closed $\mathbf{F}_1$-field K^{a} that contains K and possesses the property that any injective homomorphism K to an algebraically closed field L can be extended to an injective homomorphism $K^{\mathrm{a}} \to L$. The latter is equivalent to the property that it is a unique (up to a canonical isomorphism) algebraic extension of K such that, for every algebraic extension L of K there exists a K-homomorphism $L \to K^{\mathrm{a}}$. The multiplicative group of K^{a} is what is called the "divisible hull" of the group K^* (see [Lam, §3]).

2.6.8. Remark. The analog of the descent theorem from the theory of integral extensions of rings holds in the much stronger form of Lemma 1.4.6.

2.7. Valuation $\mathbf{F}_1$-algebras. An *ordered* $\mathbf{F}_1$*-field* is an $\mathbf{F}_1$-field Γ provided with a total ordering $\leq$ which has the following two properties: (1) $0 < f$, and (2) if $f < g$, then $fh < gh$ for all $f, g, h \in \Gamma^*$. For example, $\mathbf{R}_+$ is an ordered $\mathbf{F}_1$-field. Notice the structure of an ordered $\mathbf{F}_1$-field on Γ extends in a unique way to a similar structure on the algebraic closure Γ^{a} of Γ. (For $f, g \in (\Gamma^{\mathrm{a}})^*$, one has $f < g$ if $a < b$, where $a, b \in \Gamma^*$ are such that $f^n = a$ and $g^n = b$ for some $n \geq 1$.)

A *valuation* on an $\mathbf{F}_1$-field K is a homomorphism $K \to \Gamma : f \mapsto |f|$ to an ordered $\mathbf{F}_1$-field Γ. Notice that, for such a valuation $|\ |$ on K, the image $|K|$ is an ordered $\mathbf{F}_1$-subfield of Γ. Two valuations $|\ | : K \to \Gamma$ and $|\ | : K \to \Gamma'$ are said to be equivalent if there is an isomorphism of ordered $\mathbf{F}_1$-fields $\lambda : |K| \xrightarrow{\sim} |K|'$ such that $\lambda(|f|) = |f|'$ for all $f \in K$. A *valuation* $\mathbf{F}_1$*-field* is an $\mathbf{F}_1$-field K provided with an equivalence class of valuations. An *embedding of valuation* $\mathbf{F}_1$*-fields* $(K, |\ |) \hookrightarrow (K', |\ |)$ consists of compatible embeddings of $\mathbf{F}_1$-fields $K \hookrightarrow K'$ and of ordered $\mathbf{F}_1$-groups $|K| \hookrightarrow |K'|$. In this situation we also say that we are given an *extension of valuation* $\mathbf{F}_1$*-fields* K'/K.

A valuation $|\ | : K \to \Gamma$ is said to be *of rank zero*, or *trivial*, if $|K| = \{0, 1\}$. A valuation $|\ | : K \to \Gamma$ is said to be *of rank one* if it is nontrivial and there is an embedding of the ordered $\mathbf{F}_1$-fields $|K| \hookrightarrow \mathbf{R}_+$. If, in addition, the image of $|K^*|$ is discrete in $\mathbf{R}_+^*$ (i.e., either the valuation is trivial, or the group $|K^*|$ is infinite cyclic), the valuation is said to be *discrete*.

Furthermore, an $\mathbf{F}_1$-subalgebra A of an $\mathbf{F}_1$-field K is said to be a *valuation* $\mathbf{F}_1$*-subalgebra* if, for any element $f \in K^*$, one has either $f \in A$ or $f^{-1} \in A$. Of course, in this case K is the fraction $\mathbf{F}_1$-field of A, and A is integrally closed in K. A valuation $\mathbf{F}_1$-subalgebra A in K is said to be *trivial* if $A = K$. An abstract $\mathbf{F}_1$-algebra A is said to be a *valuation* $\mathbf{F}_1$*-algebra* if it is integral and is a valuation $\mathbf{F}_1$-subalgebra in its fraction $\mathbf{F}_1$-field. Notice that the quotient $A/\mathfrak{p}$ of such an $\mathbf{F}_1$-algebra by any Zariski prime ideal $\mathfrak{p}$ is also a valuation $\mathbf{F}_1$-algebra.

2.7.1. Lemma. *(i) If $|\ |: K \to \Gamma$ is a valuation on K, then $A = \{f \in K \big| |f| \leq 1\}$ is a valuation $\mathbf{F}_1$-subalgebra in K;*

(ii) the correspondence $|\ | \mapsto A$ gives rise to a bijection between the set of equivalence classes of valuations on K and the set of valuation $\mathbf{F}_1$-subalgebras in K.

Proof. Given a valuation $\mathbf{F}_1$-algebra A in K, the set $\Gamma = \{0\} \cup K^*/A^*$ has the structure of an ordered valuation $\mathbf{F}_1$-field: if $|f|$ denotes the image of an element $f \in K^*$ in Γ, then $|f| \leq |g|$ if $fg^{-1} \in A$. Thus, the map $|\ |: K \to \Gamma$ is a valuation whose valuation $\mathbf{F}_1$-algebra is A. ∎

2.7.2. Proposition. *Let A be an $\mathbf{F}_1$-subalgebra of an $\mathbf{F}_1$-field K. Then for any Zariski prime ideal $\mathfrak{p} \subset A$ there exists a valuation $\mathbf{F}_1$-subalgebra B in K that contains A and such that $\mathbf{m}_B \cap A = \mathfrak{p}$.*

Proof. First of all, replacing A by $A_\mathfrak{p}$, we may assume that $\mathfrak{p} = \mathbf{m}_A$. Let S be the set of all $\mathbf{F}_1$-subalgebras $A \subset B \subset K$ with $\mathbf{m}_B \cap A = \mathbf{m}_A$. The set S is nonempty since $A \in S$, and it is provided with the partial ordering for which $B \leq C$ if $B \subset C$ and $\mathbf{m}_C \cap B = \mathbf{m}_B$. This partial ordering satisfies the condition of Zorn's lemma; therefore, there exists a maximal element B in S. *We claim that B is a valuation $\mathbf{F}_1$-algebra in K.* Indeed, let $f \in K \backslash B$. Then the $\mathbf{F}_1$-algebra $B[f]$ generated by B and f does not belong to S and, therefore, $\mathbf{m}_B B[f] = B[f]$. It follows that $bcf^n = 1$ for some $b \in \mathbf{m}_B$, $c \in B$ and $n \geq 0$. Since b is not invertible in B, then $n \geq 1$ and, therefore, $(f^{-1})^n \in B$, i.e., the element f^{-1} is algebraic over B. From Lemma 1.4.4 it follows that for the $\mathbf{F}_1$-algebra $C = A[f^{-1}]$ one has $\mathbf{m}_C \cap B = \mathbf{m}$. Since B is maximal, we get $f^{-1} \in B$. ∎

2.7.3. Corollary. *Given an extension of $\mathbf{F}_1$-fields L/K, every valuation on K extends to a valuation on L. If the extension is algebraic, such an extension is unique.*

Proof. Let A be a valuation $\mathbf{F}_1$-subalgebra in K. That there exists a valuation $\mathbf{F}_1$-subalgebra B in L that contains A and such that $\mathbf{m}_B \cap K = \mathbf{m}_A$ follows from Proposition 2.7.2. Notice that in this case one has $B \cap K = A$. Assume that L is algebraic over K. *We claim that $B = \{f \in L \big| f^n \in A$ for some $n \geq 1\}$.* Indeed, since L is algebraic over K, for every $f \in L$ there exists $n \geq 1$ with $f^n \in K$. If $f \in B$, we have $f^n \in B \cap K = A$. Conversely, if $f^n \in A$, then $f^n \in B$ and, therefore, $f \in B$ because B is integrally closed in L. ∎

2.7.4. Corollary. *Let A be a $\mathbf{F}_1$-subalgebra of an $\mathbf{F}_1$-field K. Then the integral closure of A in K coincides with the intersection of all valuation $\mathbf{F}_1$-subalgebras in K that contain A.*

Proof. That the integral closure is contained in the intersection follows from the fact that valuation $\mathbf{F}_1$-algebras are integrally closed. Let f be an element of K which is not integral over A. *We claim that the Zariski ideal $\mathbf{b}$ of $C = A[f^{-1}]$ generated by $\mathbf{m}_A$ and f^{-1} is nontrivial.* Indeed, if this is not true, then either $abf^{-m} = 1$ for some $a \in \mathbf{m}_A$, $b \in A$ and $m \geq 0$, or $cf^{-n} = 1$ for some $c \in A$ and $n \geq 1$. Since a is not invertible in A, then $m \geq 1$. It follows that in both cases the element f is integral over A which is a contradiction. Thus, $\mathbf{b} \subset \mathbf{m}_C$. By Proposition 2.7.2, there exists a valuation $\mathbf{F}_1$-algebra B in K that contains C and such that $\mathbf{m}_B \cap C = \mathbf{m}_C$. It follows that $f^{-1} \in \mathbf{m}_B$ and, therefore, $f \notin B$. ∎

Let K be an $\mathbf{F}_1$-field and A an $\mathbf{F}_1$-subalgebra of K. If a nontrivial valuation ring B in K contains A, one says that B has center in A, and the Zariski prime ideal $\mathbf{m}_B \cap A$ is said to be the *center* of B in A. The set of all nontrivial valuation $\mathbf{F}_1$-subalgebras in K over A is said to be the *Zariski-Riemann space* of K over A, and it is denoted by $\mathrm{Zar}(K, A)$. The space $\mathrm{Zar}(K, A)$ is provided with a topology whose basis of open subsets is formed by sets of the form $U(f_1, \ldots, f_n) = \mathrm{Zar}(K, A[f_1, \ldots, f_n])$ with $f_1, \ldots, f_n \in K$.

2.7.5. Proposition. *The space* $\mathrm{Zar}(K, A)$ *is quasicompact.*

Proof. We have to show that, given a family S of closed subsets of $\mathrm{Zar}(K, A)$ with the property that any finite subfamily of sets from S has a nonempty intersection, the intersection of all elements of S is nonempty. By Zorn's lemma, we may assume that S is maximal with that property. This immediately implies that (1) if a closed set X contains an element of S, then $X \in S$, (2) if $X_1, \ldots, X_n \in S$, then $X_1 \cap \ldots \cap X_n \in S$, and (3) if $X_1, \ldots, X_n$ are closed and $X_1 \cup \ldots \cup X_n \in S$, then $X_i \in S$ for some $1 \leq i \leq n$. For $f \in K^*$, we set $V(f) = \{B \in \mathrm{Zar}(K, A) \big| f \in \mathbf{m}_B\}$. It is a closed subset of $\mathrm{Zar}(K, A)$ since its complement is $U(f^{-1})$. *We claim that any set* $X \in S$ *different from* $\mathrm{Zar}(K, A)$ *is contained in some* $V(f) \in S$. Indeed, the nonempty complement of X contains an open set of the form $U(f_1, \ldots, f_n)$ with $f_1, \ldots, f_n \in K^*$. By (1), the complement of the latter lies in S and, since it coincides with $V(f_1^{-1}) \cup \ldots \cup V(f_n^{-1})$, one has $V(f_i^{-1}) \in S$ for some $1 \leq i \leq n$, by (3). The claim implies that $\bigcap_{X \in S} X = \bigcap_{f \in F} V(f)$, where $F = \{f \in K^* \big| V(f) \in S\}$. Let C be the A-subalgebra of K generated by all elements of F. Since $V(f) \cap V(f^{-1}) = \emptyset$, it follows that none of the elements $f \in F$ is invertible in C, i.e., $F \subset \mathbf{m}_C$. By Proposition 2.7.2, there exists a valuation $\mathbf{F}_1$-subalgebra $B \subset K$ that contains C and such that $\mathbf{m}_B \cap C = \mathbf{m}_C$, and we get $B \in \bigcap_{X \in S} X$. ∎

Let A be a valuation $\mathbf{F}_1$-algebra.

2.7.6. Proposition. *Let* A' *be an* A*-algebra which is integral as an* A*-module and such that* $\mathbf{m}_{A'} \cap A = \mathbf{m}_A$. *Then for every* A*-module* M *the following is true:*

(i) the homomorphism of A*-modules* $M \to M' = M \otimes_A A' : m \mapsto m \otimes 1$ *is injective (and so* M *can be identified with its image in* M'*);*

(ii) for an A-submodule E of M, the A'-submodule E' of M' generated by E consists of the pairs $(a'm, b'n)$ with $m, n \in N_E$ and $(a'm, a'n)$ with $(m, n) \in E$, where $a', b' \in A'$;

(iii) one has $N_{E'} = A'N_E$;

(iv) the map $E \mapsto E'$ commutes with finite intersections.

Proof. (i) and (ii). Consider first the case when the A-module M is free. Then the A'-module M' is also free, the statement (i) holds for trivial reason, and the property (1) implies that, if $a'_1 m_1 = a'_2 m_2$ for nonzero elements $a'_1, a'_2 \in A'$ and $m_1, m_2 \in M$, then there exists $a \in A$ with either $m_1 = am_2$ and $aa'_1 = a'_2$, or $am_1 = m_2$ and $a'_1 = aa'_2$. To prove (ii), it suffices to verify that the set of pairs considered forms an ideal, and the only non-evident property to check is transitivity. It suffices to consider the following pairs (of pairs) (a) $(a'_1 m_1, a'_1 n_1)$ and $(a'_2 m_2, a'_2 n_2)$ of the second type with $a' n_1 = a'_2 m_2$, and (b) $(a'm_1, a'n_1)$ and $(b'm_2, c'n_2)$ of the second and first type, respectively, with $a'n_1 = c'm_2$. We may assume that all of the considered elements of A' and M are nonzero.

(a) By the property (1), we may assume that there exists an element $a \in A$ with $a'_1 = aa'_2$ and $an_1 = m_2$. Since $(m_1, n_1) \in E$, one has $(am_1, m_2) = (am_1, an_1) \in E$ and, since $(m_2, n_2) \in E$, it follows that $(am_1, n_2) \in E$. We get $(a'_1 m_1, a'_2 n_2) = (a'_2(am_1), a'_2 n_2) \in E$.

(b) We may assume that $m_1, n_1 \notin N_E$. By the property (1) again, there exists an element $a \in A$ with either $a' = ab'$ and $an_1 = m_2$, or $aa' = c$ and $n_1 = am_2$. By the assumption, the latter case is impossible. One therefore has $(am_1, m_2) = (am_1, an_1) \in E$ and, in particular, $am_1 \in N_E$ and $(am_1, n_2) \in E$. It follows that $(a'm_1, b'n_2) = (b'(am_1), b'n_2) \in E$.

Notice that the property (2) was not used so far.

In the general case, we take a surjective homomorphism $\varphi : P \to M$ from a free A-module P, and notice that the A'-submodule $E' = \mathrm{Ker}(P' \to M')$ is generated by the image of the A-submodule $E = \mathrm{Ker}(P \to M)$. Suppose first that $m \otimes 1 = n \otimes 1$ (in M') for some elements $m, n \in M$, and take elements $p, q \in P$ with $\varphi(p) = m$ and $\varphi(q) = n$. It follows that $(p, q) \in E'$ and, by the previous case, one has either $(p, q) = (a'p_1, b'q_1)$ with $p_1, q_1 \in N_E$ (call it the case (α)), or $(p, q) = (a'p_1, a'q_1)$ with $(p_1, q_1) \in E$ (the case (β)), where $a', b' \in A'$. If $p = a'p_1$, the property (1) implies that there exists $a \in A$ with either $a' = a$, or $aa' = 1$. In the latter case, the property (2) implies that $a \in A^*$ and $a' = a^{-1}$. All this easily implies that $m = n = 0$ in the case (α) and $m = n$ in the case (β), i.e., the statement (i) is true, and we may identify M with its image in M'. If now E is an arbitrary ideal of M, then its preimage F in P generates the preimage of E' in P'. Applying the description of F' in terms of F, we get the required description of the ideal E' in terms of E.

(iii) Suppose first that the A-module M is free. If $m' \in N_{E'}$, the statement (ii) implies that the pair $(m', 0)$ is of the first type and, therefore, $m = a'm$ for some $a' \in A'$ and $m \in N_E$. If M is arbitrary, we take a surjective homomorphism $\varphi : P \to M$ from a free A-module P. Then $N_{E'} = \varphi(N_{F'})$, where F is the ideal of P which is the preimage of E, and since $N_{F'} = A'N_F$, it follows that $N_{E'} = A'N_F$.

(iv) First of all, *we claim that, if $aa' = ba'$ for $a, b \in A$ and $a' \in A' \backslash \{0\}$, then $a = b$.* Indeed, we may assume that there exists $c \in A$ with $a = cb$ and $a' = ca'$. By the property (3), the latter equality implies that $c = 1$ and, therefore, $a = b$.

As above, the situation is easily reduced to the case when the A-module M is free. It suffices to verify that, given A-submodules E_1 and E_2 of M, the A'-submodule $E'_1 \cap E'_2$ is generated by $E_1 \cap E_2$. Let $(m', n') \in E'_1 \cap E'_2$, and consider the following three cases.

Case 1: the pair (m', n') is of the first type with respect to both E'_1 and E'_2. In this case it suffices to show that the Zariski A'-submodle $N_{E'_1} \cap N_{E'_2}$ is generated by $N_{E_1} \cap N_{E_2}$. Let $m' \in N_{E'_1} \cap N_{E'_2}$, i.e., $m' = a'_1 m_1 = a'_2 m_2$ with $a'_1, a'_2 \in A'$, $m_1 \in N_{E_1}$ and $m_2 \in N_{E_2}$. We may then assume that there exists $a \in A$ with $a'_1 = aa'_2$ and $am_1 = m_2$. It follows that $m_2 \in N_{E_1} \cap N_{E_2}$ and, therefore, $m' = a'_2 m_2 \in A'(N_{E_1} \cap N_{E_2})$.

Case 2: the pair (m', n') is of the second type with respect to both E'_1 and E'_2, i.e., $(m', n') = (a'_1 m_1, a'_1 n_1) = (a'_2 m_2, a'_2 n_2)$ with $(m_1, n_1) \in E_1$, $(m_2, n_2) \in E_2$ and $a'_1, a'_2 \in A'$. We may assume that there exists $a \in A$ with $a'_1 = aa'_2$ and $m_2 = am_1$. Furthermore, there exists $b \in A$ with either $a'_1 = ba'_2$ and $m_2 = bm_1$, or $a'_2 = ba'_1$ and $m_1 = bm_2$. In the former case, the property (3) implies that $a = b$ and, therefore, $(m_2, n_2) = (am_1, an_1) \in E_1 \cap E_2$. In the latter case, one has $a'_1 = aba'_1$ and again, by the property (3), $ab = 1$. The property (2) implies that $b = a^{-1} \in A$ and, therefore, $(m_2, n_2) = (am_1, an_1) \in E_1 \cap E_2$. In both cases, $(m', n') = (a'_1 m_1, a'_1 n_1)$ lies in the ideal generated by $E_1 \cap E_2$.

Case 3: the pair (m', n') is of the first (resp. second) type with respect to E'_1 (resp. E'_2), i.e., $(m', n') = (a'm_1, b'n_1) = (c'm_2, c'n_2)$ with $m_1, n_1 \in N_{E_1}$, $(m_2, n_2) \in E'_2$ and $a', b', c' \in A$. We may assume that there exists $a \in A$ with $a' = ac'$ and $m_2 = am_1$ and, in particular, $m_2 \in N_{E_1}$. Furthermore, there exists $b \in A$ with either $b' = bc'$ and $n_2 = bn_1$, or $c' = ba'$ and $n_1 = bn_2$. In the former case, one has $n_2 \in N_{E_1}$. In the latter case, one has $a' = aba'$ and, by the property (3), $ab = 1$. The property (2) implies that $b = a^{-1} \in A$ and, therefore, $n_2 = an_1 \in N_{E_1}$. In both cases, $(m', n') = (c'm_2, c'n_2)$ lies in the ideal generated by $E_1 \cap E_2$. ∎

2.7.7. Corollary. *In the situation of Proposition 2.7.6, the following is true:*

(i) there is a canonical isomorphism $M/E \otimes_A A' \xrightarrow{\sim} M'/E'$;

(ii) if the A-module M is integral, then the equality $a'm = b'n$ with $a', b' \in A'$ and $m, n \in M$ implies that there exists an element $a \in A$ such that either $a' = ab'$ and $am = n$, or $aa' = b'$ and $m = an$.

Proof. The statement (i) directly follows from Proposition 2.7.6(ii). Suppose that the A-module M is integral, and we are given an equality $a'm = b'n$ with $a', b' \in A'$ and $m, n \in M$. Let $\varphi : P \to M$ be an epimorphism from a free A-module P that take the standard generators of P to nonzero elements of M. Then its Zariski kernel is zero, i.e., $N_E = 0$, where $E = \mathrm{Ker}(\varphi)$. If p and q are elements of P with $\varphi(p) = m$ and $\varphi(q) = n$, then $(a'p, b'q) \in E'$, where E' is the kernel of the induced epimorphism $P' = P \otimes_A A' \to M'$. Proposition 2.7.6 implies that $N_{E'} = 0$ and, therefore, $a'p = c'p_1$ and $b'q = c'q_1$, where $c' \in A'$ and $(p_1, q_1) \in E$. The first equality implies that there exists an element $a \in A$ with either (l) $aa' = c'$ and $p = ap_1$, or (r) $a' = ac'$ and $ap = p_1$, and the second equality implies that there exists an element $b \in A$ with either (l) $bb' = c'$ and $q = bq_1$, or (r) $b' = bc'$ and $bq = q_1$. If both equalities are of type (l), we may assume that $\frac{a}{b} \in A$, and we get $\frac{a}{b}a' = b'$ and $m = \frac{a}{b}bm_1 = \frac{a}{b}m$, where $m_1 = \varphi(p_1) = \varphi(q_1)$. Furthermore, if the firs equality is of type (l) and the second one is of type (r), we get $(ab)a' = bc' = b'$ and $m = am_1 = (ab)n$. Finally, if both equalities are of type (r), we may assume that $\frac{a}{b} \in A$, and we get $a' = \frac{a}{b}(bc') = \frac{a}{b}b'$ and $b(\frac{a}{b}m) = bn$. Since M is an integral A-algebra, the latter implies that $\frac{a}{b}m = n$. ∎

For a field K with a fixed valuation $|\ |$, we denote by K° the corresponding valuation $\mathbf{F}_1$-algebra $\{f \in K \big| |f| \le 1\}$, by $K^{\circ\circ}$ its Zariski maximal ideal $\{f \in K^\circ \big| |f| < 1\}$, and by $\widetilde{K}$ its *residue field* $K^\circ/K^{\circ\circ}$. Notice that there is a canonical isomorphism of $\mathbf{F}_1$-field $K/(K^\circ)^* \xrightarrow{\sim} |K|$ and that the canonical embedding $\widetilde{K} \hookrightarrow K^\circ$ induces an isomorphism $\widetilde{K}^* \xrightarrow{\sim} (K^\circ)^*$.

Let L/K be an extension of valuation $\mathbf{F}_1$-fields. Then there is an exact sequence of groups

$$1 \longrightarrow \widetilde{L}^*/\widetilde{K}^* \longrightarrow L^*/K^* \longrightarrow |L^*|/|K^*| \longrightarrow 1$$

We say that an extension of valuation $\mathbf{F}_1$-fields L/K is *unramified* if $|L| = |K|$, i.e., $\widetilde{L}^*/\widetilde{K}^* \xrightarrow{\sim} L^*/K^*$. It is easy to see that this is equivalent to the property that the Zariski ideal $L^{\circ\circ}$ of L° is generated by $K^{\circ\circ}$. An extension of valuation $\mathbf{F}_1$-fields L/K is said to be *purely ramified* if $\widetilde{K} \xrightarrow{\sim} \widetilde{L}$, i.e., $L^*/K^* \xrightarrow{\sim} |L^*|/|K^*|$. Every extension of valuation $\mathbf{F}_1$-fields L/K has a unique maximal unramified subextension $K \subset M \subset L$ such that L is purely unramified over M. Namely, M is generated by $\widetilde{L}$ over K (i.e., $M = K \cdot \widetilde{L}$).

Let L be a finite extension of K. By Corollary 2.7.3, the valuation on K extends in a unique way to a valuation on L. The *ramification index* $e(L/K)$ is the order of the quotient group $|L^*|/|K^*|$.

The *ramification degree* $f(L/K)$ is the degree $[\widetilde{L} : \widetilde{K}]$.

2.7.8. Proposition. *In the above situation, one has* $[L : K] = f(L/K)e(L/K)$ *and, in particular,* $f(L/K) = [M : K]$ *and* $e(L/K) = [L : M]$*, where* M *is the maximal unramified subextension.*

Proof. It suffices to consider the case when L is generated by an element $\beta \in L^*$ with $\beta^p = \alpha \in K^*$ for some prime $p \geq 2$. Suppose first that $|\alpha| \in |K^*|^p$. Then $\alpha = \gamma^p \alpha'$ with $\alpha', \gamma \in K^*$ and $|\alpha'| = 1$. Replacing β by $\frac{\beta}{\gamma^p}$, we may assume that $\alpha \in (K^\circ)^*$. In this case, $\beta \in (L^\circ)^*$, the elements $1, \beta, \dots, \beta^{p-1}$ form a basis of $\widetilde{L}$ over $\widetilde{K}$ and, therefore, $f(L/K) = p$. If $g \in L^{\circ\circ}$, then $g = \gamma\beta^i$ with $0 \leq i \leq p-1$ and $\gamma \in K^{\circ\circ}$, i.e., $e(L/K) = 1$. Suppose now that $|\alpha| \notin |K^*|^p$. Then $|L^*|$ is generated by $|K^*|$ and $|\beta|$ and, therefore, $e(L/K) = p$. Furthermore, if $g \in Ł^*$ and $|g| = 1$, then $g = \gamma\beta^i$ with $0 \leq i \leq p-1$ and $\gamma \in K^*$ such that $|\alpha|^i = |\gamma^{-1}|^p$. Since $|\alpha| \notin |K^*|^p$, it follows that $i = 0$, i.e., $g \in K^*$. Thus, $\widetilde{L} = \widetilde{K}$, i.e., $f(L/K) = 1$. ∎

2.7.9. Remark. If the valuation on K is discrete, then L° is a finite K°-algebra and $e(L/K)$ is equal to the dimension of the $\widetilde{L}$-vector space $L^\circ/(K^{\circ\circ})$, where $(K^{\circ\circ})$ is the Zariski ideal of L° generated by $K^{\circ\circ}$. If the valuation on K is not discrete, then L° is not necessarily finitely generated as a K°-algebra. Indeed, let K an $\mathbf{F}_1$-field with nondiscrete valuation of rank one and such that, for some prime number p, the group $|K^*|$ is not p-divisible. Let α be a nonzero element from $K^{\circ\circ}$ with $|\alpha| \notin |K^*|^p$, and let A be the finitely generated integral K°-algebra $K^\circ(\sqrt[p]{\alpha})$. The fraction $\mathbf{F}_1$-field of A is $L = K(\sqrt[p]{\alpha})$, and the integral closure of A in L is the K°-algebra L° which is not finitely generated over K°. Indeed, suppose that L° is generated by nonzero elements $\beta_1, \dots, \beta_m$ over K°. We may assume that all of these elements do not lie in K and, since $(L^\circ)^* = (K^\circ)^*$, one has $\varepsilon = \max\limits_{1 \leq i \leq m} |\beta_i| < 1$. Then $|\beta| \leq \varepsilon$ for each element $\beta \in L^\circ \backslash K^\circ$. The latter is impossible because there exists an element $a \in K^*$ with $\varepsilon|\beta_1| < |a| < |\beta_1|$, i.e., $\varepsilon < \left|\frac{a}{\beta_1}\right| < 1$. By the way, the maximal Zariski ideal $L^{\circ\circ}$ of L° is generated by $K^{\circ\circ}$.

2.8. Finitely presented K-algebras and modules. Let K be an $\mathbf{F}_1$-field.

2.8.1. Definition. A K-algebra A is said to be *finitely presented* if it is isomorphic to a quotient of $K[T_1, \dots, T_n]$ by a finitely generated ideal.

2.8.2. Proposition. *The following properties of a* K*-algebra* A *are equivalent:*

(a) A *is finitely presented;*

(b) there exist an $\mathbf{F}_1$*-subfield* $K' \subset K$ *with finitely generated group* K'^* *and a finitely generated* K'*-subalgebra* $A' \subset A$ *such that* $A' \otimes_{K'} K \xrightarrow{\sim} A$*;*

(c) A is finitely generated and the family of the stabilizers in K^* of its nonzero elements is a finite set of finitely generated subgroups.

Proof. (a)$\Longrightarrow$(b) Assume that A is the quotient of $B = K[T_1, \dots, T_n]$ by a finitely generated ideal E, and let K' be the $\mathbf{F}_1$-subfield of K which is generated by coefficients of all term components from a finite set of generators of E. Let also E' be the ideal of $B' = K'[T_1, \dots, T_n]$ generated by the same system of generators, and set $A' = B'/E'$. Then the ideal of B generated by E' coincides with E and, by Corollary 2.8.2, $B'/E' \otimes_{K'} K \stackrel{\sim}{\to} B/E$. It follows that A' is a finitely generated K'-subalgebra of A and $A' \otimes_{K'} K \stackrel{\sim}{\to} A$.

(b)$\Longrightarrow$(c) Since the family of stabilizers of nonzero elements does not change after tensoring with a bigger $\mathbf{F}_1$-field, we may assume that the group K^* is finitely generated. This gives, by the way, the fact that the stabilizers are finitely generated groups, and we have to show that there are at most finitely many distinct among them. For this we take a finite system of generators $g_1, \dots, g_n$ of A over K, and consider the map

$$\mathbf{Z}_+^n \to A : \mu = (\mu_1, \dots, \mu_n) \mapsto g^\mu = g_1^{\mu_1} \cdot \ldots \cdot g_n^{\mu_n} \ .$$

We provide $\mathbf{Z}_+^n$ with the partial ordering with respect to which $\mu \le \nu$ if and only if $\mu_i \le \nu_i$ for all $1 \le i \le n$, and notice that, if $\mu \le \nu$ and $g^\mu \ne 0$, then g^μ divides g^ν and, therefore, the stabilizer $G(g^\mu)$ of g^μ is contained in $G(g^\nu)$.

2.8.3. Lemma. *For every sequence $\mu^{(1)}, \mu^{(2)}, \dots$ of elements of $\mathbf{Z}_+^n$, there is a strictly increasing sequence of positive integers $k_1 < k_2 < \dots$ such that $\mu^{(k_1)} \le \mu^{(k_2)} \le \dots$.*

An equivalent way to formulate Lemma 2.8.3 is to say that any subset of $\mathbf{Z}_+^n$ has at most a finite number of minimal elements.

Proof. If $n = 1$, the statement is trivial. Suppose that $n \ge 2$ and that the statement is true for $n - 1$. By the case $n = 1$, we can replace our sequence by a subsequence and assume that their first coordinates do not decrease: $\mu_1^{(1)} \le \mu_2^{(2)} \le \dots$. Consider now the following sequence in $\mathbf{Z}_+^{n-1}$: $\nu^{(i)} = (\mu_2^{(i)}, \dots, \mu_n^{(i)})$. By the induction hypothesis, there is a strictly increasing sequence of positive integers $k_1 < k_2 < \dots$ such that $\nu^{(k_1)} \le \nu^{(k_2)} \le \dots$, and we get $\mu^{(k_1)} \le \mu^{(k_2)} \le \dots$. ∎

Suppose now that there is an infinite sequence of nonzero elements $f_1, f_2, \dots$ such that their stabilizers $G(f_1), G(f_2), \dots$ are pairwise distinct subgroups of K^*. Choose a representation of each f_i in the form $g^{\mu^{(i)}}$ with $\mu^{(i)} \in \mathbf{Z}_+^n$. By Lemma 2.8.3, we can replace our sequence by a subsequence and assume that $\mu^{(1)} \le \mu^{(2)} \le \dots$. We get an increasing sequence of abelian subgroups

$G(f_1) \subset G(f_2) \subset \ldots$ of the finitely generated abelian group K^*. But any such sequence stabilizes, and this contradicts the assumption that all of the subgroups $G(f_1), G(f_2), \ldots$ are pairwise distinct.

(c)$\Longrightarrow$(a) Consider a surjective homomorphism of K-algebras $B = K[T_1, \ldots, K_n] \to A : T_i \mapsto g_i$, and denote by E its kernel. Furthermore, choose a Gröbner basis of E and denote by E' the ideal of B generated by it. Lemma 1.3.1 implies that the canonical surjective homomorphism $B/E' \to A = B/E$ induces a bijection between the sets of cyclic Zariski K-submodules of both K-algebras. Furthermore, let $\{G_i\}_{i\in I}$ be the finite set of finitely generated stabilizers of nonzero elements of A, and, for every $i \in I$, fix a finite system $\{\lambda_{ij}\}_{j\in J_i}$ of generators of G_i. As in the proof of the previous implication, we consider the map $\mathbf{Z}_+^n \to A : \mu \mapsto g^\mu$ and $\mathbf{Z}_+^n \to B : \mu \mapsto T^\mu$. For $i \in I$, let Σ_i be the set of all $\mu \in \mathbf{Z}_+^n$ such that the element g^μ is nonzero and its stabilizer $G(g^\mu)$ coincides with G_i, and let $\{\mu^{(il)}\}_{l\in L_i}$ be the finite set of minimal elements of Σ_i (see the remark after the formulation of Lemma 2.8.3). *We claim that the ideal E is generated by E' and the set of pairs* $\{(T^{\mu^{(il)}}, \lambda_{ij}T^{\mu^{(il)}})\}_{i\in I, j\in J_i, l\in L_i}$. Indeed, let E'' denote the latter ideal of B. Since the canonical surjective homomorphism $B/E'' \to A = B/E$ induces a bijection between the sets of cyclic K-vector subspaces of both K-algebras, to establish the equality $E'' = E$, it suffices to verify that $(T^\mu, \lambda T^\mu) \in E''$ for all $i \in I$, $\lambda \in G_i$ and $\mu \in \Sigma_i$. Let $\mu^{(il)}$ be a minimal element of Σ_i with $\mu^{(il)} \le \mu$. Since $(T^{\mu^{(il)}}, \lambda T^{\mu^{(il)}}) \in E''$, it follows that $(T^\mu, \lambda T^\mu) = (T^{\mu^{(il)}} \cdot T^{\mu-\mu^{(il)}}, \lambda T^{\mu^{(il)}} \cdot T^{\mu-\mu^{(il)}}) \in E''$. ∎

2.8.4. Corollary. *(i) The kernel of any epimorphism $B \to A$ from a finitely generated K-algebra B to a finitely presented K-algebra A is a finitely generated ideal of B;*

(ii) the quotient A/E of any finitely presented K-algebra A by a finitely generated ideal E is a finitely presented K-algebra;

(iii) given homomorphisms from a finitely generated K-algebra A to finitely presented K-algebras B and C, the K-algebra $B \otimes_A C$ is finitely presented;

(iv) given a finitely presented algebra A and a K-field K', $A \otimes_K K'$ is a finitely presented K'-algebra, and $\mathrm{Zspec}(A) \xrightarrow{\sim} \mathrm{Zspec}(A \otimes_K K')$;

(v) given a homomorphism of finitely presented K-algebras $\varphi : A \to B$, there exist an $\mathbf{F}_1$-subfield $K' \subset K$ and finitely generated K'-subalgebras $A' \subset A$ and $B' \subset B$ with the properties of Proposition 2.8.2(b) and such that φ is induced by a homomorphism of K'-algebras $A' \to B'$.

Proof. The statements (ii), (iv) and (v) are trivial.

(i) Consider first the case when $B = K[T_1, \ldots, T_n]$. Let K' be an $\mathbf{F}_1$-subfield of K with finitely generated group K'^* and A' be a finitely generated K'-subalgebra of A with $A' \otimes_{K'} K$. We can increase the $\mathbf{F}_1$-subfield K' and assume that the induced homomorphism $B' = K'[T_1, \ldots, T_n] \to A'$

is surjective. Its kernel E' is finitely generated, and *we claim that E is generated by E'.* Indeed, let E'' be the ideal of B generated by E'. Then there is a canonical isomorphism $B'/E' \otimes_{K'} K \stackrel{\sim}{\to} B/E''$. But the left hand side of the latter is $A' \otimes_{K'} K$ which is A. This implies that the canonical epimorphism $B/E'' \to B/E = A$ is bijective and, therefore, $E'' = E$, i.e., the claim is true. In the general case, we take an arbitrary surjective homomorphism of K-algebras $C = K[T_1, \ldots, T_n] \to B$. By the above case, the kernel of the induced epimorphism $C \to A$ is finitely generated. Since it is the preimage of the kernel E of the epimorphism $B \to A$, it follows that E is finitely generated.

(iii) First of all, if $A = K$, the statement follows from the property (b) (or (c)) of Proposition 2.8.2. If A is arbitrary, $B \otimes_A C$ is a quotient of $B \otimes_K C$ by the ideal generated by elements of the form $(ab \otimes c, b \otimes ac)$ for $a \in A$, $b \in B$ and $c \in C$. It remains to notice that it suffices to take the elements a, b and c from finite systems of generators of A, B and C, respectively. ∎

2.8.5. Corollary. *Let A be a finitely presented K-algebra. Then for any Zariski prime ideal $\mathfrak{p} \subset A$, the following is true:*

(i) the localization $A_\mathfrak{p}$ is a finitely presented K-algebra;

(ii) for any Zariski ideal $\mathbf{a} \subset \mathfrak{p}$, the ideal $E = \mathrm{Ker}(A \to A_\mathfrak{p}/\mathbf{a}A_\mathfrak{p})$ is finitely generated; in particular, the ideal $\Pi_\mathfrak{p}$ is finitely generated;

(iii) the kernel and cokernel of the canonical homomorphism $K^* \to \kappa(\mathfrak{p})^*$ are finitely generated.

Proof. (i) Let $f_1, \ldots, f_n$ be generators of A over K, and assume that $f_i \notin \mathfrak{p}$ for $1 \le i \le m$, and $f_i \in \mathfrak{p}$ for $m+1 \le i \le n$. Then $A_\mathfrak{p}$ is generated over A by the elements $\frac{1}{f_1}, \ldots, \frac{1}{f_m}$, i.e., it is a finitely generated K-algebra. That it is finitely presented follows from Proposition 2.8.2.

(ii) We may assume that we are in the situation of Proposition 2.8.2(b) and that the Zariski ideal $\mathbf{a}$ is generated by $\mathbf{a}' = \mathbf{a} \cap A'$. Since the $\mathbf{F}_1$-algebra A' is finitely generated, it is noetherian, and so it suffices to verify that the ideal E is generated by the ideal $E' = \mathrm{Ker}(A' \to A'_{\mathfrak{p}'}/\mathbf{a}'A'_{\mathfrak{p}'})$ of A', where $\mathfrak{p}' = \mathfrak{p} \cap A'$. Let $(f, g) \in E \backslash (\mathbf{a} \times \mathbf{a})$. Then there exists an element $h \notin \mathfrak{p}$ with $fh = gh$. One has $f = af'$, $g = bg'$ and $h = ch'$ for some $a, b, c \in K^*$ and $f', g', h' \in A' \backslash \mathbf{a}'$. It follows that $af'h' = bg'h'$. By Proposition 2.8.2, this implies that $b = \lambda a$ for $\lambda \in K'^*$. Replacing g' by $\lambda g'$, we get $f'h' = g'h'$, i.e., $(f', g') \in E'$, and $(f, g) = (af', ag')$.

(iii) Since $\kappa(\mathfrak{p})^* = (A_\mathfrak{p})^*$, (i) allows us to replace A by $A_\mathfrak{p}$, and so it suffices to show that the kernel and cokernel of the homomorphism $A \to A^*$ are finitely generated. The kernel is the stabilizer of $1 \in A$ and, therefore, it is finitely generated. Furthermore, let $f_1, \ldots, f_n$ be generators of A over K, and assume that $f_i \in A^*$ for $1 \le i \le m$, and $f_i \notin A^*$ for $m+1 \le i \le n$. Then the group A^* is generated by the elements $f_1, \ldots, f_m$ and the image of the group K^*. It follows that the cokernel of the homomorphism $K^* \to A^*$ is finitely generated. ∎

Let A be an $\mathbf{F}_1$-algebra.

2.8.6. Definition. An A-module M is said to be *finitely presented* if it is isomorphism to a quotient of a free A-module of finite rank $A^{(n)}$ by a finitely generated A-submodule.

The proof of the following is a natural extension of the proof of Proposition 2.8.2.

2.8.7. Proposition. *Let A be a finitely presented K-algebra. Then the following properties of a finite A-module M are equivalent:*

(a) M is finitely presented;

(b) there exist K' and A', as in Proposition 2.8.2(b), and a finite Zariski A'-submodule $M' \subset M$ such that $M' \otimes_{K'} K \xrightarrow{\sim} M$;

(c) the set of stabilizers of nonzero elements of M in K^* is a finite set of finitely generated subgroups.

Proof. (a)$\Longrightarrow$(b). We represent M as a quotient of $A^{(m)}$ by a finitely generated A-submodule, and A as a quotient of $B = K[T_1, \dots, T_n]$ by a finitely generated ideal. Then M can be considered as an B-module, which is a quotient of $B^{(m)}$ by a finitely generated B-submodule E. Let K' be the K'-subfield of K generated by the coefficients of all term components from a finite set of generators of E, $B' = K'[T_1, \dots, T_n]$, E' the B'-submodule of $B'^{(m)}$ generated by the same set of generators, and $M' = B'^{(m)}/E'$. Then $M' \otimes_{K'} K \xrightarrow{\sim} M$.

(b)$\Longrightarrow$(c). As in the proof of the corresponding implication of Proposition 2.8.2, we may assume that the group K^* is finitely generated. Fix finite systems of generators $g_1, \dots, g_n$ of A over K and $e_1, \dots, e_l$ of M over A. Suppose there is an infinite sequence $m_1, m_2, \dots$ of elements of M such that their stabilizers $G(m_1), G(m_2), \dots$ are pairwise distinct. Replacing this sequence by a subsequence, we may assume that, for some $1 \le j \le l$, each m_i is of the form $g^{\mu^{(i)}} e_j$ with $\mu^{(i)} \in \mathbf{Z}_+^n$. By Lemma 2.8.3, we can again replace this sequence by a subsequence so that $\mu^{(1)} \le \mu^{(2)} \le \dots$ and, in particular, $G(m_1) \subset G(m_2) \subset \dots$. Any such sequence of subgroups of K^* stabilizes, and this contradicts the assumption that all of the groups $G(m_i)$ are pairwise distinct.

(c)$\Longrightarrow$(a). Fix a surjective homomorphism of K-algebras $B = K[T_1, \dots, T_n] \to A : T_i \mapsto g_i$ and a finite system $e_1, \dots, e_p$ of generators of M over A, and denote by E the kernel of the induced homomorphism of B-modules $B^{(m)} \to M$. Furthermore, choose a Gröbner basis of E and denote by E' the B-submodule of $B^{(m)}$ generated by it. By Lemma 1.3.1, the canonical surjective homomorphism $B^{(m)}/E' \to M = B^{(m)}/E$ induces a bijection between the sets of cyclic Zariski K-submodules. Furthermore, let $\{G_i\}_{i \in I}$ be the finite set of finitely generated stabilizers of nonzero elements of M and, for every $i \in I$, fix a finite system $\{\lambda_{ij}\}_{j \in J_j}$ of generators of G_i. For $i \in I$ and $1 \le k \le l$, let Σ_{ik} be the set of all $\mu \in \mathbf{Z}_+^n$ such that the element $g^\mu e_k$ is nonzero and its stabilizer $G(g^\mu e_k)$ coincides with G_i, and let $\{\mu^{(ikl)}\}_{l \in L_{ik}}$ be the set of minimal elements of Σ_{ik}. Then the reasoning from the proof of the corresponding implication of Proposition 2.8.2 shows that the B-submodule E is generated by E' and the set of pairs $(T^{\mu^{(ikl)}} e_k, \lambda_{ij} T^{\mu^{(ikl)}} e_k)_{i \in I, j \in J_i, l \in L_{ik}}$. ∎

2.8.8. Corollary. *Let $A \to B$ be a homomorphism of K-algebras, and assume that A is a finitely presented K-algebra. Then the following are equivalent:*

(a) B is a finitely presented K-algebra and a finite A-module;

(b) B is a finitely presented A-module. ∎

The following property corresponds to the commutative algebra property of a coherent ring.

2.8.9. Corollary. *Let A be a finitely presented K-algebra. Then any finitely generated Zariski ideal of A is a finitely presented A-module.* ∎

Let A be a finitely presented K-algebra, and M a finitely presented A-module.

2.8.10. Proposition. *(i) The A-module M is decomposable, and it admits a minimal primary decomposition $\Delta(M) = \bigcap_{i=1}^n E_i$ with finitely generated A-submodules E_i;*

(ii) the set $\mathrm{ass}(M)$ is finite, and the map $\mathrm{ass}(M) \to \mathrm{Zass}(M)$ is surjective;

(iii) there is a chain of Zariski A-submodules $N_0 = 0 \subset N_1 \subset \ldots \subset N_k = M$ such that each quotient N_i/N_{i-1} is isomorphic to an A-module of the form A/Π, where Π is a prime ideal of A;

(iv) the radical $\mathbf{r}(E)$ of any finitely generated A-submodule E of M is a finitely generated ideal and, in particular, the nilradical $\mathbf{n}(M)$ of M is a finitely generated ideal.

Proof. By Proposition 2.8.7, there exist an $\mathbf{F}_1$-subfield $K' \subset K$ with finitely generated group K'^*, a finitely generated K'-subalgebra $A' \subset A$ and a finitely generated Zariski A'-submodule $M' \subset M$ such that $A' \otimes_{K'} K \xrightarrow{\sim} A$ and $M' \otimes_{K'} K \xrightarrow{\sim} M$. Then A' is a finitely generated $\mathbf{F}_1$-algebra. This implies A' and M' are noetherian and, by Proposition 2.4.11, the properties (i)-(iii) hold for A' and M'.

We claim that for any primary A'-submodule E' of M' the A-submodule E of M generated by E' is also primary. Indeed, we have to show that, if (fm, fn) is a nonzero pair in E with $f \in A$ and $m, n \in M$, then either $(m, n) \in E$, or $f \in \mathrm{z}\mathbf{n}(M/E)$. We may assume that $f \in A'$, and let $m = \alpha m'$ and $n = \beta n'$ for $\alpha, \beta \in K^*$ and $m', n' \in M'$. By Proposition 2.7.6(ii), the pair $(fm, fn) = (\alpha f m', \beta f n')$ is of one of two types. If $(\alpha f m', \beta f n') = (a m'', b n'')$ with $m'', n'' \in N_{E'}$ and $a, b \in K$, we get $fm' = \frac{a}{\alpha} m''$ and $fn' = \frac{b}{\beta} n''$. It follows that $\frac{a}{\alpha}, \frac{b}{\beta} \in K'^*$ and, since the Zariski

A-submodule $N_{E'}$ of M' is primary, it follows that either $m', n' \in N_{E'}$, or $f \in \mathrm{z}\mathbf{n}(M'/N_{E'}) \subset \mathrm{z}\mathbf{n}(M'/E') \subset \mathrm{z}\mathbf{n}(M/E)$. If $(\alpha f m', \beta f n') = (am'', an'')$ with $(m'', n'') \in E'$ and $a \in K$, we get $(fm', fn') = (\frac{a}{\alpha}m'', \frac{a}{\beta}n'')$. It follows that $\frac{a}{\alpha}, \frac{b}{\beta} \in K'^*$ and, therefore, $\frac{\alpha}{\beta} \in K'^*$. One also has $(f(\frac{\alpha}{\beta}m'), fn') = (\frac{a}{\alpha}m'', \frac{a}{\alpha}n'') \in E'$ and, therefore, either $(\frac{\alpha}{\beta}m', fn') \in E'$, or $f \in \mathrm{z}\mathbf{n}(M'/E') \subset \mathrm{z}\mathbf{n}(M/E)$. If the former inclusion holds, it implies the inclusion $(m, n) \in E$, and the claim follows.

The claim and Propositions 2.7.6(iv) imply that, any minimal primary decomposition $\Delta(M') = \bigcap_{i=1}^n E'_i$ (which exists, by Proposition 2.4.11(i)) gives rise to a minimal primary decomposition $\Delta(M) = \bigcap_{i=1}^n E_i$, where E_i is the ideal of A generated by E'_i, i.e., we get the statement (i). It follows also that the canonical surjective map $\mathrm{Fspec}(A') \to \mathrm{Fspec}(A)$ gives rise to a bijection $\mathrm{ass}(M) \stackrel{\sim}{\to} \mathrm{ass}(M')$. Since the canonical bijective map $\mathrm{Zspec}(A) \stackrel{\sim}{\to} \mathrm{Zspec}(A')$ gives rise to a bijection $\mathrm{Zass}(M) \stackrel{\sim}{\to} \mathrm{Zass}(M')$, Proposition 2.4.11(ii) implies the statement (ii). The statement (iii) follows from Proposition 2.4.11(iii). Finally, increasing the above $\mathbf{F}_1$-subfield K' of K, we may assume that the A-submodule E is generated by an A'-submodule E' of M'. Then it is easy to see that the radical $\mathbf{r}(E)$ of E is generated by the radical $\mathbf{r}(E')$ of E', and (iv) follows. ∎

For an integer $n \geq 1$, let A^n denote the K-subalgebra of A generated by elements of the form f^n for $f \in A$.

2.8.11. Lemma. *(i) A^n is a finitely presented K-algebra;*

(ii) A is a finitely presented A^n-module;

(iii) there exists $d \geq 1$ such that, for every $n \geq 1$ divisible by d, the K-algebra A^n is reduced.

Proof. (i) and (ii). Let elements $f_1, \dots, f_m$ generate the K-algebra A. Then the K-algebra A^n is generated by the elements $f_1^n, \dots, f_m^n$, i.e., it is finitely generated. It is finitely presented since the condition (c) of Proposition 2.8.2 is satisfied. Furthermore, the A^n-module A is generated by elements of the form $f_1^{i_1} \cdot \ldots \cdot f_m^{i_m}$ with $0 \leq i_1, \dots, i_m \leq m-1$. It is therefore a finitely presented A^n-module, by Corollary 2.8.8.

(iii) Consider first the case when $K = \mathbf{F}_1$. Then the $\mathbf{F}_1$-algebra is noetherian and, in particular, the ideal $E = \{(f, g) \in A \times A \big| f^n = g^n \text{ for some } n \geq 1\}$ is finitely generated. It follows that there exists $d \geq 1$ such that for any $(f, g) \in E$ one has $f^d = g^d$. *We claim that for any $n \geq 1$ divisible by d the $\mathbf{F}_1$-algebra A^n is reduced.* Indeed, let $(f^n, g^n) \in \mathbf{n}(A^n)$. Then $(f, g) \in E$ and, therefore, $f^d = g^d$. Since n is divisible by d, it follows that $f^n = g^n$.

Consider now the case when the group K^* is finitely generated. Then A can be viewed as a finitely generated $\mathbf{F}_1$-algebra, and let B denote the latter. Then $A^n = KB^n$, and the validity of the required fact for B implies its validity for A.

Finally, consider the general case. Let K' be an $\mathbf{F}_1$-subfield of K, and A' a finitely generated K'-subalgebra of A with $A = A' \otimes_{K'} K$. Then $A^n = {A'}^n \otimes_{K'} K$, and the required fact for A follows from the previous case. ∎

2.8.12. Corollary. *Let $A \to B$ be a homomorphism of finitely presented K-algebras, and assume that $B/\mathbf{n}(B)$ is a finitely presented A-module. Then B is a finitely presented A-module.*

Proof. The situation is easily reduced to the case when $K = \mathbf{F}_1$. Let n be a positive integer such that the $\mathbf{F}_1$-algebras A^n and B^n are reduced. In particular, the map $f \mapsto f^n$ gives rise to an isomorphism $B/\mathbf{n}(B) \xrightarrow{\sim} B^n$. The assumption implies that B^n is a finitely presented A^n-module. Since B is a finitely presented B^n-module, it follows that B is a finitely presented A^n-module and, therefore, B is a finitely presented A-module. ∎

2.8.13. Remarks. The statement of Proposition 2.8.10(i) is not true if the K-algebra A is only finitely generated but not finitely presented. For example, assume that there is a sequence of subgroups $G_0 = \{1\} \subset G_1 \subset G_2 \subset \ldots \subset K^*$ with $G_i \neq G_{i+1}$ for all $i \geq 0$, and let A be the quotient of $K[T]$ by the ideal E consisting of the zero pair $(0,0)$ and all pairs of the form $(\lambda T^i, \mu T^i)$ with $\lambda, \mu \in K^*$ and $\lambda\mu^{-1} \in G_i$. If t is the image of T in A, each nonzero element of A is equal either to t^n, or to λt^n with $n \geq 0$ and $\lambda \notin G_n$. There are two Zariski prime ideals $\mathfrak{p} = 0$ and $\mathbf{m} = At$. The only $\mathfrak{p}$-primary ideal is $\Pi_\mathfrak{p}$, which is generated by the pairs $(\lambda, 1)$ with $\lambda \in G = \bigcup_{i=0}^\infty G_i$, and each $\mathbf{m}$-primary ideal contains the $\mathbf{m}$-primary ideal $E_n = \mathrm{Ker}(A \to A/\mathbf{m}^n)$ for some $n \geq 1$. The intersection $\Pi_\mathfrak{p} \cap E_n$ contains the elements $(\lambda t^n, t^n)$ with $\lambda \in G \backslash G_n$, and so it is strictly bigger than $\Delta(A)$.

§3. Topology on the spectrum of an $\mathbf{F}_1$-algebra

3.1. Definition and basic properties. Both spectra $\mathrm{Zspec}(A)$ and $\mathrm{Fspec}(A)$ of an $\mathbf{F}_1$-algebra A are provided with topology as follows.

A base of topology on $\mathrm{Zspec}(A)$ is formed by sets of the form $D(f) = \{\mathfrak{p} \in \mathrm{Zspec}(A) \big| f \notin \mathfrak{p}\}$. The family of closed subsets consists of sets of the form $V(\mathbf{a}) = \{\mathfrak{p} \in \mathrm{Zspec}(A) \big| \mathbf{a} \subset \mathfrak{p}\}$, where $\mathbf{a}$ is a Zariski ideal of A. (One has $V(\mathbf{a}) \cup V(\mathbf{b}) = V(\mathbf{a} \cap \mathbf{b})$ and $D(f) \cap D(g) = D(fg)$.) The Zariski spectrum $\mathrm{Zspec}(A)$ is in fact not interesting as a topological space. For example, the maximal Zariski ideal $\mathbf{m}_A$ is a unique closed point of $\mathrm{Zspec}(A)$, and any open neighborhood of $\mathbf{m}_A$ coincides with the whole space. We will mostly consider $\mathrm{Zspec}(A)$ as a partially ordered set with respect to the partially ordering introduced in §1.2 (it is opposite to the inclusion relation). For example, if A is a valuation $\mathbf{F}_1$-algebra, $\mathrm{Zspec}(A)$ is a totally ordered set.

Furthermore, $\mathrm{Fspec}(A)$ is provided with the weakest topology in which sets of the form $D(f,g) = \{\Pi \in \mathrm{Fspec}(A) \big| (f,g) \notin \Pi\}$, where $f,g \in A$, are open. A base of this topology is forms by finite intersections $\bigcap_{i=1}^{n} D(f_i, g_i)$ with $f_i, g_i \in A$. The canonical map $\mathrm{Fspec}(A) \to \mathrm{Zspec}(A) : \Pi \mapsto \mathfrak{p}_\Pi = \{f \in A \big| (f,0) \in \Pi\}$ is evidently continuous. The restriction of the topology of $\mathrm{Fspec}(A)$ to the fiber of the above map at a Zariski prime ideal $\mathfrak{p}$ is such that its base of open sets consists of collections of subgroups of $\kappa(\mathfrak{p})^*$ that do not intersect with a fixed finite set of elements (see Lemma 1.2.5). In particular, $\Pi_\mathfrak{p}$ is the generic point of the fiber, and $\Pi_{(\mathfrak{p})}$ is a unique closed point of the fiber.

Notice that sets of the form $V(E) = \{\Pi \in \mathrm{Fspec}(A) \big| E \subset \Pi\}$, where E is an ideal of A, are closed in $\mathrm{Fspec}(A)$ and, in fact, the topology on $\mathrm{Fspec}(A)$ is the weakest one with respect such sets are closed. We will say a subset of $\mathrm{Fspec}(A)$ is *strongly closed* if it is a finite union of sets of the form $V(E)$. The family of strongly closed subsets is preserved under finite (but not arbitrary) intersections. It follows that every closed subset of $\mathrm{Fspec}(A)$ is the intersection of a filtered family of strongly closed subsets. Indeed, if Σ is a closed subset, every point $x \notin \Sigma$ has an open neighborhood $\mathcal{U} = \bigcap_{i=1}^{n} D(f_i, g_i)$ that does not intersect with Σ. Since the complement of $\mathcal{U}$ is the strongly closed set $\bigcup_{i=1}^{n} V(E_i)$, where E_i is the ideal generated by the pair (f_i, g_i), the claim follows.

Every homomorphism $\varphi : A \to B$ gives rise to continuous maps $\mathrm{Fspec}(B) \to \mathrm{Fspec}(A) : \Pi \mapsto \varphi^{-1}(\Pi)$ and $\mathrm{Zspec}(B) \to \mathrm{Zspec}(A) : \mathfrak{p} \mapsto z\varphi^{-1}(\mathfrak{p})$. If φ is surjective, these maps induce homeomorphisms of $\mathrm{Zspec}(B)$ and $\mathrm{Fspec}(B)$ with their images in $\mathrm{Zspec}(A)$ and $\mathrm{Fspec}(A)$, respectively, and the image of $\mathrm{Fspec}(B)$ is closed whereas the image of $\mathrm{Zspec}(B)$ is not necessarily closed.

If B is a commutative ring with unity, then every homomorphism of $\mathbf{F}_1$-algebras $A \to B^\cdot$ gives rise to a continuous map $\mathrm{Spec}(B) \to \mathcal{X} = \mathrm{Fspec}(A)$. Namely, it takes a prime ideal $\mathfrak{q}$ of B to the prime ideal $\mathrm{Ker}(A \to (B/\mathfrak{q})^\cdot)$ of A. For example, there is a canonical continuous map $\mathrm{Spec}(B) \to \mathrm{Fspec}(B^\cdot)$ whose composition with the canonical map $\mathrm{Fspec}(B^\cdot) \to \mathrm{Zspec}(B^\cdot)$ is injective.

We will denote points of $\mathcal{X}$ by letters x, y and so on. For a point $x \in \mathcal{X}$, we denote by Π_x and $\mathfrak{p}_x$ the corresponding prime and Zariski prime ideals, and by $\kappa(x)$ the fraction $\mathbf{F}_1$-field of the integral domain A/Π_x. The image of an element $f \in A$ in $\kappa(x)$ will be denoted by $f(x)$. For example, for $f, g \in A$, one has $D(f,g) = \{x \in \mathcal{X} \big| f(x) \neq g(x)\}$.

3.1.1. Theorem. *The spectrum* $\mathrm{Fspec}(A)$ *is a quasi-compact sober topological space.*

Recall that a topological space is called sober if every irreducible closed subset has a unique generic point. Notice that the similar statement for the Zariski spectrum $\mathrm{Zspec}(A)$ is trivial.

Proof. Step 1. For a (usual) field k, let $k[A]$ denote the set of finite sums $\sum_{i=1}^{n} \lambda_i f_i$ with $\lambda_i \in k$ and $f_i \in A$. The set $k[A]$ provided with the evident addition and multiplication is a commutative k-algebra. The canonical homomorphism of $\mathbf{F}_1$-algebras $A \to k[A]^{\cdot}$ gives rise to a continuous map $\tau : \mathrm{Spec}(k[A]) \to \mathrm{Fspec}(A)$. *Then the image of τ contains all prime ideals of the form $\Pi_{(\mathfrak{p})}$ for Zariski prime ideals $\mathfrak{p} \subset A$.* Indeed, the kernel of the homomorphism of $\mathbf{F}_1$-algebras $A \to k[A/\Pi_{(\mathfrak{p})}]^{\cdot} = k$ coincides with $\Pi_{(\mathfrak{p})}$. Thus, if $\mathfrak{q}$ is the kernel of the homomorphism $k[A] \to k[A/\Pi_{(\mathfrak{p})}] = k$, then its image in $\mathrm{Fspec}(A)$ is the ideal $\Pi_{(\mathfrak{p})}$.

Step 2. *Given points $x, y \in \mathrm{Fspec}(A)$, one has $x \in \overline{\{y\}}$ if and only if $\Pi_y \subset \Pi_x$.* Indeed, one has $x \in \overline{\{y\}}$ if and only if every open neighborhood of the point x contains the point y. Suppose first that $x \in \overline{\{y\}}$, and let $(f, g) \in \Pi_y$. If $(f, g) \notin \Pi_x$, then $x \in D(f, g)$ and, therefore, $y \in D(f, g)$, i.e., $(f, g) \notin \Pi_x$ which contradicts the assumption. Conversely, suppose that $\Pi_y \subset \Pi_x$, and let $\mathcal{U}$ be an open neighborhood of the point x of the form $\bigcap_{i=1}^{n} D(f_i, g_i)$. Then $(f_i, g_i) \notin \Pi_x$ and, therefore, $(f_i, g_i) \notin \Pi_y$ for all $1 \le i \le n$, i.e., $y \in \mathcal{U}$. The claim implies that, given a Zariski prime ideal $\mathfrak{p} \subset A$, any open set that contains the $\mathfrak{p}$-prime ideal $\Pi_{(\mathfrak{p})}$ contains all $\mathfrak{p}$-prime ideals.

Step 3. *The space $\mathrm{Fspec}(A)$ is quasi-compact.* Indeed, let $\{\mathcal{U}_i\}_{i \in I}$ be an open covering of $\mathrm{Fspec}(A)$. Then for any field k, $\{\tau^{-1}(\mathcal{U}_i)\}_{i \in I}$ is an open covering of the quasi-compact space $\mathrm{Spec}(k[A])$ and, therefore, there is a finite subset $J \subset I$ such that $\mathrm{Spec}(k[A]) = \bigcup_{i \in J} \tau^{-1}(\mathcal{U}_i)$. *We claim that* $\mathrm{Fspec}(A) = \bigcup_{i \in J} \mathcal{U}_i$. Indeed, let Π is a prime ideal of A over a Zariski prime ideal $\mathfrak{p} \subset A$. By Step 1, there exists $i \in J$ with $\Pi_{(\mathfrak{p})} \in \mathcal{U}_i$ and, by Step 2, one has $\Pi \in \mathcal{U}_i$.

Step 4. *The space $\mathrm{Fspec}(A)$ is sober.* Indeed, let V be a nonempty closed irreducible subset of $\mathcal{X}$. *We claim that the ideal E of A consisting of the pairs (f, g) with $f(x) = g(x)$ for all $x \in V$ is prime.* Indeed, suppose that $(fh, gh) \in E$, i.e., $f(x)h(x) = g(x)h(x)$ for all $x \in V$. The equality implies that the set V lies in the union of the closed sets $\mathcal{X} \backslash D(f, g)$ and $\mathcal{X} \backslash D(h, 0)$. Since V is irreducible, one has either $V \subset \mathcal{X} \backslash D(f, g)$, or $V \subset \mathcal{X} \backslash D(h, 0)$. In the former case, we get $(f, g) \in E$ and, in the latter case, we get $(h, 0) \in E$, i.e., E is a prime ideal. *We now claim that $V = \overline{\{x_0\}}$, where x_0 is the point of $\mathcal{X}$ with $\Pi_{x_0} = E$.* Indeed, (i) implies that $V \subset \overline{\{x_0\}}$, and so it suffices to verify that $x_0 \in V$. Suppose that $x_0 \notin V$. Then there exists an open neighborhood $\mathcal{U} = \bigcap_{i=1}^{n} D(f_i, g_i)$ of x_0 with $\mathcal{U} \cap V = \emptyset$. It follows that $V \subset \bigcup_{i=1}^{n} \mathcal{X} \backslash D(f_i, g_i)$. Irreducibility of V implies that $V \subset \mathcal{X} \backslash D(f_i, g_i)$ for some $1 \le i \le n$, i.e., $f_i(x) = g_i(x)$ for all $x \in V$. This means that $f_i(x) = g_i(x)$ for all $x \in V$, i.e., $(f_i, g_i) \in \Pi_{x_0}$, which contradicts the inclusion $x_0 \in D(f_i, g_i)$. ∎

The following is a consequence of the proof of Theorem 3.1.1.

3.1.2. Corollary. *(i) The map $x \mapsto \overline{\{x\}}$ gives rise to a bijection between $\mathrm{Fspec}(A)$ and the* *set of closed irreducible subsets of* $\mathrm{Fspec}(A)$*, and one has* $\mathrm{Fspec}(A/\Pi_x) \xrightarrow{\sim} \overline{\{x\}}$*;*

(ii) the map $\mathfrak{p} \mapsto \Pi_{(\mathfrak{p})}$ *gives rise to a bijection between* $\mathrm{Zspec}(A)$ *and the set of closed points of* $\mathrm{Fspec}(A)$*.* ∎

3.1.3. Proposition. *The following properties of a Zariski prime ideal* $\mathfrak{p} \subset A$ *are equivalent:*

(a) the set $\mathrm{Fspec}(A_{\mathfrak{p}})$ *is open in* $\mathrm{Fspec}(A)$*;*

(b) $\mathrm{Fspec}(A_{\mathfrak{p}})$ *is a neighborhood of the prime ideal* $\Pi_{(\mathfrak{p})}$ *in* $\mathrm{Fspec}(A)$*;*

(c) $A_{\mathfrak{p}} = A_f$ *for some element* $f \in A \backslash \mathfrak{p}$*.*

Proof. The implications (c)$\Longrightarrow$(a)$\Longrightarrow$(b) are trivial. Suppose that (b) is true. Then there exist elements $a_1, b_1, \ldots, a_n, b_n \in A$ such that $\Pi_{(\mathfrak{p})} \in D(a_1, b_1) \cap \ldots \cap D(a_n, b_n) \subset \mathrm{Fspec}(A_{\mathfrak{p}})$. It follows that, for every $1 \leq i \leq n$, either $a_i \notin \mathfrak{p}$ and $b_i \in \mathfrak{p}$, or $a_i \in \mathfrak{p}$ and $b_i \notin \mathfrak{p}$. We may assume that $a_i \notin \mathfrak{p}$ and $b_i \in \mathfrak{p}$ for all $1 \leq i \leq n$. *We claim that* $A_{\mathfrak{p}} = A_f$ *for* $f = a_1 \cdot \ldots \cdot a_n$. Indeed, since $f \notin \mathfrak{p}$, then $A_f \subset A_{\mathfrak{p}}$. Let $\mathfrak{q}$ be the maximal Zariski ideal of A that does not contain any powers of f. Then $\mathfrak{q}$ is a Zariski prime ideal that contains $\mathfrak{p}$, and one has $A_f = A_{\mathfrak{q}}$. Since $a_i \notin \mathfrak{q}$ and $b_i \in \mathfrak{p} \subset \mathfrak{q}$, it follows that $\Pi_{(\mathfrak{q})} \in D(a_1, b_1) \cap \ldots \cap D(a_n, b_n) \subset \mathrm{Fspec}(A_{\mathfrak{p}})$. This implies that $\mathfrak{q} = \mathfrak{p}$, i.e. $A_{\mathfrak{p}} = A_f$ ∎

We now consider an example. Let I be an idempotent $\mathbf{F}_1$-algebra. Then a base of topology on $\mathrm{Fspec}(A)$ is formed by sets of the form $D(f) \cap D(g_1, 1) \cup \ldots D(g_n, 1)$ (which are also, by the way, closed subsets). Indeed, this follows from the equalities $D(f, g) = (D(f) \cap D(g, 1)) \cup (D(g) \cap D(f, 1))$ and $D(f) \cap D(g) = D(fg)$. Since the canonical map $\mathrm{Fspec}(I) \to \mathrm{Zspec}(I)$ is a bijection, we consider the spectrum $\mathrm{Fspec}(I)$ with the induced partial ordering, i.e., $\Pi_{\mathfrak{p}} \leq \Pi_{\mathfrak{q}}$ if $\mathfrak{p} \leq \mathfrak{q}$ (i.e., $\mathfrak{q} \subset \mathfrak{p}$). Notice the set of pairs $(\Pi_{\mathfrak{p}}, \Pi_{\mathfrak{q}})$ with $\Pi_{\mathfrak{p}} \leq \Pi_{\mathfrak{q}}$ is closed in $\mathrm{Fspec}(I) \times \mathrm{Fspec}(I)$.

3.1.4. Proposition. *(i)* $\mathrm{Fspec}(I)$ *is a profinite space, and the image of* $\check{I}$ *in it (under the map* $e \mapsto \Pi_e$*) is dense;*

(ii) the image of $\check{I}$ *in* $\mathrm{Zspec}(I)$ *(under the map* $e \mapsto \mathfrak{p}_e$*) consists of the Zariski prime ideals* $\mathfrak{p} \subset I$ *such that* $\mathrm{Fspec}(I_{\mathfrak{p}})$ *is open in* $\mathrm{Fspec}(I)$*;*

(iii) the correspondence $E \mapsto S_E = \{\Pi_{\mathfrak{p}} \in \mathrm{Fspec}(I) \big| E \subset \Pi_{\mathfrak{p}}\}$ *gives rise to a bijection between the set of ideals of* I *and the set of closed subsets* $S \subset \mathrm{Fspec}(I)$ *such that the infimum of any family of elements of* S *belongs to* S*;*

(iv) the subsets $S \subset \mathrm{Fspec}(I)$ *that correspond to Zariski ideals are characterized by the stronger property: if* $\Pi_{\mathfrak{p}} \leq \Pi_{\mathfrak{q}} \in S$*, then* $\Pi_{\mathfrak{p}} \in S$*;*

(v) the subsets $S \subset \mathrm{Fspec}(I)$ *that correspond to finitely generated ideals are characterized by the property that they are also open sets.*

3.1.5. Lemma. *The canonical bijections in Lemma 1.4.4 are homeomorphisms.*

Proof. The statement follows from the fact that, for the open subsets $D(f) \subset \mathrm{Fspec}(A)$ and $\bigcap_{k=1}^{n} D(f_k, g_k)$, we can find $i \in I$ such that the elements f and f_k, g_k for all $1 \le k \le n$) come from A_i and, therefore, they are the preimages of the corresponding open subsets of $\mathrm{Zspec}(A_i)$ and $\mathrm{Fspec}(A_i)$, respectively. ∎

Proof of Proposition 3.1.4. (i) The idempotent $\mathbf{F}_1$-algebra I is the union of its finite $\mathbf{F}_1$-subalgebras. It remains to notice that, if I is finite, the map $\check{A} \to \mathrm{Fspec}(A) : e \mapsto \Pi_e$ from §1.3 is a bijection, and one has $\{\Pi_e\} = D(e) \cap D(f_1, 1) \cap \dots, D(f_n, 1)$ with $\mathfrak{p}_e = \{f_1, \dots, f_n\}$.

(ii) First of all, if $e \in \check{I}$, then $I_{\mathfrak{p}_e} = I_e$ and, therefore, $\mathrm{Fspec}(I_{\mathfrak{p}_e})$ is open in $\mathrm{Fspec}(I)$. Conversely, suppose that, for a Zariski prime ideal $\mathfrak{p} \subset I$, $\mathrm{Fspec}(I_{\mathfrak{p}})$ is open in $\mathrm{Fspec}(I)$. By Proposition 3.1.3, there exists an element $e \in I \backslash \mathfrak{p}$ with $I_{\mathfrak{p}} = I_e$. This implies that e is the maximal element in $I \backslash \mathfrak{p}$ and, therefore, $\mathfrak{p} = \mathfrak{p}_e$.

(iii) Since $S_E = \mathrm{Zspec}(I/E)$, the set S_E evidently possesses the required properties, and Corollary 1.6.2(ii) implies that $E_{S_E} = E$. Furthermore, for a subset $S \subset \mathrm{Zspec}(I)$, the inclusion $S \subset S_{E_S}$ is trivial. Assume now that S possesses the above properties, and consider first the case when I is finite. We have to verify that, if $\Pi_{\mathfrak{p}} \notin S$, then $E_S \not\subset \Pi_{\mathfrak{p}}$. Suppose first that $\Pi_{\mathfrak{p}} \not\le \Pi_{\mathfrak{q}}$ for all $\Pi_{\mathfrak{q}} \in S$. For every $\Pi_{\mathfrak{q}} \in S$, take an idempotent $e_{\mathfrak{q}} \in \mathfrak{q} \backslash \mathfrak{p}$. Then the idempotent $e = \prod_{\Pi_{\mathfrak{q}} \in S} e_{\mathfrak{q}}$ lies in the intersection of all such $\mathfrak{q}$, but not in $\mathfrak{p}$. It follows that $(e, 0) \in E_S \backslash \Pi_{\mathfrak{p}}$. Suppose now that $\Pi_{\mathfrak{p}} \le \Pi_{\mathfrak{q}}$ for some $\Pi_{\mathfrak{q}} \in S$. We may assume that $\Pi_{\mathfrak{q}}$ is the unique minimal with that property. Let e and f be the maximal idempotents in $A \backslash \mathfrak{p}$ and $A \backslash \mathfrak{q}$, respectively. Since $\mathfrak{q} \subset \mathfrak{p}$, then $f \in \mathfrak{p}$ and, therefore, the pair (e, f) does not lie in $\Pi_{\mathfrak{p}}$. *We claim that (e, f) lies in E_S.* Indeed, let $\mathfrak{r}$ be an element of S. If $\Pi_{\mathfrak{q}} \le \Pi_{\mathfrak{r}}$, then $\Pi_{\mathfrak{p}} \le \Pi_{\mathfrak{r}}$. It follows that $e, f \notin \mathfrak{r}$ and, therefore, $(e, f) \in \Pi_{\mathfrak{r}}$. If $\Pi_{\mathfrak{q}} \not\le \Pi_{\mathfrak{r}}$, then $\Pi_{\mathfrak{p}} \not\le \Pi_{\mathfrak{r}}$ since $\Pi_{\mathfrak{q}}$ is minimal with $\Pi_{\mathfrak{p}} \le \Pi_{\mathfrak{q}}$. It follows that $e, f \in \mathfrak{r}$ and, therefore, $(e, f) \in \Pi_{\mathfrak{r}}$.

Consider now the general case. We have to verify that every prime ideal $\Pi_{\mathfrak{p}}$ that contains E_S belongs to S. Let $\{I_k\}_{k \in K}$ be the filtered family of finite $\mathbf{F}_1$-subalgebras of I and, for $k \in K$, let S_k be the image of S under the canonical map $\mathrm{Fspec}(I) \to \mathrm{Fspec}(I_k)$, and set $E_k = E_S \cap (I_k \times I_k)$. By the previous case, one has $S_k = S_{E_k}$ and, therefore, $\Pi_{\mathfrak{p}} \cap (I_k \times I_k) \in S_k$. The assumption on closeness of S implies that $S \stackrel{\sim}{\to} \varprojlim S_k$. It follows that $\Pi_{\mathfrak{p}} \in S$.

(iv) That the sets S_E corresponding to Zariski ideals possesses the required property is trivial. Suppose that S is a subset with that property, and let E be the corresponding ideal of I, i.e., such that $S = S_E$. We set $\mathbf{a} = \bigcap_{\Pi_{\mathfrak{p}} \in S} \mathfrak{p}$ and $E' = E_{\mathbf{a}} = \Delta(I) \cup (\mathbf{a} \times \mathbf{a})$. *We claim that $E = E'$.* Indeed, by (iii), it suffices to verify that $S_E = S_{E'}$. By the construction, one has $\mathbf{a} = \{f \in I \big| (f, 0) \in E$ and, in particular, $E' \subset E$, i.e., $S_E \subset S_{E'}$. Suppose that $\Pi_\mathfrak{p} \in S_{E'} \backslash S_E$, i.e., $\mathbf{a} \subset \mathfrak{p}$ and $E \not\subset \Pi_\mathfrak{p}$. It follows that there exists a pair $(e, f) \in E \backslash \Pi_\mathfrak{p}$. We may assume that $e \in \mathfrak{p}$ and $f \notin \mathfrak{p}$. If $\Pi_\mathfrak{q} \in S$, then $\Pi_{\mathfrak{p} \cup \mathfrak{q}} \leq \Pi_\mathfrak{p}$ and, by the property of S, we get $\Pi_{\mathfrak{p} \cup \mathfrak{q}} \in S$ and, therefore, $(e, f) \in E \subset \Pi_{\mathfrak{p} \cup \mathfrak{q}}$. Since $e \in \mathfrak{p}$, it follows that $f \in \mathfrak{q}$ for all $\Pi_\mathfrak{q} \in S$. This implies that $f \in \mathbf{a} \subset \mathfrak{p}$, which is a contradiction.

(v) That the sets that correspond to finitely generated ideals are open is trivial. Suppose that the set S_E of an ideal $E \subset I \times I$ is open. Since it is closed, it is compact and, therefore, it is a finite union of open sets of the form $\bigcap_{i=1} D(f_i, g_i)$ with $f_i, g_i \in I$. Let I' be a finite $\mathbf{F}_1$-subalgebra of A that contains all of the elements f_i, g_i, and let S' be the image of S with respect to the canonical morphism $\mathcal{X} \to \mathcal{X}' = \mathrm{Fspec}(I')$. Then S' is preserved by the infimum operation and, by (iii), one has $S' = S_{E'}$ for an ideal E' of I'. Since S is the preimage of S', it follows easily that the ideal E is generated by E', i.e., E is finitely generated. ∎

In §1.6 we constructed an injective homomorphism $I \to \mathrm{Hom}_{\mathrm{inf}}(\mathrm{Zspec}(I), \{0, 1\})$ that takes $e \in I$ to the map $\varphi_e : \mathrm{Zspec}(I) \to \{0, 1\}$ defined by $\varphi_e(\mathfrak{p}) = 1$, if $e \notin \mathfrak{p}$, and $\varphi_e(\mathfrak{p}) = 0$, if $e \in \mathfrak{p}$. Since $\mathrm{Fspec}(I) \xrightarrow{\sim} \mathrm{Zspec}(I)$, we may consider any element of $\mathrm{Hom}_{\mathrm{inf}}(\mathrm{Zspec}(I), \{0, 1\})$ as a map $\mathrm{Fspec}(I) \to \{0, 1\}$.

3.1.6. Proposition. *The image of $\check{I}$ in* $\mathrm{Hom}_{\mathrm{inf}}(\mathrm{Zspec}(I), \{0, 1\})$ *consists of the elements φ for which the induced map* $\mathrm{Fspec}(I) \to \{0, 1\}$ *is continuous.*

Proof. First of all, let $\varphi \in \mathrm{Hom}_{\mathrm{inf}}(\mathrm{Zspec}(I), \{0, 1\})$. Then there is a unique minimal element $\mathfrak{p} \in \mathrm{Zspec}(I)$ with $\varphi(\mathfrak{p}) = 1$. If $\varphi(\mathfrak{q}) = 0$, then $\mathfrak{q} \not\subset \mathfrak{p}$ and, therefore, given an element $e \in \mathfrak{q} \backslash \mathfrak{p}$ one has $\varphi(\mathfrak{q}) = 0$ for any $\mathfrak{q} \in D(e, 1)$. This means that the set $\varphi^{-1}(1)$ is always closed. If $e \in \check{I}$, then $\varphi_e^{-1}(1) = D(e)$ and, therefore, the map φ_e is continuous. Conversely, suppose that an φ is continuous, i.e., the set $\varphi^{-1}(1)$ is open. It follows that a nonempty open set $\mathcal{U} = D(e) \cap D(f_1, 1) \cap \ldots \cap D(f_n, 1)$ contains the prime ideal $\Pi_\mathfrak{p}$ for the above Zariski ideal $\mathfrak{p}$ and lies in $\varphi^{-1}(1)$. *We claim that $\varphi = \varphi_e$.* Indeed, since $\mathfrak{p} \in D(e)$, it follows that $e \notin \mathfrak{p}$ and, therefore, $\mathfrak{p} \subset \mathfrak{p}_e$. It follows also that $\Pi_e \in \mathcal{U}$ and, therefore, $\varphi(\mathfrak{p}_e) = 1$. Since $\mathfrak{p}$ is the minimal element of $\mathrm{Zspec}(I)$ with the latter property, we get $\mathfrak{p} = \mathfrak{p}_e$. ∎

3.2. Irreducible components of the spectrum. Let A be an $\mathbf{F}_1$-algebra and $\mathcal{X} = \mathrm{Fspec}(A)$. For a Zariski prime ideal $\mathfrak{p} \subset A$, we set $\mathcal{X}_\mathfrak{p} = \{x \in \mathcal{X} \big| f(x) = 0 \text{ for all } x \in \mathfrak{p}\}$, $\check{\mathcal{X}}_\mathfrak{p} = \{x \in \mathcal{X}_\mathfrak{p} \big| f(x) \neq \text{ for } f \notin \mathfrak{p}\}$, and $\mathcal{X}^{(\mathfrak{p})} = \overline{\check{\mathcal{X}}_\mathfrak{p}}$. We also set $A^{(\mathfrak{p})} = A/\Pi_\mathfrak{p}$. By Corollary 3.1.2, one has $\mathcal{X}^{(\mathfrak{p})} = \mathrm{Fspec}(A^{(\mathfrak{p})}) = \overline{\{\Pi_\mathfrak{p}\}}$. In particular, each set $\mathcal{X}^{(\mathfrak{p})}$ lies in only one connected component of $\mathcal{X}$. Furthermore, the map $\Pi_{\mathfrak{p}} \mapsto \mathcal{X}^{(\mathfrak{p})}$ gives rise to a bijection between the set of minimal prime ideals of A and the set of irreducible components of $\mathrm{Fspec}(A)$. If the set of irreducible components of $\mathcal{X}$ is finite, $\mathcal{X}$ is a locally connected space and, in particular, all its quasi-components are connected components.

3.2.1. Lemma. *The set of irreducible components of $\mathcal{X}$ is finite if and only if the $\mathbf{F}_1$-algebra A is weakly decomposable. Furthermore, in this case the idempotent $\mathbf{F}_1$-algebra I_A is finite.*

Proof. If $\mathcal{X}^{(\mathfrak{p}_1)}, \ldots, \mathcal{X}^{(\mathfrak{p}_n)}$ are all of the irreducible components of $\mathcal{X}$ then, for every prime ideal Π, the irreducible closed subset $\mathrm{Fspec}(A/\Pi)$ lies in some $\mathcal{X}^{(\mathfrak{p}_i)}$ and, therefore, Π contains the ideal $\Pi_{\mathfrak{p}_i}$. Proposition 2.2.4 implies that $\mathbf{n}(A) = \bigcap_{i=1}^n \Pi_{\mathfrak{p}_i}$, i.e., A is weakly decomposable. Conversely, suppose that A is weakly decomposable, i.e., $\mathbf{n}(A) = \bigcap_{i=1}^n \Pi_{\mathfrak{p}_i}$. Then every prime ideal Π contains the above intersection, and Lemma 1.2.7(ii) implies that there exists a nonempty subset $J \subset \{1, \ldots, n\}$ such that $\Pi \supset \Pi_{\mathfrak{q}}$, where $\mathfrak{q} = \bigcup_{i \in J} \mathfrak{p}_i$. It follows that $\Pi \in \mathcal{X}^{(\mathfrak{q})}$ and, therefore, the set of irreducible components of $\mathcal{X}$ is finite. By Proposition 2.5.5, to verify the last statement we may assume that A is reduced. In this case it is embedded in the finite direct product of integral $\mathbf{F}_1$-algebras $\prod_{i=1}^n A^{(\mathfrak{p}_i)}$ whose idempotent $\mathbf{F}_1$-subalgebra is finite. ∎

3.2.2. Theorem. *Let $A \to B$ be a homomorphism of $\mathbf{F}_1$-algebras that induces a map $\varphi : \mathcal{Y} = \mathrm{Fspec}(B) \to \mathcal{X} = \mathrm{Fspec}(A)$. Then*

(i) if the homomorphism $A \to B$ is injective, then for every irreducible component $\mathcal{X}^{(\mathfrak{p})}$ of $\mathcal{X}$ there exists an irreducible component $\mathcal{Y}^{(\mathfrak{q})}$ of $\mathcal{Y}$ such that $\varphi(\check{\mathcal{Y}}_{\mathfrak{q}}) = \check{\mathcal{X}}_{\mathfrak{p}}$;

(ii) if the map φ has dense image and A is reduced, the homomorphism $A \to B$ is injective.

Proof. (i) The homomorphism from $A_{\mathfrak{p}}$ to the localization of B with respect to $A \backslash \mathfrak{p}$ is injective. We may therefore replace A by $A_{\mathfrak{p}}$ and B by that localization and assume that $\mathfrak{p} = \mathbf{m}_A$.

Step 1. *One has $\mathbf{m}_B \cap A = \mathbf{m}_A$.*

Case 1: A and B are finitely generated over $\mathbf{F}_1$. We may assume that A and B are reduced. By Propositions 2.4.11(i) and 2.4.9(iii), there is a minimal primary decomposition $\Delta(A) = \bigcap_{i=1}^n \Pi_{\mathfrak{p}_i}$ with $\Pi_{\mathfrak{p}_i}$'s lying in the set of minimal prime ideals of A. Lemma 1.2.7(ii) implies that every minimal prime ideal of A coincides with $\Pi_{\mathfrak{p}}$ for some $\mathfrak{p} = \mathfrak{p}_{i_1} \cup \ldots \cup \mathfrak{p}_{i_k}$. Since $\Delta(B) = \bigcap \Pi_{\mathfrak{q}}$, then $\Delta(A) = \bigcap (\Pi_{\mathfrak{q}} \cap (A \times A))$, where the intersections are taken over the finite set of Zariski prime ideals of B. It follows that, for every $1 \le i \le n$, $\mathfrak{p}_i$ lies in the image of $\mathrm{Zspec}(B)$. This implies that each Zariski prime ideal $\mathfrak{p}$ of the form $\mathfrak{p}_{i_1} \cup \ldots \cup \mathfrak{p}_{i_k}$ also lies in the image of $\mathrm{Zspec}(B)$. By Corollary 1.2.6, $\mathbf{m}_A$ is of such form, and so there exists a Zariski prime ideal $\mathfrak{q} \subset B$ with $\mathbf{m}_A = \mathfrak{q} \cap A$. This implies that $\mathbf{m}_B \cap A = \mathbf{m}_A$.

Case 2: A is finitely generated over $\mathbf{F}_1$. Let $\{B_i\}_{i\in I}$ be the filtered family of finite generated A-subalgebras of B. By the previous case, one has $\mathbf{m}_{B_i}\cap A=\mathbf{m}_A$. Since $\mathbf{m}_B=\bigcup_{i\in I}\mathbf{m}_{B_i}$, we get the required fact.

Case 3: A and B are arbitrary. Let $\{A_i\}_{i\in I}$ be the filtered family of finite generated $\mathbf{F}_1$-subalgebras of A. By the previous case, one has $\mathbf{m}_B\cap A_i=\mathbf{m}_{A_i}$. Since $\mathbf{m}_A=\bigcup_{i\in I}\mathbf{m}_{A_i}$, we get the required fact.

Step 2. *The statement (i) is true.* By Step 1, we have $\mathbf{m}_B\cap A=\mathbf{m}_A$. It follows that $\Pi_{\mathbf{m}_B}\cap(A\times A)=\Pi_{\mathbf{m}_A}$. Let $\Pi_{\mathfrak{q}}$ be a minimal prime ideal of B which lies in $\Pi_{\mathbf{m}_B}$. By the assumption, $\Pi_{\mathbf{m}_A}$ is a minimal prime ideal of A and, therefore, $\Pi_{\mathfrak{q}}\cap(A\times A)=\Pi_{\mathbf{m}_A}$. In particular, $\mathcal{Y}^{(\mathfrak{q})}$ is an irreducible component of $\mathcal{Y}$ and $\varphi(\check{\mathcal{Y}}^{(\mathfrak{q})})\subset\mathcal{X}^{(\mathbf{m})}$. Since $\Pi_{\mathfrak{q}}\subset\Pi_{\mathbf{m}_B}$, it follows that the homomorphism $A^*\to\kappa(\mathfrak{q})^*$ is injective, and we get the required equality $\varphi(\check{\mathcal{Y}}^{(\mathfrak{q})})=\mathcal{X}^{(\mathbf{m})}$.

(ii) It suffices to consider the case when A is a finitely generated $\mathbf{F}_1$-algebra. In this case, take a minimal primary decomposition $\Delta(A)=\bigcap_{i=1}^n\Pi_{\mathfrak{p}_i}$ with $\Pi_{\mathfrak{p}_i}$ lying in the set of minimal prime ideals of A. Then A embeds in the direct product $\prod_{i=1}^n A/\Pi_{\mathfrak{p}_i}$. Let F_i be the ideal of B generated by $\Pi_{\mathfrak{p}_i}$. Since the set of all points in $\mathcal{X}^{(\mathfrak{p}_i)}$ that do not lie in other irreducible components of $\mathcal{X}$ is nonempty and open, it follows that the image of the map $\mathrm{Fspec}(B/F_i)\to\mathrm{Fspec}(A/\Pi_{\mathfrak{p}_i})$ is dense. Thus, replacing A by $A/\Pi_{\mathfrak{p}_i}$ and B by B/F_i, we may assume that A is integral. Let E be the kernel of the homomorphism $A\to B$. Then the image of $\mathcal{Y}$ in $\mathcal{X}$ lies in the closed subset $\mathrm{Fspec}(A/E)$. Since it is dense, we get $\mathrm{Fspec}(A/E)=\mathrm{Fspec}(A)$. This immediately implies that $E=\Delta(A)$, i.e., the homomorphism $A\to B$ is injective. ∎

3.3. Connected components of the spectrum. Recall that the connected component of a point x of a topological space X is the maximal connected subset that contain the point x. The connected components of points define a partition of X by closed subsets. The set of connected components is denoted by $\pi_0(X)$ and provided with the quotient topology induced by that of X. Recall also that the quasi-component of x is the intersection of all open-closed subsets that contain x. The quasi-component of x is a closed subset that contains the connected component of x but does not coincide with it in general. If X is locally connected (i.e., every point has a fundamental system of open connected neighborhoods), all quasi-components of X are connected open subsets.

3.3.1. Theorem. *(i) The canonical map $\mathcal{X}=\mathrm{Fspec}(A)\to\mathrm{Fspec}(I_A)$ is surjective, its fibers are connected, and it induces a homeomorphism $\pi_0(\mathcal{X})\stackrel{\sim}{\to}\mathrm{Fspec}(I_A)$;*

(ii) given $\mathfrak{p},\mathfrak{q}\in\mathrm{Zspec}(A)$, the connected component of $\mathcal{X}^{(\mathfrak{p}\cup\mathfrak{q})}$ depends only on the connected components of $\mathcal{X}^{(\mathfrak{p})}$ and $\mathcal{X}^{(\mathfrak{q})}$;

(iii) the bijection $\pi_0(\mathcal{X}) \xrightarrow{\sim} \mathrm{Zspec}(I_A)$, induced by the homeomorphism from (i), is an isomorphism of posets.

The partial ordering on the set $\pi_0(\mathcal{X})$ is defined as follows: $\mathcal{U} \le \mathcal{V}$ if $\mathfrak{p} \le \mathfrak{q}$ for some $\mathfrak{p}$ and $\mathfrak{q}$ with $\mathcal{X}^{(\mathfrak{p})} \subset \mathcal{U}$ and $\mathcal{X}^{(\mathfrak{q})} \subset \mathcal{V}$. (By the statement (iii), it is a well defined partial ordering.) Notice that the statement (i) implies that all quasi-components of $\mathcal{X}$ are in fact connected components.

3.3.2. Lemma. *(i) For any pair of Zariski prime ideals $\mathfrak{p}, \mathfrak{q} \subset A$, one has $\mathcal{X}^{(\mathfrak{p})} \cap \mathcal{X}^{(\mathfrak{q})} \subset \mathcal{X}^{(\mathfrak{p}\cup\mathfrak{q})}$; (ii) if $\mathfrak{p} \subset \mathfrak{q}$, then $\mathcal{X}^{(\mathfrak{p})} \cap \mathcal{X}^{(\mathfrak{q})} = \mathrm{Fspec}(A^{(\mathfrak{p})}/\mathbf{a}_{\mathfrak{p}\mathfrak{q}})$, where $\mathbf{a}_{\mathfrak{p}\mathfrak{q}}$ is the image of $\mathfrak{q}$ in $A^{(\mathfrak{p})}$.*

Proof. (i) It suffices to show that the ideal of A generated by $\Pi_{\mathfrak{p}}$ and $\mathfrak{q}$ contains the ideal $\Pi_{\mathfrak{p}\cup\mathfrak{q}}$. The latter consists of pairs (f, g) with either $f, g \in \mathfrak{p} \cup \mathfrak{q}$, or $fh = gh$ for some element $h \notin \mathfrak{p} \cup \mathfrak{q}$. In the former case, the pair (f, g) lies in the ideal generated by $\Pi_{\mathfrak{p}}$ and $\Pi_{\mathfrak{q}}$ and, in the latter case, (f, g) lies in the intersection $\Pi_{\mathfrak{p}} \cap \Pi_{\mathfrak{q}}$.

(ii) Notice that, if a pair $(f, g) \in \Pi_{\mathfrak{q}}$ is such that $fh = gh$ for an element $h \notin \mathfrak{q}$, then $(f, g) \in \Pi_{\mathfrak{p}}$. This implies that a prime ideal Π lies in $\mathcal{X}^{(\mathfrak{p})} \cap \mathcal{X}^{(\mathfrak{q})}$ if and only if it contains the ideal generated by $\Pi_{\mathfrak{p}}$ and $\mathfrak{q}$. The required fact follows. ∎

3.3.3. Corollary. *Given Zariski prime ideals $\mathfrak{p} \subset \mathfrak{q} \subset A$ with $\mathcal{X}^{(\mathfrak{p})} \cap \mathcal{X}^{(\mathfrak{q})} \neq \emptyset$, one has $\mathcal{X}^{(\mathfrak{p}\cup\mathfrak{r})} \cap \mathcal{X}^{(\mathfrak{q}\cup\mathfrak{r})} \neq \emptyset$ for any Zariski prime ideal $\mathfrak{r} \subset A$.*

Proof. By Lemma 3.3.2(ii), the assumption implies that the Zariski ideal $\mathbf{a}_{\mathfrak{p}\mathfrak{q}} \subset A^{(\mathfrak{p})}$, generated by the image of $\mathfrak{q}$, is nontrivial, i.e., $(f, 1) \notin \Pi_{\mathfrak{p}}$ for all elements $f \in \mathfrak{q}$. Suppose that $(f, 1) \in \Pi_{\mathfrak{p}\cup\mathfrak{r}}$ for some element $f \in \mathfrak{q} \cup \mathfrak{r}$. Since $f \notin \mathfrak{r}$, it follows that $f \in \mathfrak{q}$. The inclusion $(f, 1) \in \Pi_{\mathfrak{p}\cup\mathfrak{r}}$ means that there exists an element $g \notin \mathfrak{p} \cup \mathfrak{r}$ with $fg = g$. This implies that $(f, 1) \in \Pi_{\mathfrak{p}}$, which is a contradiction. ∎

The *strong connected component* of a point $x \in \mathcal{X}$ is the set of all points $y \in \mathcal{X}$ for which there exist irreducible components $\mathcal{Y}_1, \ldots, \mathcal{Y}_n$, $n \ge 0$, of $\mathcal{X}$ such that $x \in \mathcal{Y}_1$, $y \in \mathcal{Y}_n$ and $\mathcal{Y}_i \cap \mathcal{Y}_{i+1} \neq \emptyset$ for all $1 \le i \le n-1$. Such a strong connected component is connected and, therefore, it lies in a connected component of $\mathcal{X}$ and, if a set $\mathcal{X}^{(\mathfrak{p})}$ has nonempty intersection with a strong connected component, it is entirely contained in that component. Notice that, if the set of irreducible components of $\mathcal{X}$ is finite (i.e., A is weakly decomposable), then every strong connected component is open and, in particular, it is a quasi-component. (The latter is not true in general, see Remark 3.2.10.)

We say that a Zariski prime ideal $\mathfrak{p}$ of A is *marked* if it is of the form $\mathfrak{q}_1 \cup \ldots \cup \mathfrak{q}_n$, where every $\mathfrak{q}_i$ is a Zariski prime ideal for which $\Pi_{\mathfrak{q}_i}$ is a minimal prime ideal of A.

3.3.4. Corollary. *Let $\mathfrak{p}$ and $\mathfrak{q}$ be Zariski prime (resp. and marked) ideals of A. If the sets $\mathcal{X}^{(\mathfrak{p})}$ and $\mathcal{X}^{(\mathfrak{q})}$ lie in one strong connected component, then there exist chains of Zariski prime (resp. and marked) ideals $\mathfrak{p}_0 = \mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_m$ and $\mathfrak{q}_0 = \mathfrak{q} \subset \mathfrak{q}_1 \subset \ldots \subset \mathfrak{q}_m$ with $\mathfrak{p}_m = \mathfrak{q}_m$ and such that $\mathcal{X}^{(\mathfrak{p}_i)} \cap \mathcal{X}^{(\mathfrak{p}_{i+1})} \neq \emptyset$ and $\mathcal{X}^{(\mathfrak{q}_i)} \cap \mathcal{X}^{(\mathfrak{q}_{i+1})} \neq \emptyset$ for all $0 \le i \le m-1$.*

Proof. By the assumption, there exist irreducible components $\mathcal{X}^{(\mathfrak{r}_1)}, \ldots, \mathcal{X}^{(\mathfrak{r}_n)}$ such that $\mathcal{X}^{(\mathfrak{r}_i)} \cap \mathcal{X}^{(\mathfrak{r}_{i+1})} \neq \emptyset$ for all $0 \le i \le n$, where $\mathfrak{r}_0 = \mathfrak{p}$ and $\mathfrak{r}_{n+1} = \mathfrak{q}$. *We claim that the following chains of Zariski prime ideals possess the required property:* $\mathfrak{r}_0 = \mathfrak{p} \subset \mathfrak{r}_0 \cup \mathfrak{r}_1 \subset \ldots \subset \mathfrak{r}_0 \cup \ldots \cup \mathfrak{r}_{n+1}$ *and* $\mathfrak{r}_{n+1} = \mathfrak{q} \subset \mathfrak{r}_n \cup \mathfrak{r}_{n+1} \subset \ldots \subset \mathfrak{r}_0 \cup \ldots \cup \mathfrak{r}_{n+1}$. Indeed, by Lemma 3.3.2(ii), one has $\mathcal{X}^{(\mathfrak{r}_i)} \cap \mathcal{X}^{(\mathfrak{r}_{i+1})} \subset \mathcal{X}^{(\mathfrak{r}_i \cup \mathfrak{r}_{i+1})}$ and, therefore, Corollary 3.3.3 implies that $\mathcal{X}^{(\mathfrak{r}_i)} \cap \mathcal{X}^{(\mathfrak{r}_i \cup \mathfrak{r}_{i+1})} \neq \emptyset$ and $\mathcal{X}^{(\mathfrak{r}_{i+1})} \cap \mathcal{X}^{(\mathfrak{r}_i \cup \mathfrak{r}_{i+1})} \neq \emptyset$. Applying Corollary 3.3.3 and the latter properties to the shorter sequence $\mathfrak{r}_0 \cup \mathfrak{r}_1, \ldots, \mathfrak{r}_n \cup \mathfrak{r}_{n+1}$, we get the claim. ∎

3.3.5. Corollary. *For any pair $\mathfrak{p}, \mathfrak{q} \in \mathrm{Zspec}(A)$, the strong connected component of $\mathcal{X}^{(\mathfrak{p} \cup \mathfrak{q})}$ depends only on the strong connected components of $\mathcal{X}^{(\mathfrak{p})}$ and $\mathcal{X}^{(\mathfrak{q})}$.*

Proof. Suppose that $\mathfrak{p}$ and $\mathfrak{p}'$ lie in one strong component. By Corollary 3.3.4, there exist chains $\mathfrak{p}_0 = \mathfrak{p} \subset \mathfrak{p}_1 \subset \ldots \subset \mathfrak{p}_m$ and $\mathfrak{p}'_0 = \mathfrak{p}' \subset \mathfrak{p}'_1 \subset \ldots \subset \mathfrak{p}'_m$ with $\mathfrak{p}_m = \mathfrak{p}'_m$ and such that $\mathcal{X}^{(\mathfrak{p}_i)} \cap \mathcal{X}^{(\mathfrak{p}_{i+1})} \neq \emptyset$ and $\mathcal{X}^{(\mathfrak{p}'_i)} \cap \mathcal{X}^{(\mathfrak{p}'_{i+1})} \neq \emptyset$ for all $0 \le i \le m-1$. Corollary 3.3.3implies that there are similar chains $\mathfrak{p}_0 \cup \mathfrak{q} = \mathfrak{p} \cup \mathfrak{q} \subset \mathfrak{p}_1 \cup \mathfrak{q} \subset \ldots \subset \mathfrak{p}_m \cup \mathfrak{q}$ and $\mathfrak{p}'_0 \cup \mathfrak{q} = \mathfrak{p}' \cup \mathfrak{q} \subset \mathfrak{p}'_1 \cup \mathfrak{q} \subset \ldots \subset \mathfrak{p}'_m \cup \mathfrak{q}$ and, therefore, $\mathcal{X}^{(\mathfrak{p} \cup \mathfrak{q})}$ and $\mathcal{X}^{(\mathfrak{p}' \cup \mathfrak{q})}$ lie in one strong connected component. ∎

Proof of Theorem 3.3.1. If $B = A/\mathbf{n}(A)$, then $\mathrm{Fspec}(B) \xrightarrow{\sim} \mathrm{Fspec}(A)$ and, by Proposition 2.2.5, $I_A \xrightarrow{\sim} I_B$. We may therefore always replace A by $A/\mathbf{n}(A)$ and assume that A is reduced.

Particular case: A is finitely generated over $\mathbf{F}_1$. In this case A is decomposable. As was already noticed, in this case all quasi-components of $\mathcal{X}$ are open connected subsets and, in particular, (i) is true, and $\pi_0(\mathcal{X})$ is a finite discrete space. The validity of (iii) follows from Corollary 3.3.5 and, therefore, $\pi_0(\mathcal{X})$ is a finite poset with the infimum operation. Furthermore, Corollary 1.6.2(ii) implies that, for a nonzero idempotent $e \in I_A$, the ideal F_e generated by the prime ideal Π_e is nontrivial and, in particular, the map in (ii) is surjective. Notice that the preimage of Π_e with respect to the canonical map $\mathrm{Fspec}(A) \to \mathrm{Fspec}(I_A)$ coincides with $\mathrm{Fspec}(A/F_e)$.

We claim that the idempotent $\mathbf{F}_1$-algebra of A/F_e coincides with $\{0, 1\}$. Indeed, suppose that the image of an element $f \in A$ in A/F_e is an idempotent. This means that $(f, f^2) \in F_e$. By Lemma 1.6.1(i), one has either $fe = f^2e$, or $fe \in \mathfrak{p}_e A$. The later inclusion means that the image of f in A/F_e is zero. Assume therefore that $fe \notin \mathfrak{p}_e A$ and $fe = f^2 e$. Then $(fe)^2 = fe$, i.e., $fe \in I_A$. Since $(fe)e = fe$, then $e \le fe$. If $fe \neq e$, then $fe \in \mathfrak{p}_e$, i.e., the image of f in A/F_e is zero. If $fe = e$, then $(f,1) \in F_e$, i.e., the image of f in A/F_e is 1, and the claim follows.

By the above claim, replacing A by A/F_e we may assume that $I_A = \{0,1\}$ (and reduced), and we have to show that the space $\mathrm{Fspec}(A)$ is connected. Suppose this is not true. Let $\mathcal{U}$ be a maximal connected component, and let $\mathfrak{p}$ be the maximal Zariski prime ideal with the property $\mathcal{X}^{(\mathfrak{p})} \subset \mathcal{U}$ (i.e., the minimal element of $\mathrm{Zspec}(A)$ with that property). Since $\mathcal{U}$ is a maximal element of $\pi_0(\mathcal{X})$, it follows that $\mathfrak{p} \neq \mathbf{m}_A$. Lemma 3.3.2(ii) implies that, for every strictly bigger Zariski prime ideal $\mathfrak{q} \supset \mathfrak{p}$, the Zariski ideal of $A^{(\mathfrak{p})} = A/\Pi_{\mathfrak{p}}$ generated by the image of $\mathfrak{q}$ is trivial. This means that there exists an element $f_{\mathfrak{q}} \in \mathfrak{q}\backslash\mathfrak{p}$ whose image in $A^{(\mathfrak{p})}$ is 1, i.e., $(f_{\mathfrak{q}},1) \in \Pi_{\mathfrak{p}}$. Let f be the product of these elements $f_{\mathfrak{q}}$. Then $f \notin \mathfrak{p}$ and $f \neq 1$. *We claim that f is an idempotent* (and this will contradict the assumption). Indeed, since $\Delta(A)$ coincides with the intersection $\bigcap_{\mathfrak{q}} \Pi_{\mathfrak{q}}$ taken over all Zariski prime ideals of A, it suffices to verify that $(f,f^2) \in \Pi_{\mathfrak{q}}$ for all $\mathfrak{q} \in \mathrm{Zspec}(A)$. First of all, $(f,1) \in \Pi_{\mathfrak{p}}$ by the construction, i.e., there exists an element $h \notin \mathfrak{p}$ with $fh = h$. If $\mathfrak{q} \subset \mathfrak{p}$, the latter equality implies that $(f,1) \in \Pi_{\mathfrak{q}}$ and, therefore, $(f,f^2) \in \Pi_{\mathfrak{q}}$. Furthermore, if $\mathfrak{q} \not\subset \mathfrak{p}$, then $f \in (\mathfrak{p} \cup \mathfrak{q})\backslash\mathfrak{p}$ and, therefore, $f \in \mathfrak{q}$. It follows that $(f,f^2) \in \Pi_{\mathfrak{q}}$.

General case. By Lemma 3.1.5, one has $\mathcal{X} \widetilde{\to} \varprojlim \mathrm{Fspec}(A_i)$, where $\{A_i\}_{i\in I}$ is the filtered system of $\mathbf{F}_1$-subalgebras finitely generated over $\mathbf{F}_1$. We set $\mathcal{X}_i = \mathrm{Fspec}(A_i)$.

3.3.6. Lemma. *Every open-closed subset $\mathcal{U}$ of $\mathcal{X}$ is the preimage of an open-closed subset $\mathcal{U}_i$ of $\mathcal{X}_i$ for some $i \in I$.*

Proof. Let $\mathcal{U}$ be an open-closed subset of $\mathcal{X}$, and set $\mathcal{V} = \mathcal{X}\backslash\mathcal{U}$. Of course, we may assume that both $\mathcal{U}$ and $\mathcal{V}$ are nonempty. Since both sets are open and quasi-compact, they are finite unions of sets of the form $\bigcap_{k=1}^n D(a_k,b_k)$. We can therefore find $i \in I$ such that all such elements a_k and b_k lie in A_i. If $\mathcal{U}_i$ and $\mathcal{V}_i$ are the corresponding subsets of $\mathcal{X}_i$, then $\mathcal{U}$ and $\mathcal{V}$ are their preimages in $\mathcal{X}$, respectively. By the construction, $\mathcal{U}_i$ and $\mathcal{V}_i$ are open subsets. If $\mathcal{U}_i \cap \mathcal{V}_i \neq \emptyset$ then, by Theorem 3.2.2, there is a point in $\mathcal{X}$ whose image in $\mathcal{X}_i$ lies in the intersection. But this is impossible since $\mathcal{U} \cap \mathcal{V} = \emptyset$. Thus, $\mathcal{U}_i \cap \mathcal{V}_i = \emptyset$.

Furthermore, we notice that, if a Zariski prime ideal $\mathfrak{p} \subset A_i$ that lies in the image of the composition map $\mathcal{U} \to \mathrm{Zspec}(A) \to \mathrm{Zspec}(A_i)$, then $\check{\mathcal{X}}_{i,\mathfrak{p}} \subset \mathcal{U}_i$. Indeed, suppose that $\mathfrak{q}$ is a Zariski prime ideal of A such that $\check{\mathcal{X}}_{\mathfrak{p}} \cap \mathcal{U} \neq \emptyset$ and whose image in $\mathrm{Zspec}(A_i)$ is $\mathfrak{p}$. Since $\mathcal{U}$ is open-closed, it follows that $\mathcal{X}^{(\mathfrak{q})} \subset \mathcal{U}$ and, in particular, $\Pi_{(\mathfrak{q})} \in \mathcal{U}$. The latter ideal corresponds to the whole group $\kappa(\mathfrak{q})^*$ (see Lemma 1.2.5) and, therefore, its image in $\mathcal{X}_i$ corresponds to the whole group $\kappa(\mathfrak{p})^*$, i.e., it is the prime ideal $\Pi_{(\mathfrak{p})}$. It follows that $\Pi_{(\mathfrak{p})} \in \mathcal{U}_i$. Since $\mathcal{U}_i$ is open, we get $\check{\mathcal{X}}_{i,\mathfrak{p}} \subset \mathcal{U}_i$. For the same reason, if a Zariski prime ideal $\mathfrak{p} \subset A_i$ that lies in the image of the composition map

$\mathcal{V} \to \mathrm{Zspec}(A) \to \mathrm{Zspec}(A_i)$, then $\check{\mathcal{X}}_{i,\mathfrak{p}} \subset \mathcal{V}_i$. In particular, the images of $\mathcal{U}_i$ and $\mathcal{V}_i$ in $\mathrm{Zspec}(A_i)$ do not intersect. Notice that the union of those images coincides with the image of $\mathrm{Zspec}(A)$ in $\mathrm{Zspec}(A_i)$.

Finally, since the Zariski spectrum $\mathrm{Zspec}(A_i)$ is finite, we can find $j \geq i$ such that the image of $\mathrm{Zspec}(A_j)$ in $\mathrm{Zspec}(A_i)$ coincides with that of $\mathrm{Zspec}(A)$. It follows that, for the preimages $\mathcal{U}_j$ and $\mathcal{V}_j$ of $\mathcal{U}_i$ and $\mathcal{V}_i$ in $\mathcal{X}_j$, respectively, one has $\mathcal{U}_i \cup \mathcal{V}_j = \mathcal{X}_j$. This means that $\mathcal{U}_j$ and $\mathcal{V}_j$ are disjoint open-closed subsets that cover $\mathcal{X}_j$. ∎

Let Σ be the quasi-component of a point $x \in \mathcal{X}$. For $i \in I$, let x_i be the image of x in $\mathcal{X}_i$, and let Σ_i be its connected component (which coincides with its quasi-component). It is clear that, for $i \leq j$, the image of Σ_j in $\mathcal{X}_i$ lies in Σ_i. *We claim that* $\Sigma \tilde{\to} \varprojlim \Sigma_i$. Indeed, both are subsets of $\mathcal{X}$, the set on the left hand side lies in that on the right hand side and, to show that they coincide, it suffices to verify that every open-closed subset $\mathcal{U}$ that contains Σ also contains the set on the right hand side. But this follows from Lemma 3.3.6 because such subset $\mathcal{U}$ is the preimage of an open-closed subset $\mathcal{U}_i$ of $\mathcal{X}_i$ for some $i \in I$. Since $x_i \in \Sigma_i \cap \mathcal{U}_i$, we get $\Sigma_i \subset \mathcal{U}_i$, and the claim follows.

By the particular case, for every $i \in I$ there is a unique nonzero idempotent $e_i \in I_{A_i}$ such that Σ_i is the preimage of the prime ideal Π_{e_i} of I_{A_i} with respect to the canonical map $\mathrm{Fspec}(A_i) \to \mathrm{Fspec}(I_{A_i})$ and, therefore, $\Sigma_i = \mathrm{Fspec}(A_i/F_{e_i})$, where F_{e_i} is the ideal of A generated by Π_{e_i}. If $j \geq i$, the image of Σ_j in $\mathcal{X}_i$ lies in Σ_i, and this implies that the image of Π_{e_j} with respect to the canonical map $\mathrm{Fspec}(I_{A_j}) \to \mathrm{Fspec}(I_{A_i})$ is the prime ideal Π_{e_i}. In particular, e_i is the maximal idempotent in $\check{I}_{A_i}$ with $e_i \leq e_j$ and $F_{e_i} \subset F_{e_j} \cap (A_i \times A_i)$. If $B = \varinjlim A_i/F_{e_i}$ then, by Lemma 3.1.5 and the previous claim, we get $\Sigma \tilde{\to} \mathrm{Fspec}(B)$. Since the idempotent algebra of each quotient A_i/F_{e_i} consists of 0 and 1, it follows that $I_B = \{0, 1\}$. Furthermore, since each A_i/F_{e_i} is a noetherian $\mathbf{F}_1$-algebra, we can find $j \geq i$ such that the image of A_i/F_{e_i} in A_j/F_{e_j} is canonically isomorpic to its image in B. If B_i denote the latter, then B is the union of all such B_i's, and one has $\Sigma = \mathrm{Fspec}(B) \tilde{\to} \varprojlim \mathrm{Fspec}(B_i)$. We can now show that the set Σ is connected (which implies that Σ is a connected component). Namely, let $\mathcal{U}$ be a nonempty open-closed subset of Σ. By Lemma 3.3.6, there exists $i \in I$ such that $\mathcal{U}$ is the preimage of an open-closed subset of $\mathrm{Fspec}(B_i)$. But since $I_{B_i} = \{0, 1\}$, the latter space is connected and, therefore, $\mathcal{U}_i = \mathrm{Fspec}(B_i)$. This implies that $\mathcal{U} = \Sigma$, and this gives the statement (i).

To prove the statement (ii), we consider a related description of the $\mathbf{F}_1$-algebra B, which follows from the construction. Namely, we consider the Zariski prime ideal $\mathfrak{p} = \bigcup_{i \in I} \mathfrak{p}_{e_i}$ of I_A

(recall that $\mathfrak{p}_{e_i} = \{f \in I_{A_i} \big| f \not\leq e_i\}$). If $F_{\mathfrak{p}}$ is the ideal of A generated by the corresponding prime ideal $\Pi_{\mathfrak{p}}$ of I_A (i.e., $\Pi_{\mathfrak{p}} = \{(e,f) \in I_A \times I_A \big|$ either $e, f \in \mathfrak{p}$, or $e, f \notin \mathfrak{p}\}$), then $A/F_{\mathfrak{p}} \stackrel{\sim}{\to} B$. Since $\mathrm{Fspec}(B) = \Sigma$ is connected, we get the statement (ii). The statements (iii) and (iv) easily follow from the particular case. ∎

3.3.7. Remark. Let A be the quotient of $\mathbf{F}_1[T_1, T_2, \ldots]$ by the ideal generated by the pairs (T_iT_{i+1}, T_{i+1}) for $i \geq 1$, and let f_i be the image of T_i in A. The $\mathbf{F}_1$-algebra A has no zero divisors, i.e., (0) is a Zariski prime ideal, and each nonzero Zariski prime ideal is of the form $\mathfrak{p}_n = \bigcup_{i=n+1}^{\infty} Af_i$ for $n \geq 0$. Then $\mathcal{X}^{(\mathfrak{p}_0)} \subset \mathcal{X}^{(\mathfrak{p}_1)}$, $\mathcal{X}^{(\mathfrak{p}_n)} \cap \mathcal{X}^{(\mathfrak{p}_{n+1})} \neq \emptyset$ for $n \geq 1$, but all other pairwise distinct pairs do not intersect. This means that $\mathcal{X}$ has two strong components $\mathcal{X}^{(\mathfrak{p}_0)} \cup \mathcal{X}^{(\mathfrak{p}_1)} \cup \ldots$ and $\mathcal{X}^{(0)}$. On the other hand, the space $\mathcal{X}$ is connected.

3.4. Disconnected sums of $\mathbf{F}_1$-algebras.

3.4.1. Definition. A map of $\mathbf{F}_1$-algebras $\varphi : A \to B$ is said to be a *quasi-homomorphism* if it takes 0_A and 1_A to 0_B and 1_B, respectively, and possesses the following property: if $ab \neq 0$ in A, then $\varphi(ab) = \varphi(a)\varphi(b)$.

3.4.2. Examples. (i) If A has no zero divisors, any such map is a homomorphism.

(ii) Let $\mathbf{a} \subset \mathbf{b}$ be Zariski ideals of A. Then the map $A \to A$ that takes an element $a \notin \mathbf{b}$ to a and all elements from $\mathbf{b}$ to zero gives rise to a quasi-homomorphism $A/\mathbf{b} \to A/\mathbf{a}$.

(iii) Given an $\mathbf{F}_1$-subalgebra $I \subset I_A$ and an idempotent $e \in \check{I}$, we set $A^{(e)} = A/F_e$, where F_e is the ideal of A generated by the prime ideal Π_e of I. By Lemma 1.6.1, one has $A^{(e)} = A_e/\mathfrak{p}_e A_e$ (where A_e is the localization of A with respect to e). Given two nonzero idempotents $e \leq f$, the composition of the quasi-homomorphism $A_e/\mathfrak{p}_e A_e \to A_e$ (from (ii)) with the canonical homomorphism $A_e \to A_f$ gives rise to a quasi-homomorphism $\nu_{e,f} : A^{(e)} \to A^{(f)}$.

3.4.3. Definition. A *disconnected sum datum* (*of* $\mathbf{F}_1$-*algebras*) is a tuple $\{\check{I}, A_i, \nu_{ij}\}$ consisting of an inf-poset $\check{I}$, a system of $\mathbf{F}_1$-algebras $\{A_i\}_{i \in \check{I}}$ and, for every pair $i \leq j$ in $\check{I}$, a quasi-homomorphism $\nu_{ij} : A_i \to A_j$, which possesses the following properties:

(0) ν_{ii} is the identity map on A_i;

(1) if $i \leq j \leq k$ and $a \in A_i$ are such that $\nu_{ij}(a) \neq 0$, then $\nu_{jk}(\nu_{ij}(a)) = \nu_{ik}(a)$;

(2) if $i \leq j = \inf(J)$ for a subset $J \subset \check{I}$ and $a \in A_i$ are such that $\nu_{ik}(a) \neq 0$ for all $k \in J$, then $\nu_{ij}(a) \neq 0$.

Given a disconnected sum datum $\{\check{I}, A_i, \nu_{ij}\}$, let $\coprod_{\check{I}}^{\nu} A_i$ denote the subset of the direct product $\prod_{i \in \check{I}} A_i$ consisting of the tuples $(a_i)_{i \in \check{I}}$ with the following properties:

(a) if $i \le j$ and $a_i \neq 0$, then $\nu_{ij}(a_i) = a_j$;

(b) given a subset $J \subset \check{I}$, if $a_i \neq 0$ for all $i \in J$, then $a_{\inf(J)} \neq 0$.

We set $A = \coprod_{\check{I}}^{\nu} A_i$. For $i \in \check{I}$ and $a \in A_i$, let $\nu_i(a)$ be the element of $\prod_{i \in \check{I}} A_i$ with j's component $\nu_{ij}(a_i)$, if $i \le j$, and 0, otherwise. The property (0) implies the i'th component of $\nu_i(a)$ is equal a and, in particular, the map $\nu_i : A_i \to A$ is injective. We also set $e_i = \nu_i(1)$.

3.4.4. Proposition. *In the above situation, the following is true:*

(i) A is an $\mathbf{F}_1$-subalgebra of $\prod_{i \in \check{I}} A_i$;

(ii) $\nu_i(A_i) \subset A$, $A = \bigcup_{i \in \check{I}} \nu_i(A_i)$ and $\nu_i(A_i) \cap \nu_j(A_j) = 0$ for all $i \neq j$;

(iii) for $i, j \in \check{I}$, one has $e_i e_j = e_{\sup(i,j)}$, if $\sup(i,j)$ exists, and $e_i e_j = 0$, otherwise; in particular, the correspondence $i \mapsto e_i$ identifies the set $I = \{0\} \cup \check{I}$ with an $\mathbf{F}_1$-subalgebra of I_A;

(iv) for $i \in \check{I}$, the composition of ν_i with the canonical epimorphism $A \to A^{(e_i)}$ gives rise to an isomorphism $A_i \xrightarrow{\sim} A^{(e_i)}$;

(v) if $i \le j$, the quasi-homomorphism ν_{ij} is compatible with ν_{e_i, e_j} from Example 3.4.2(iii).

(vi) if the poset $\check{I}$ is noetherian, then $\mathcal{X} = \coprod_{i \in \check{I}} \mathcal{X}_i$, where $\mathcal{X} = \mathrm{Fspec}(A)$ and $\mathcal{X}_i = \mathrm{Fspec}(A_i)$.

The quotient $\mathbf{F}_1$-algebra $A^{(e_i)}$ of A is defined here as in Example 3.4.2 (i.e., with respect to the idempotent $\mathbf{F}_1$-subalgebra I). The $\mathbf{F}_1$-algebra $A = \coprod_{\check{I}}^{\nu} A_i$ will be said to be the *disconnected sum of A_i's with respect to ν_{ij}'s.*

Proof. (i) Since the set A contains 0 and 1 of the direct product, it suffices to verify that, for $(a_i)_{i \in \check{I}}, (b_i)_{i \in \check{I}} \in A$, the tuple $(a_i b_i)_{i \in \check{I}}$ possesses the properties (a) and (b).

(a) Suppose that $a_i b_i \neq 0$. By the definition of a quasi-homomorphism, for every $j \ge i$ one has $\nu_{ij}(a_i b_i) = \nu_{ij}(a_i)\nu_{ij}(a_j)$. Since $a_i \neq 0$ and $b_i \neq 0$, the property (a) for the given tuples implies that $\nu_{ij}(a_i) = a_j$ and $\nu_{ij}(a_i) = a_j$ and, therefore, $\nu_{ij}(a_i b_i) = a_j b_j$.

(b) Suppose that, for a subset $J \subset \check{I}$, one has $a_i b_i \neq 0$ for all $i \in J$, and set $j = \inf(J)$. It follows that $a_i \neq 0$ and $b_i \neq 0$ for all $i \in J$ and, therefore, $a_j \neq 0$ and $b_j \neq 0$, i.e., $a_j b_j \neq 0$.

Notice that we did not use the properties (0)-(2) so far.

(ii) First of all, we verify the first inclusion, i.e., validity of (a) and (b) for every element $\nu_i(a)$ with $a \in A_i$.

(a) Suppose that $j \le k$ and $\nu_i(a)_j \neq 0$. It follows that $i \le j$ and $\nu_{ij}(a) = \nu_i(a)_j \neq 0$. By the property (1), we get $\nu_{jk}(\nu_{ij}(a)) = \nu_{ik}(a) = \nu_i(a)_k$.

(b) Suppose that, for a subset $J \subset \check{I}$, one has $\nu_i(a)_k \neq 0$ for all $k \in J$, and set $j = \inf(J)$. Then $i \le k$ and $\nu_i(a)_k = \nu_{ik}(a) \neq 0$ for all $k \in J$. It follows that $i \le j$ and $\nu_i(a)_j = \nu_{ij}(a) \neq 0$, by the property (2).

Furthermore, for a nonzero tuple $(a_i)_{i\in\check{I}} \in A$, let J be the subset of all $i \in \check{I}$ with $a_i \neq 0$, and set $j = \inf(J)$. The property (b) implies that $j \in J$ and, by the properties (0) and (a), we get $(a_i)_{i\in\check{I}} = \nu_j(a_j)$, i.e., $A = \bigcup_{i\in\check{I}} \nu_i(A_i)$.

The last statement in (ii) directly follows from the definition of ν_i's.

The statement (iii) is trivial.

(iv) If the images of elements $a, b \in A_i$ coincide in $A^{(e_i)}$, then $(\nu_i(a), \nu_i(b)) \in F_{e_i}$, i.e., either $\nu_i(a)e_i = \nu_i(b)e_i$, or $\nu_i(a)e_i, \nu_i(b)e_i \in \mathfrak{p}_{e_i}A$. In the former case, the equality of i'th components of both sides implies that $a = b$. As for the latter case, we recall that $\mathfrak{p}_{e_i}$ is the Zariski prime ideal of I whose nonzero elements are e_j's with $j \not\leq i$. This implies that, if $\nu_i(a)e_i \in \mathfrak{p}_i A$, then $\nu_i(a)e_i = \alpha e_j$ for some $\alpha \in A$ and $j \not\leq i$. The i-th component of the left hand side is a, but that of the right hand side is 0. Thus, in the second case, one has $a = b = 0$ and, therefore, the map $A_i \to A^{(e_i)}$ is injective. That it is surjective follows from the facts that $(\alpha, \alpha e_i) \in F_{e_i}$ and $\alpha e_i \in \nu_i(A_i)$ for all elements $\alpha \in A$, and that it is a homomorphism follows from the definition of the multiplication.

The statement (v) follows from (iv) and the facts that, for $i \leq j$, the multiplication by e_j takes $\nu_i(A_i)$ to $\nu_j(A_j)$ and coincides with ν_{ij} on $A_i \xrightarrow{\sim} \nu_i(A_i)$.

(vi) It suffices to verify that every Zariski prime ideal $\mathfrak{p} \subset A$ is the Zariski preimage of a Zariski prime ideal of some A_i. Since $\check{I}$ is noetherian, there is a unique maximal element $i \in \check{I}$ with $e_i \notin \mathfrak{p}$. *We claim that $\mathfrak{p}$ is the Zariski preimage of a Zariski prime ideal of A_i.* Indeed, it suffices to verify that, if $a \in \mathfrak{p}$ and $(a, b) \in F_{e_i}$, then $b \in \mathfrak{p}$. Suppose first that $ae_i = be_i$. Then $be_i \in \mathfrak{p}$ and, since $e_i \notin \mathfrak{p}$, it follows that $b \in \mathfrak{p}$. Suppose now that $ae_i, be_i \in \mathfrak{p}_{e_i}A$. Since $\mathfrak{p}_{e_i} \subset \mathfrak{p}$, it follows that $be_i \in \mathfrak{p}$ and, therefore, $b \in \mathfrak{p}$. ∎

Suppose we are given disconnected sum data $\{\check{I}, A_i, \nu_{ij}\}$ and $\{\check{I}', A_{i'}, \nu_{i'j'}\}$, an inf-map $\check{I} \to \check{I}' : i \mapsto i'$ and, for every $i \in \check{I}$, a homomorphism $f_i : A_{i'} \to A_i$. Then there is an induced homomorphism of $\mathbf{F}_1$-algebras $\prod_{i'\in\check{I}'} A_{i'} \to \prod_{i\in\check{I}} A_i : (a_{i'})_{i'\in I'} \mapsto (f_i(a_{i'}))_{i\in I}$.

3.4.5. Lemma. *The above homomorphism gives rise to a homomorphism of $\mathbf{F}_1$-algebras $f : \coprod^{\nu}_{\check{I}'} A_{i'} \to \coprod^{\nu}_{\check{I}} A_i$ if and only if the following holds:*

(1) if J is a subset of $\check{I}$ and $k = \inf(J)$, then $\mathrm{Zker}(f_k) \subset \bigcup_{i\in J} \mathrm{Zker}(f_i \circ \nu_{k'i'})$;

(2) for every $i \leq j$ in $\check{I}$, the following diagram is commutative outside $\mathrm{Zker}(f_i)$:

$$\begin{array}{ccc} A_i & \xrightarrow{\nu_{ij}} & A_j \\ \uparrow f_i & & \uparrow f_j \\ A_{i'} & \xrightarrow{\nu_{i'j'}} & A_{j'} \end{array}$$

Proof. *Direct implication.* (1) For an element $a \in \mathrm{Zker}(f_k)$, the k-th coordinate of the element

$f(\nu_{k'}(a))$ is zero. The property (b) from the construction of A implies that its i-th coordinate is zero for some $i \in J$. Since the latter is equal to $f_i(\nu_{k'i'}(a))$, it follows that $a \in \mathrm{Zker}(f_i \circ \nu_{k'i'})$.

(2) Suppose that $f_i(a) \neq 0$ for an element $a \in A_i$. Then $a \neq 0$, and so the j'-th coordinate of the element $\nu_{i'}(a)$ is equal to $\nu_{i'j'}(a)$. It follows that the j-th coordinate of the element $f(\nu_{i'}(a))$ is equal to $f_j(\nu_{i'j'}(a))$. But $f(\nu_{i'}(a)) \in \coprod^{\nu}_{\check{I}} A_i$, and the property (a) from the construction of A implies that the same j-th coordinate is equal to $\nu_{ij}(f_i(a))$.

Converse implication. Let $a' = (a_{i'})_{i' \in \check{I}'}$ be an element of $A' = \coprod^{\nu}_{\check{I}'} A_{i'}$. It suffices to show that $a = (a_i)_{i \in \check{I}}$, defined by $a_i = f_i(a_{i'})$, is an element of $A = \coprod^{\nu}_{\check{I}} A_i$. For this we have to verify the properties (a) and (b) from the construction of A.

(a) Let $i \leq j$, and suppose that $a_i = f_i(a_{i'}) \neq 0$. Then $a_{i'} \neq 0$ and, by the property (a) for a', we have $a_{j'} = \nu_{i'j'}(a_{i'})$. Since $a_{i'} \notin \mathrm{Zker}(f_i)$, the property (2) implies that $a_j = f_j(a_{j'}) = \nu_{ij}(f_i(a_{i'}) = \nu_{ij}(a_i)$.

(b) Let $k = \inf(J)$, and suppose that $a_i = f_i(a_{i'}) \neq 0$ for all $i \in J$. Then $a_{i'} \neq 0$ for all $i \in J$. By the property (b) for a', we have $a_{k'} \neq 0$ and, by the property (a), we have $a_{i'} = \nu_{k'i'}(a_{k'})$. The assumption and the property (1) imply that $a_k = f_k(a_{k'}) \neq 0$. ∎

A *morphism of disconnected sum data* $f : \{\check{I}', A_{i'}, \nu_{i'j'}\} \to \{\check{I}, A_i, \nu_{ij}\}$ consists of an inf-map $\check{I} \to \check{I}' : i \mapsto i'$ and, for every $i \in \check{I}$, a homomorphism $f_i : A'_{i'} \to A_i$ so that the conditions (1) and (2) of Lemma 3.4.5 are satisfied. Lemma 3.4.5 implies that disconnected sum data form a category with respect to the above morphisms (it is denoted by $\mathcal{D}sd$), and that the correspondence $\{\check{I}, A_i, \nu_{ij}\} \mapsto A = \coprod^{\nu}_{\check{I}} A_i$ defines a functor $\mathcal{D}sd \to \mathbf{F}_1\text{-}\mathcal{A}lg$.

Let now (A, I) be a pair consisting of an $\mathbf{F}_1$-algebra A and an idempotent $\mathbf{F}_1$-subalgebra $I \subset A$ such that the poset $\check{I}$ is an inf-poset. It is easy to see that the system of quasi-homomorphisms $\nu_{e,f} : A^{(e)} \to A^{(f)}$ for $e \leq f$ in $\check{I}$ from Example 3.4.2(iii) possesses the properties (0)-(2) of Definition 3.4.3, i.e., $\{\check{I}, A^{(e)}, \nu_{e,f}\}$ is a disconnected sum datum.

3.4.6. Proposition. *In the above situation, suppose that the idempotent* $\mathbf{F}_1$*-algebra* I *is almost finite. Then the canonical homomorphism* $A \to \prod_{e \in \check{I}} A^{(e)}$ *induces an isomorphism of* $\mathbf{F}_1$*-algebras* $A \xrightarrow{\sim} \coprod^{\nu}_{\check{I}} A^{(e)}$.

Proof. Let μ_e denote the canonical homomorphism $A \to A^{(e)}$. The Zariski kernel of μ_e coincides with $\mathfrak{p}_e A$. If $a \notin \mathfrak{p}_e A$, then $\mu_e(a) = \mu_e(ae)$ and, therefore, $\nu_{e,f}(\mu_e(a)) = \mu_f(a)$ for all $e \leq f$ in $\check{I}$. Given a subset of $J \subset \check{I}$, suppose that $\mu_e(a) \neq 0$ for all $e \in J$. Then $a \notin \mathfrak{p}_e A$ for all $e \in \check{I}$. By Lemma 1.6.3, one has $\mathfrak{p}_f = \bigcup_{e \in J} \mathfrak{p}_e$, where $f = \inf(J)$. It follows that, if $\mu_f(a) = 0$, then $a \in \mathfrak{p}_f A$ and, therefore, $a \in \mathfrak{p}_e A$ for some $e \in J$, which is impossible. Thus, the image of

A in the direct product lies in the disconnected sum. Injectivity of the homomorphism considered follows from Lemma 1.6.2(ii). Finally, let $(a_e)_{e\in\check{I}}$ be an element from the disconnected sum. By the property (2), there exists a unique minimal $e \in \check{I}$ with $a_e \neq 0$. Let a be a preimage of a_e in A. It is easy to see that the tuple considered is the image of the element ae. ∎

Let $\mathbf{F}_1$-$\mathcal{A}lg_{afid}$ denote the category of pairs (A, I) consisting of an $\mathbf{F}_1$-algebra A and an almost finite idempotent $\mathbf{F}_1$-subalgebra $I \subset A$. Let also $\mathcal{D}sd_{noet}$ denote the full subcategory of $\mathcal{D}sd$ consisting of the disconnected sum data $\{\check{I}, A_i, \nu_{ij}\}$ with noetherian inf-poset $\check{I}$. Lemma 1.6.5 and Proposition 3.4.6 imply the following fact.

3.4.7. Corollary. *(i) The correspondence* $\{\check{I}, A_i, \nu_{ij}\} \mapsto (A, I)$ *with* $A = \coprod_{\check{I}}^{\nu} A_i$ *and* $I = \{0\} \cup \check{I}$ *gives rise to an equivalence of categories* $\mathcal{D}sd_{noet} \xrightarrow{\sim} \mathbf{F}_1$-$\mathcal{A}lg_{afid}$*;*

(ii) the above functor induces an equivalence between the full subcategory of $\mathcal{D}sd_{noet}$ *consisting of* $\{\check{I}, A_i, \nu_{ij}\}$ *such that all of the spectra* $\mathrm{Fspec}(A_i)$ *are connected and the category of* $\mathbf{F}_1$*-algebras* A *with almost finite idempotent* $\mathbf{F}_1$*-subalgebra* I_A*;*

(iii) the above functor induces an equivalence between the full subcategory of $\mathcal{D}sd_{noet}$ *consisting of* $\{\check{I}, A_i, \nu_{ij}\}$ *with finite* $\check{I}$ *and local artinian* A_i*'s and the category of artinian* $\mathbf{F}_1$*-algebras.* ∎

Let $\{\check{I}, A_i, \nu_{ij}\}$ be a disconnected sum datum, and suppose we are given homomorphisms of $\mathbf{F}_1$-algebras $\alpha_i : B \to A_i$ with the following properties:

(1) if $i \leq j$ and $b \in B_i$ are such that $\alpha_i(b) \neq 0$, then $\nu_{ij}(\alpha_i(b)) = \alpha_j(b)$;

(2) given a subset $J \subset \check{I}$ and $b \in B$ such that $\alpha_i(b) \neq 0$ for all $i \in J$, then $\alpha_{\min(J)}(b) \neq 0$.

3.4.8. Proposition. *In the above situation, the following is true:*

(i) there is a unique structure of a B*-algebra on* $A = \coprod_{\check{I}}^{\nu} A_i$*, which is compatible with those on* A_i*'s;*

(ii) if the set $\check{I}$ *is finite and all* A_i *are finite (resp. finitely generated)* B*-algebras, then so is* A*.*

Proof. (i) The properties (1)-(2) imply that, for every element $b \in B$, the tuple $(\alpha_i(b))$ belongs to A, and it is easy to see that the map $B \to A : b \mapsto (\alpha_i(b))$ is a homomorphism which defines the required structure of a B-algebra on A.

(ii) It is enough to notice that the images of elements of A_i's that generate them as a B-module (resp. B-algebra) generate A also as a B-module (resp. B-algebra). ∎

3.4.9. Remark. It is not true in general that, for a disconnected sum datum $\{\check{I}, A_i, \nu_{ij}\}$ with Zariski artinian idempotent $\mathbf{F}_1$-algebra $I = \{0\} \cup \check{I}$ and local artinian $\mathbf{F}_1$-algebras A_i, the disconnected sum $A = \coprod_{\check{I}}^{\nu} A_i$ is Zariski artinian. Indeed, let $\check{I}$ be the poset $\{-\infty, \dots, -2, -1\}$.

The corresponding idempotent $\mathbf{F}_1$-algebra $I = \{0, e_{-\infty} = 1, \dots, e_{-2}, e_{-1}\}$ with $e_{-i}e_{-j} = e_{-\min(i,j)}$ is Zariski artinian. For $n \geq 1$, let A_{-n} be the quotient of $\mathbf{F}_1[T]$ by the Zariski ideal generated by T^n (which is a local artinian $\mathbf{F}_1$-algebra) and, for $m \geq n$, let $\nu_{-m,-n}$ be the canonical surjective homomorphism $A_{-m} \to A_{-n}$. We also set $A_{-\infty} = \mathbf{F}_1$ and, for $n \geq 1$, denote by $\nu_{-\infty,-n}$ the canonical homomorphism $A_{-\infty} \to A_{-n}$. Then $\{\check{I}, A_i, \nu_{ij}\}$ is a disconnected sum datum. Furthermore, for $n \geq 1$, let $\mathbf{a}_n$ be the Zariski ideal of $A = \coprod_{\check{I}}^{\nu}$ with elements of the form $\nu_{-m}(t^k)$ with $m \geq k \geq n$ (t is the image of T in A_{-m}). Then the descending chain of Zariski ideals $\mathbf{a}_1 \supset \mathbf{a}_2 \supset \dots$ does not stabilize.

3.5. Disconnected sums of A-modules. Let A be an $\mathbf{F}_1$-algebra.

3.5.1. Definition. A map of A-modules $\varphi : M \to N$ is said to be a *quasi-homomorphism* if it takes 0_M to 0_N and possesses the following property: if $am \neq 0$ for $a \in A$ and $m \in M$, then $\varphi(am) = a\varphi(m)$.

For example, let $P \subset Q$ be Zariski A-submodules of M. Then the map $M \to M$ that takes an element $m \notin Q$ to m and all elements from Q to zero gives rise to a quasi-homomorphism $M/Q \to M/P$.

3.5.2. Definition. A *disconnected sum datum* (*of A-modules*) is a tuple $\{\check{I}, M_i, \nu_{ij}\}$ consisting of an inf-poset $\check{I}$, a system of A-modules $\{M_i\}_{i \in \check{I}}$ and, for every pair $i \leq j$ in $\check{I}$, a quasi-homomorphism $\nu_{ij} : M_i \to M_j$, which possesses the properties (0)-(2) of Definition 3.4.3.

Given a disconnected sum datum of A-modules $\{\check{I}, M_i, \nu_{ij}\}$, let $\coprod_{\check{I}}^{\nu} M_i$ denote the subset of $\prod_{i \in \check{I}} M_i$ consisting of the tuples $(m_i)_{i \in \check{I}}$ possessing the properties (a) and (b) from the construction of the disconnected sum of $\mathbf{F}_1$-algebras. For $i \in \check{I}$ and $m \in M_i$, let also $\nu_i(m)$ be the element of $\prod_{i \in \check{I}} M_i$ with j's component $\nu_{ij}(m)$, if $i \leq j$, and 0, otherwise. It is easy to see that $M = \coprod_{\check{I}}^{\nu} M_i$ is an A-submodule of $\prod_{i \in \check{I}} M_i$, and one has $\nu_i(M_i) \subset M$, $M = \bigcup_{i \in \check{I}} \nu_i(M_i)$, and $\nu_i(M_i) \cap \nu_j(M_j) = 0$ for all $i \neq j$.

3.5.3. Examples. (i) Suppose that $A = \coprod_{\check{I}}^{\nu} A_i$ for a disconnected sum datum of $\mathbf{F}_1$-algebras $\{\check{I}, A_i, \nu_{ij}\}$. If $i \leq j$ in $\check{I}$, the quasi-homomorphisms ν_{ij} induces a quasi-homomorphism $\nu_{ij} : M_i = M \otimes_A A_i \to M_j = M \otimes_A A_j$. Then the tuple $\{\check{I}, M_i, \nu_{ij}\}$ is a disconnected sum datum of A-modules, and there is a canonical isomorphism $M \xrightarrow{\sim} \coprod_{\check{I}}^{\nu} M_i$. Indeed, verification of the property (1) of Definition 3.4.3 is trivial. Furthermore, by Proposition 3.4.4, one can identify the set $I = \check{I} \cup \{0\}$ with an idempotent $\mathbf{F}_1$-subalgebra of A and, by Lemma 1.6.1(i), if $i \in \check{I}$ corresponds to an idempotent e, one has $M_i = M^{(e)} = M_e/\mathfrak{p}_e M_e$, where M_e is the localization of

M with respect to e. Suppose that $e \leq f = \inf\{g_j\}_{j\in J}$ for a subset $J \subset \check{I}$ and that the image of an element $m \in M_e$ in $M^{(f)}$ is zero, i.e., $m \in \mathfrak{p}_f M_f$. Since $\mathfrak{p}_f = \bigcup_{j\in J} \mathfrak{p}_{g_j}$, it follows that there exists $j \in J$ with $m \in \mathfrak{p}_{g_j} M_{g_j}$. This implies the property (2) of Definition 3.4.3, i.e., $\{\check{I}, M_i, \nu_{ij}\}$ is a disconnected sum datum. That the image of the map $M \to \prod_{i\in\check{I}} M_i$ coincides with $\coprod_{\check{I}}^{\nu} M_i$ is trivial. That this map is injective follows from Lemma 1.6.1(ii).

(ii) Suppose that an A-module M is a direct sum of a family of Zariski A-submodules $\{M_i\}_{i\in\check{I}}$. We provide $\check{I}$ with the structure of an inf-poset (e.g., by providing it with the structure of a well ordered set). For $i \leq j$ in $\check{I}$, let ν_{ij} be the zero homomorphism $M_i \to M_j$. Then $\{\check{I}, M_i, \nu_{ij}\}$ is a disconnected sum datum of A-modules, and one has $M \xrightarrow{\sim} \coprod_{\check{I}}^{\nu} M_i$.

§4. Affine schemes over $\mathbf{F}_1$

4.1. Affine schemes and weak open affine subschemes. The category of *affine schemes over* $\mathbf{F}_1$ is, by definition, the category $\mathcal{A}sch_{\mathbf{F}_1}$ anti-equivalent to the category of $\mathbf{F}_1$-algebras. We refer to an affine scheme $\mathcal{X} = \mathrm{Fspec}(A)$ by the letter $\mathcal{X}$. We also set $I_{\mathcal{X}} = I_{\mathcal{A}}$. We call $\mathcal{X}$ *decomposable*, *weakly decomposable*, *reduced*, *integral*, *quasi-integral*, *irreducible*, *idempotent*, *noetherian*, *Zariski noetherian*, *artinian*, or *finitely presented over an* $\mathbf{F}_1$*-field* K if the $\mathbf{F}_1$-algebra A possesses the corresponding property.

4.1.1. Definition. An open subset $\mathcal{U}$ of an affine scheme $\mathcal{X} = \mathrm{Fspec}(A)$ is said to be a *weak open affine subscheme* if there is a homomorphism of $\mathbf{F}_1$-algebras $A \to A_{\mathcal{U}}$ such that

(1) the image of $\mathrm{Fspec}(A_{\mathcal{U}})$ in $\mathcal{X}$ lies in $\mathcal{U}$;

(2) any homomorphism of $\mathbf{F}_1$-algebras $A \to B$ such that the image of $\mathrm{Fspec}(B)$ in $\mathcal{X}$ lies in $\mathcal{U}$ goes through a unique homomorphism $A_{\mathcal{U}} \to B$.

For a subset $\mathcal{U} \subset \mathcal{X}$, we set $\mathcal{I}(\mathcal{I}) = \{\mathfrak{p} \in \mathrm{Zspec}(A) \big| \check{\mathcal{X}}_{\mathfrak{p}} \cap \mathcal{U} \neq \emptyset\}$.

4.1.2. Lemma. *Let $\mathcal{U}$ be a weak open affine subscheme of $\mathcal{X}$. Then*

(i) the homomorphism $A \to A_{\mathcal{U}}$ is unique up to a unique isomorphism;

(ii) the map $\mathrm{Fspec}(A_{\mathcal{U}}) \to \mathcal{U}$ is bijective, and $\kappa(x) \xrightarrow{\sim} \kappa(y)$ for every point $x \in \mathcal{U}$, where y is its preimage in $\mathrm{Fspec}(A_{\mathcal{U}})$;

(iii) the map $\mathrm{Zspec}(A_{\mathcal{U}}) \to \mathrm{Zspec}(A)$ is injective, and its image coincides with $\mathcal{I}(\mathcal{U})$;

(iv) $\check{\mathcal{X}}_{\mathfrak{p}} \subset \mathcal{U}$ for every $\mathfrak{p} \in \mathcal{I}(\mathcal{U})$, and $\inf(J) \in \mathcal{I}(\mathcal{U})$ for every subset $J \subset \mathcal{I}(\mathcal{U})$;

(v) for any weak open affine subscheme $\mathcal{V} \subset \mathcal{X}$, the intersection $\mathcal{U} \cap \mathcal{V}$ is a weak open affine subscheme with respect to the homomorphism $A \to A_{\mathcal{U}} \otimes_A A_{\mathcal{V}}$;

(vi) if the map $\mathrm{Fspec}(A_{\mathcal{U}}) \to \mathcal{U}$ *is a homeomorphism, then for any weak open affine subscheme* $\mathcal{V}$ *of* $\mathrm{Fspec}(A_{\mathcal{U}})$ *its image in* $\mathcal{X}$ *is a weak open affine subscheme;*

(vii) for any morphism of affine schemes $\varphi : \mathcal{Y} = \mathcal{M}(B) \to \mathcal{X}$, $\varphi^{-1}(\mathcal{U})$ *is a weak open affine subscheme of* $\mathcal{Y}$ *with respect to the homomorphism* $B \to B \otimes_A A_{\mathcal{U}}$.

Proof. The statements (i) and (v)-(vii) are trivial.

(ii) By the property (2), the homomorphism $A \to \kappa(x)$ goes through a unique homomorphism $A_{\mathcal{U}} \to \kappa(x)$, and so x is the image of a unique point $y \in \mathrm{Fspec}(A_{\mathcal{U}})$, and $\kappa(x) \xrightarrow{\sim} \kappa(y)$.

(iii) and (iv). That the image of $\mathrm{Zspec}(A_{\mathcal{U}})$ coincides with $\mathcal{I}(\mathcal{U})$ follows from (ii). *We claim that* $\check{\mathcal{X}}_{\mathfrak{p}} \subset \mathcal{U}$ *for every* $\mathfrak{p} \in \mathcal{I}(\mathcal{U})$. Indeed, from (vii) it follows that $\check{\mathcal{X}}_{\mathfrak{p}} \cap \mathcal{U}$ is a weak open affine subscheme of $\check{\mathcal{X}}_{\mathfrak{p}} = \mathrm{Fspec}(\kappa(\mathfrak{p}))$. We may therefore assume that A is an $\mathbf{F}_1$-field. Let G be the kernel of the homomorphism $A^* \to A^*_{\mathcal{U}}$. Then the set $\mathcal{U}$ consists of the points that correspond to the subgroups of A^* which contain G. It follows that the set $\mathcal{U}$ is closed. Since $\mathrm{Fspec}(A)$ is irreducible, this implies that $G = \{1\}$, and the claim follows. The remaining statements easily follow. ∎

4.1.3. Lemma. *(i) For an element* $f \in A$, *the set* $D(f) = \{x \in \mathcal{X} \big| f(x) \neq 0\}$ *is a weak open affine subscheme (called a* principal open subset*), and it corresponds to the homomorphism* $A \to A_f$*;*

(ii) there is a bijection between the set of nonempty principal open subsets of $\mathcal{X}$ *and the set of Zariski prime ideals* $\mathfrak{p} \subset A$ *with the property that* $A_{\mathfrak{p}} = A_f$ *for some* $f \in A$.

4.1.4. Remark. (i) For a principal open subset the map from Lemma 4.1.2(ii) is a homeomorphism.

(ii) The map in the statement (ii) takes an element $f \in A$ to the maximal Zariski ideal that does not contain any positive powers of f.

(iii) If the Zariski spectrum $\mathrm{Zspec}(A)$ is finite (e.g., A is finitely generated over an $\mathbf{F}_1$-field), then image of the map in (ii) coincides with $\mathrm{Zspec}(A)$.

Proof. The statement (i) is trivial.

(ii) Let $\mathfrak{p}$ be the maximal Zariski ideal of A that does not contain any positive powers of an element $f \in A$. (It is a Zariski prime ideal of A.) For any element $g \in A \backslash \mathfrak{p}$, the Zariski ideal generated by $\mathfrak{p}$ and g contains a power of f, i.e., $gh = f^n$ for some $h \in A$ and $n \geq 1$. It follows that $\frac{1}{g} = \frac{h}{f^n}$, i.e., $A_{\mathfrak{p}} = A_f$. Since $\mathrm{Fspec}(A_{\mathfrak{p}}) = D(f)$, the map considered is bijective. ∎

4.1.5. Lemma. *Suppose that* A *is an idempotent* $\mathbf{F}_1$*-algebra. Then* *(i) for any finitely generated ideal E of A, the set* $\mathcal{U} = \{x \in \mathcal{X} \big| e(x) = f(x) \text{ for all } (e,f) \in E\}$ *is a weak open affine subscheme, and it corresponds to the homomorphism $A \to A/E$;*

(ii) every weak open affine subscheme of $\mathcal{X}$ is of the form (i).

Proof. (i) By the assumption, the set $\mathcal{U}$ is a finite intersection of sets of the form $\mathcal{X}(e,f) = \{x \in \mathcal{X} \big| e(x) = f(x)\}$. Since $\mathcal{X}(e,f) = (D(e,0) \cap D(f,0)) \cup (D(e,1) \cap D(f,1))$ is an open set, $\mathcal{U}$ is open in $\mathcal{X}$. Furthermore, suppose we are given a homomorphism $\varphi : A \to B$ to an $\mathbf{F}_1$-algebra B such that the image of $\mathcal{Y} = \mathrm{Fspec}(B)$ in $\mathcal{X}$ lies in $\mathcal{U}$. Then, given a pair of idempotents $(e,f) \in E$, one has $\varphi(e)(y) = \varphi(f)(y)$ for all $y \in \mathcal{Y}$. It follows that $\varphi(e) = \varphi(f)$. Thus, the homomorphism φ goes through a unique homomorphism $A/E \to B$.

(ii) The set $\mathcal{U}$ is open and, by Lemma 4.1.2(ii), it is compact and, therefore, it is also closed. Since $\mathcal{U}$ is preserved by the infimum operation, Proposition 3.1.4(iii) implies that it is of the form $\mathrm{Fspec}(A/E)$ for some ideal E of A. The statement (v) of the same proposition then implies that the ideal E is finitely generated, i.e., $\mathcal{U}$ is of the form (i). ∎

Lemma 4.1.5 implies that, for an arbitrary $\mathbf{F}_1$-algebra A and a finitely generated ideal E of the idempotent subalgebra I_A of A the set $\mathcal{X}(E) = \{x \in \mathcal{X} \big| e(x) = f(x) \text{ for all } (e,f) \in E\}$ is a weak open affine subscheme (called *an idempotent open subset*), and it corresponds to the homomorphism $A \to A(E) = A/F$, where F is the ideal of A generated by E. Notice that the map $\mathrm{Fspec}(A(E)) \to \mathcal{X}(E)$ is a homeomorphism.

4.2. Elementary open subsets and elementary families.

4.2.1. Definition. An open subset of $\mathcal{X} = \mathrm{Fspec}(A)$, which is an idempotent open subset of a principal open subset $D(f)$ associated to a finitely generated Zariski ideal $\mathbf{a} \subset I_{A_f}$, is said to be *elementary.*

Lemma 4.1.2(v) implies that every elementary open subset of $\mathcal{X}$ is a weak open affine subscheme, and Theorem 3.3.1 implies that, if $\mathcal{X}$ is connected, then any elementary open subset that has a nonempty intersection with $\mathcal{X}_{\mathbf{m}}$ coincides with $\mathcal{X}$. If $\mathcal{X}$ is idempotent, elementary open subsets are also closed subsets, and they form a basis of topology of $\mathcal{X}$. Notice also that, given a morphism of affine schemes $\varphi : \mathcal{Y} = \mathcal{M}(B) \to \mathcal{X}$, the preimage of an elementary open subset of $\mathcal{X}$ is an elementary open subset of $\mathcal{Y}$.

4.2.2. Proposition. *Let $\mathcal{U}$ be a nonempty elementary open subset of $\mathcal{X}$, and let $D(f)$ and* $\mathbf{a}$ *be the corresponding principal open subset of $\mathcal{X}$ and Zariski ideal of A_f from Definition 4.2.1. Then* *(i) $D(f)$ is a unique minimal principal open subset that contains $\mathcal{U}$, $\mathbf{a} = \{e \in I_{A_f} \big| e \in \mathfrak{p}A_f$ for all $\mathfrak{p} \in \mathcal{I}(\mathcal{U})\}$, and $\mathcal{I}(\mathcal{U}) = \{\mathfrak{p} \in \mathrm{Zspec}(A) \big| f \notin \mathfrak{p}$ and $\mathfrak{p} \supset \mathbf{a}\}$ (we set $D_{\mathcal{U}} = D(f)$, $\mathfrak{p}^{(\mathcal{U})} = \mathfrak{p}^{(D_{\mathcal{U}})}$, and $\mathbf{a}_{\mathcal{U}} = \mathbf{a}$);*

(ii) if $\mathcal{U} \cap \mathcal{V} \neq \emptyset$ for an elementary open subset $\mathcal{V}$ of $\mathcal{X}$, then $\mathcal{U} \cap \mathcal{V}$ is an elementary open subset, $\mathfrak{p}^{(\mathcal{U}\cap\mathcal{V})} = \sup(\mathfrak{p}^{(\mathcal{U})}, \mathfrak{p}^{(\mathcal{V})})$ and $\mathbf{a}_{\mathcal{U}\cap\mathcal{V}} = \mathbf{a}_{\mathcal{U}} I_{D_{\mathcal{U}\cap\mathcal{V}}} \cap \mathbf{a}_{\mathcal{V}} I_{D_{\mathcal{U}\cap\mathcal{V}}}$;

(iii) if $\mathcal{V}$ is an elementary open subset of $\mathcal{U}$, then it is an elementary open subset of $\mathcal{X}$;

(iv) if $\mathcal{U}$ is closed in $\mathcal{X}$, then it is the preimage of an elementary open subset of $\mathrm{Fspec}(I_A)$.

Proof. (i) *We claim that $\mathcal{U}_{\mathbf{m}} = D(f)_{\mathbf{m}}$.* Indeed, if the sets considered do not coincide, they are disjoint. This implies that some idempotent from $\mathbf{a}$ does not vanish at $D(f)_{\mathbf{m}}$ and, therefore, it does not vanish at all points of $D(f)$, which contradicts nonemptyness of $\mathcal{U}$. It follows that the preimage of $\mathbf{m}_{A_f}$ in A (i.e., $\mathfrak{p}^{(D(f))}$) coincides with the preimage of $\mathbf{m}_{A_{\mathcal{U}}}$ in A, and the minimality property of $D(f)$ follows. Furthermore, let $e \in I_{A_f}$. If $e \in \mathbf{a}$, it is clear that $e \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{I}(\mathcal{U})$. Conversely, if $e \in \mathfrak{p}$ for all $\mathfrak{p} \in \mathcal{I}(\mathcal{U})$, then $e\big|_{\mathcal{U}} = 0$, i.e., e lies in the Zariski kernel of the homomorphism $A_f \to A_{\mathcal{U}}$, which coincides with the Zariski ideal of A_f generated by $\mathbf{a}$. The inclusion $e \in \mathbf{a}$ then follows from Corollary 1.6.2(iii). Finally, suppose a Zariski prime ideal $\mathfrak{p} \subset A_f$ contains $\mathbf{a}$. Then $e\big|_{\check{\mathcal{X}}_{\mathfrak{p}}} = 0$ for all $e \in \mathbf{a}$, i.e., $\mathfrak{p} \in \mathcal{I}(\mathcal{U})$.

(ii) Suppose that $\mathcal{V}$ is an idempotent open subset of a principal open subset $D(g)$ associated to a finitely generated Zariski ideal $\mathbf{b} \subset I_{A_g}$, and let $\mathbf{c}$ be the Zariski ideal of $I_{A_{fg}}$ generated by the images of $\mathbf{a}$ and $\mathbf{b}$. Then $\mathcal{U} \cap \mathcal{V} = \{x \in \mathcal{X} \big| (fg)(x) \neq 0$ and $e(x) = 0$ for all $e \in \mathbf{c}\}$ and, in particular, $\mathcal{U} \cap \mathcal{V}$ is an elementary open subset. Since the latter set is nonempty, it follows that it contains the set $D(fg)_{\mathbf{m}}$. This implies the required equalities. It remains to notice that, since $\mathcal{U} \cap \mathcal{V}$ is a weak open affine subscheme, it is quasicompact, and this easily implies that the Zariski ideal $\mathbf{a}_{\mathcal{U}\cap\mathcal{V}}$ of $I_{D_{\mathcal{U}\cap\mathcal{V}}}$ is finitely generated.

The statement (iii) follows from (i).

(iv) That $\mathcal{U}$ is the preimage of an open-closed subset $\mathcal{U}'$ of $\mathrm{Fspec}(I_A)$ follows from Theorem 3.3.1(i). It is also clear that $\mathcal{U}'$ is the image of $\mathcal{U}$ in $\mathrm{Fspec}(I_A)$. Let $\mathfrak{r}$ be the maximal Zariski ideal of I_A that does not contain any powers of f, i.e., $\mathfrak{r} = \mathfrak{p}^{(\mathcal{U})} \cap I_A$, and let $\mathfrak{q}$ be the maximal Zariski ideal $\mathfrak{q}$ of A with $\mathfrak{q} \cap I_A = \mathfrak{r}$. Then $\mathfrak{p}^{(\mathcal{U})}$ and $\mathfrak{q}$ are the maximal Zariski prime ideals $\mathfrak{p}$ of A for which $\check{\mathcal{X}}_{\mathfrak{p}}$ has a nonempty intersection with $\mathcal{U}$ and the preimage of $\mathcal{U}'$, respectively. Since both coincide, it follows that $\mathfrak{p}^{(\mathcal{U})} = \mathfrak{q}$. Furthermore, since $\Pi_{(\mathfrak{r})} = \Pi_{\mathfrak{r}} \in \mathcal{U}'$, Proposition 3.1.3 implies that $\mathfrak{r} = \mathfrak{p}_e$ for some element $e \in I_A \backslash \mathfrak{r}$. It follows that the principal open subset $D(e)$ of $\mathcal{X}$ coincides with $D(f)$. Finally, the canonical homomorphism of idempotent $\mathbf{F}_1$-algebras $I_A \to I_{A_f} = I_{A_e}$ is surjective and its kernel is the ideal generated by the pair $(e, 1)$. Let $e_1, \dots, e_n$ be elements of I_A whose images in

I_{A_f} generate the Zariski ideal $\mathbf{a}$. Then $\mathcal{U}' = \{x \in \mathrm{Fspec}(I_A) \big| e(x) = 1 \text{ and } e_1(x) = \ldots = e_n(x) = 0\}$ and, therefore, $\mathcal{U}'$ is an elementary open subset of $\mathrm{Fspec}(I_A)$. The required statement follows. ■

We will denote the principal open subset $D(f)$ and the Zariski ideal $\mathbf{a}$ of an elementary open subset $\mathcal{U}$ by $D_\mathcal{U}$ and $\mathbf{a}_\mathcal{U}$, respectively. Given elementary open subsets $\mathcal{U}$ and $\mathcal{V}$, we write $\mathcal{U} \le \mathcal{V}$ if $D_\mathcal{U} \supset D_\mathcal{V}$.

4.2.3. Definition. An *elementary family* is a finite family S of pairwise disjoint open subsets of $\mathcal{X}$ such that the above partial ordering makes S an inf-poset with the following property: if $\mathcal{W} = \inf(\mathcal{U}, \mathcal{V})$ in S, then for every idempotent $e \in \mathbf{a}_\mathcal{W}$ one has either $e\big|_\mathcal{U} = 0$, or $e\big|_\mathcal{V} = 0$.

4.2.4. Example. Let I be a finite $\mathbf{F}_1$-subalgebra of $I_\mathcal{X} = I_A$. Then the family of fibers of the canonical map $\mathcal{X} \to \mathrm{Fspec}(I)$ is an elementary family with S isomorphic to the poset $\check{I}$. Indeed, the preimage of the prime ideal Π_e for $e \in \check{I}$ is the elementary set $\mathcal{V}^{(e)}$ defined by the equalities $e(x) = 1$ and $f(x) = 0$ for $f \in \mathfrak{p}_e = \{f \in I \big| f \not\le e\}$, i.e., $D_{\mathcal{V}^{(e)}} = D(e)$ and $\mathbf{a}_{\mathcal{V}^{(e)}} = \mathfrak{p}_e A_e = \mathbf{m}_{A_e}$. One has $V^{(e)} \le \mathcal{V}^{(f)}$ if and only if $D(e) \supset D(f)$, i.e., $e \le f$. If $g = \inf(e, f)$, then $\mathcal{V}^{(g)} = \inf(\mathcal{V}^{(e)}, \mathcal{V}^{(f)})$ and, since $\mathfrak{p}_g = \mathfrak{p}_e \cup \mathfrak{p}_f$, for every $e \in \mathbf{a}_{\mathcal{V}^{(g)}}$ one has either $e\big|_{\mathcal{V}^{(e)}} = 0$, or $e\big|_{\mathcal{V}^{(f)}} = 0$. We say that the elementary family S is *associated to* I. Notice that, if the whole idempotent $\mathbf{F}_1$-subalgebra $I_\mathcal{X}$ is finite, the elementary family associated to it is the family of connected components of $\mathcal{X}$.

4.2.5. Proposition. *Let S be an elementary family of open subsets of $\mathcal{X}$, and let $\mathcal{W} = \inf(\mathcal{U}, \mathcal{V})$ in S. Then*

(i) for every pair $\mathfrak{p} \in \mathcal{I}(\mathcal{U})$ and $\mathfrak{q} \in \mathcal{I}(\mathcal{V})$, one has $\inf(\mathfrak{p}, \mathfrak{q}) \in \mathcal{I}(\mathcal{W})$;

(ii) if $\mathcal{U} \not\le \mathcal{V}$, then $D_\mathcal{U} \cap \mathcal{V} = \emptyset$;

(iii) for any elementary family T on $\mathcal{X}$, the family of nonempty intersections $\mathcal{U} \cap \mathcal{V}$ with $\mathcal{U} \in S$ and $\mathcal{V} \in T$ is an elementary family;

(iv) for any morphism $\varphi : \mathcal{Y} = \mathcal{M}(B) \to \mathcal{X}$, the family of nonempty open subsets $\mathcal{U}' \subset \mathcal{Y}$ of the form $\varphi^{-1}(\mathcal{U})$ with $\mathcal{U} \in S$ is an elementary family of open subsets of $\mathcal{Y}$;

(v) if sets from S cover the whole space $\mathcal{X}$, then S is associated to a finite idempotent $\mathbf{F}_1$-subalgebra of A.

Proof. (i) If $e \in \mathbf{a}_\mathcal{W}$, then either $e\big|_{D_\mathcal{U}} \in \mathbf{a}_\mathcal{U}$, or $e\big|_{D_\mathcal{V}} \in \mathbf{a}_\mathcal{V}$ and, therefore, either $e\big|_{D_\mathcal{U}} \in \mathfrak{p}A_{D_\mathcal{U}}$, or $e\big|_{D_\mathcal{V}} \in \mathfrak{q}A_{D_\mathcal{V}}$. It follows that $e \in (\mathfrak{p} \cup \mathfrak{q})A_{D_\mathcal{W}}$, and Proposition 4.2.2(i) implies the required fact.

(ii) Assume that the statement is not true. If $D_\mathcal{U} = D(f)$ and $D_\mathcal{V} = D(g)$ for some $f, g \in A$, then $D_\mathcal{U} \cap \mathcal{V} \subset D(fg) \ne \emptyset$. Since every idempotent from $\mathbf{a}_\mathcal{V}$ equals to zero at $D_\mathcal{U} \cap \mathcal{V}$, it follows that it equals to zero at $D(fg)_\mathbf{m}$, i.e., $D(fg)_\mathbf{m} \subset \mathcal{V}$. Recall that for every $e \in \mathbf{a}_\mathcal{W}$ one has either

$e\big|_{\mathcal{U}} = 0$, or $e\big|_{\mathcal{V}} = 0$. In the latter case it follows that e equals to zero at $D(fg)_{\mathbf{m}}$ and, therefore, it equals to zero at $D(f)_{\mathbf{m}} \subset \mathcal{U}$. This implies that $D(f)_{\mathbf{m}} \subset \mathcal{U} \cap \mathcal{W}$, which is impossible.

(iii) Let R denote the family considered. Suppose that $\mathcal{U}_1 \cap \mathcal{V}_1 \neq \emptyset$ and $\mathcal{U}_2 \cap \mathcal{V}_2 \neq \emptyset$ for some $\mathcal{U}_1, \mathcal{U}_2 \in S$ and $\mathcal{V}_1, \mathcal{V}_2 \in T$, and set $\mathcal{W}_1 = \inf(\mathcal{U}_1, \mathcal{U}_2)$ and $\mathcal{W}_2 = \inf(\mathcal{V}_1, \mathcal{V}_2)$. Since $\mathfrak{p}^{(\mathcal{U}_i \cap \mathcal{V}_i)} \in \mathcal{I}(\mathcal{U}_i) \cap \mathcal{I}(\mathcal{V}_i)$ for $i = 1, 2$, the statement (i) implies that $\inf(\mathfrak{p}^{(\mathcal{U}_1 \cap \mathcal{V}_1)}, \mathfrak{p}^{(\mathcal{U}_2 \cap \mathcal{V}_2)}) \in \mathcal{I}(\mathcal{W}_1) \cap \mathcal{I}(\mathcal{W}_2)$. It follows that the intersection $\mathcal{W}_1 \cap \mathcal{W}_2$ is nonempty and coincides with $\inf(\mathcal{U}_1 \cap \mathcal{V}_1, \mathcal{U}_2 \cap \mathcal{V}_2)$ in R. Furthermore, Proposition 4.2.2(ii) implies that, for every $e \in \mathbf{a}_{\mathcal{W}_1 \cap \mathcal{W}_2}$, one has $e = e_1 f_1 = e_2 f_2$, where $e_i \in I_{D_{\mathcal{W}_i}}$ and $f_i \in I_{D_{\mathcal{W}_1 \cap \mathcal{W}_2}}$. By Definition 4.2.3, one has either $e_1\big|_{\mathcal{U}_1} = 0$ or $e_1 \big|_{\mathcal{U}_2} = 0$, and either $e_2\big|_{\mathcal{V}_1} = 0$ or $e_2 \big|_{\mathcal{V}_2} = 0$. This implies that either $e\big|_{\mathcal{U}_1 \cap \mathcal{V}_1} = 0$, or $e\big|_{\mathcal{U}_2 \cap \mathcal{V}_2} = 0$ and, therefore, R is an elementary family.

(iv) Let $\mathcal{U}'$ and $\mathcal{V}'$ be sets from the from the family T considered. It is easy to see that $\mathcal{U}' \leq \mathcal{V}'$ if and only if $\mathcal{U} \leq \mathcal{V}$, and so the partial ordering on T is induced by that on S. To show that it admits the infimum operation, we have to verify that the preimage $\mathcal{W}'$ of $\mathcal{W} = \inf(\mathcal{U}, \mathcal{V})$ in $\mathcal{Y}$ is nonempty. For this we notice that, if $\mathfrak{p}' \in \mathcal{I}(\mathcal{U}')$ and $\mathfrak{q}' \in \mathcal{I}(\mathcal{V}')$, then for their images in $\mathrm{Zspec}(A)$ one has $\mathfrak{p} \in \mathcal{I}(\mathcal{U})$ and $\mathfrak{q} \in \mathcal{I}(\mathcal{V})$, and (i) implies that $\mathfrak{r} = \inf(\mathfrak{p}, \mathfrak{q}) = \mathfrak{p} \cup \mathfrak{q} \in \mathcal{I}(\mathcal{W})$. Since $\mathfrak{r}$ is the image of the Zariski prime ideal $\mathfrak{r}' = \inf(\mathfrak{p}', \mathfrak{q}')$, it follows that $\mathfrak{r}' \in \mathcal{I}(\mathcal{W}')$ and, in particular, $\mathcal{W}' \neq \emptyset$. The required property of $\mathbf{a}_{\mathcal{W}'}$ from Definition 4.2.3 easily follows.

(v) Notice that every set $\mathcal{U} \in S$ is closed. Proposition 4.2.2(iv) implies that $\mathcal{U}$ is the preimage of an elementary open subset of $\mathrm{Fspec}(I_A)$, i.e., $\mathcal{U} = \{x \in \mathcal{X} \big| e(x) = 1 \text{ and } f(x) = 0 \text{ for all } f \in \mathbf{a}\}$, where $e = e_{\mathcal{U}} \in I_A$ and $\mathbf{a}$ is a finitely generated Zariski ideal of I_A. *We claim that* $I = \{0\} \cup \{e_{\mathcal{U}} \big| \mathcal{U} \in S\}$ *is an* $\mathbf{F}_1$*-subalgebra of* I_A. Indeed, let $\mathcal{U}, \mathcal{V} \in S$. Then $D(e_{\mathcal{U}}) \cap D(e_{\mathcal{V}}) = D(e_{\mathcal{U}} e_{\mathcal{V}})$. The latter is nonempty if and only if $f = e_{\mathcal{U}} e_{\mathcal{V}} \neq 0$. To show that $f \in I$, we may assume that $f \neq 0, e_{\mathcal{U}}, e_{\mathcal{V}}$ and, in particular, $\mathcal{U} \not\leq \mathcal{V}$ and $\mathcal{V} \not\leq \mathcal{U}$. Since $D_{\mathcal{U}} \cap \mathcal{V} = \emptyset$ and $D_{\mathcal{V}} \cap \mathcal{U} = \emptyset$, it follows that $\Pi_f \notin \mathcal{U} \cup \mathcal{V}$ and, in particular, $f\big|_{\mathcal{U}} = 0$ and $f\big|_{\mathcal{V}} = 0$. Let $\mathcal{W} \in S$ contain Π_f. Assume that $\mathcal{U} \not\leq \mathcal{W}$. Since $D_{\mathcal{U}} \cap \mathcal{W} \neq \emptyset$, it follows that $\mathcal{W} < \mathcal{U}$ and, therefore, $f\big|_{\mathcal{W}} = 0$, which contradicts the inclusion $\Pi_f \in \mathcal{W}$. Thus, $\mathcal{U} \leq \mathcal{W}$. For the same reason, one has $\mathcal{V} \leq \mathcal{W}$ and, therefore, $\mathcal{W} \subset D(e_{\mathcal{U}}) \cap D(e_{\mathcal{V}}) = D(f)$. It follows that $e_{\mathcal{W}} = f$ and, in particular, $f \in I$. It remains to notice that every $\mathcal{U} \in S$ coincides with the open set $\{x \in D_{\mathcal{U}} \big| e_{\mathcal{V}}(x) = 0 \text{ for all } \mathcal{V} < \mathcal{U} \text{ in } S\}$, i.e., with the fiber of the map $\mathcal{X} \to \mathrm{Fspec}(I)$ over $\Pi_{e_{\mathcal{U}}}$. ∎

4.2.6. Proposition. *Every open covering of $\mathcal{X}$ by elementary open subsets admits a refinement which is an elementary family associated to a finite idempotent $\mathbf{F}_1$-subalgebra of A.*

Proof. Let $\mathcal{U} = \{\mathcal{U}_i\}_{i \in I}$ be such a covering. *The covering $\mathcal{U}$ has a refinement which is the* *preimage of an elementary family of open subsets of* $\mathrm{Fspec}(I_A)$. Indeed, since $\mathcal{X}$ is quasicompact, we may assume that $\mathcal{U}$ is finite, i.e., $I = \{1, \dots, n\}$, and we may assume that $\mathcal{X}_{\mathbf{m}} \subset \mathcal{U}_1$. Then $\mathcal{U}_1$ is defined by equalities $e_1(x) = \dots = e_k(x) = 0$ with $e_1, \dots, e_k \in I_A$. The claim is trivial if $n = 1$ or $k = 0$. Suppose that $n \geq 1$ and $k \geq 1$ and that the claim is true for all coverings in which one of these numbers is strictly smaller. Then $\mathcal{X} = \mathcal{X}' \coprod \mathcal{X}''$, where $\mathcal{X}' = \{x \in \mathcal{X} \big| e_1(x) = 0\}$ and $\mathcal{X}'' = \{x \in \mathcal{X} \big| e_1(x) = 1\}$. For the covering $\mathcal{U} \cap \mathcal{X}' = \{\mathcal{U}_i \cap \mathcal{X}'\}_{1 \leq i \leq n}$ of $\mathcal{X}'$, the set $\mathcal{U}_1 \cap \mathcal{X}'$ is defined by the equalities $e_2(x) = \dots = e_n(x) = 0$, and one has $\mathcal{X}'' \subset \bigcup_{i=2}^{n} \mathcal{U}_i$. Since the homomorphism $I_A \to I_{\mathcal{X}'} = I_A / eI_A$ and $I_A \to I_{\mathcal{X}''} = (I_A)_e$ are surjective, the claim follows from the induction hypothesis applies to $\mathcal{X}'$ and $\mathcal{X}''$.

The previous claim reduces the situation to the case when A is an idempotent $\mathbf{F}_1$-algebra. In this case, we notice that each of the elementary open subset $\mathcal{U}_i$ is defined by a finite number of elements from A. Let A' be the $\mathbf{F}_1$-subalgebra of A generated by all such idempotents. Then A' is finite, and the covering $\mathcal{U}$ is the preimage of a family $\mathcal{U}'$ of $\mathcal{X}' = \mathrm{Fspec}(A')$ by elementary open subsets of $\mathcal{X}'$. Since the map $\mathcal{X} \to \mathcal{X}'$ is surjective, it follows that $\mathcal{U}'$ is a covering of $\mathcal{X}'$. The covering of $\mathcal{X}'$ by its points is an elementary family which is a refinement of $\mathcal{U}'$. This implies the required fact. ∎

4.3. Open affine subschemes. Let $\mathcal{X} = \mathrm{Fspec}(A)$ be an affine schemes over $\mathbf{F}_1$.

4.3.1. Definition. An *open affine subscheme of* $\mathcal{X}$ is an open subset which admits a covering by an elementary family of open subsets.

The following statement easily follow from the properties of elementary families established in the previous subsection.

4.3.2. Proposition. *Let $\mathcal{U}$ be an open affine subscheme of $\mathcal{X}$. Then*

(i) for any open affine subscheme $\mathcal{V}$, the intersection $\mathcal{U} \cap \mathcal{V}$ is an open affine subscheme;

(ii) for any morphism $\varphi : \mathcal{Y} \to \mathcal{X}$, the preimage $\varphi^{-1}(\mathcal{U})$ is an open affine subscheme of $\mathcal{Y}$;

(iii) there exists a finitely generated $\mathbf{F}_1$-subalgebra A' of A such that $\mathcal{U}$ is the preimage of an open affine subscheme of $\mathrm{Fspec}(A')$*;*

(iv) if $\mathcal{U}$ is closed in $\mathcal{X}$, it is the preimage of an open affine subscheme of $\mathrm{Fspec}(I_A)$. ∎

4.3.3. Example. Every idempotent open subset $\mathcal{U}$ of $\mathcal{X}$ is an open affine subscheme. Indeed, by Lemma 4.1.5 one has $\mathcal{U} = \mathrm{Fspec}(A/E)$, where E is a finitely generated ideal of the idempotent $\mathbf{F}_1$-subalgebra I_A. This reduces the situation to the case when A is an idempotent $\mathbf{F}_1$-algebra. Furthermore, we can find a finite $\mathbf{F}_1$-subalgebra A' of A such that E is a generated by an ideal E'

of A'. This reduces the situation to the case when A is a finite idempotent $\mathbf{F}_1$-algebra. In this case the claim follows from Proposition 3.1.4(iii).

4.3.4. Theorem. *Every open affine subscheme $\mathcal{U}$ of $\mathcal{X}$ is a weak open affine subscheme.*

Proof. (i) Let S be an elementary family of open subsets that cover $\mathcal{U}$. If $\mathcal{V} \le \mathcal{W}$ in S, then the map $A_{D_\mathcal{V}} \to A_{D_\mathcal{W}}$, that takes all elements from $\mathbf{a}_\mathcal{V} A_{D_\mathcal{V}}$ to zero and every element $a \notin \mathbf{a}_\mathcal{V} A_{D_\mathcal{V}}$ to $a\big|_{D_\mathcal{W}}$ gives rise to a quasi-homomorphism $\nu_{\mathcal{V}\mathcal{W}} : A_\mathcal{V} \to A_\mathcal{W}$. *We claim that the triple $\{S, A_\mathcal{V}, \nu_{\mathcal{V}\mathcal{W}}\}$ is a disconnected sum datum.* Indeed, validity of the properties (0) and (1) of Definition 3.4.3 is trivial. Suppose we are given elementary open subsets $\mathcal{Y} < \mathcal{Z} = \inf(\mathcal{V}, \mathcal{W})$ in S and an element $a \in A_\mathcal{Y}$ with $\nu_{\mathcal{Y}\mathcal{V}}(a) \neq 0$ and $\nu_{\mathcal{Y}\mathcal{W}}(a) \neq 0$. We have to verify that $\nu_{\mathcal{Y}\mathcal{Z}}(a) \neq 0$. For this we may assume that a is an element in $A_{D_\mathcal{Y}} \backslash \mathbf{a}_\mathcal{Y} A_{D_\mathcal{Y}}$. The assumption mean that $a\big|_\mathcal{V} \neq \emptyset$ and $a\big|_\mathcal{W} \neq \emptyset$. If $\nu_{\mathcal{Y}\mathcal{Z}}(a) = 0$, then $a\big|_{D_\mathcal{Z}} = eb$ for some $e \in \mathbf{a}_\mathcal{Z}$ and $b \in A_{D_\mathcal{Z}}$. By Definition 4.2.3, this implies that either $e\big|_\mathcal{V} = 0$, or $e\big|_\mathcal{W} = 0$, which contradicts the assumption.

As above, one verifies that the image of the canonical homomorphism $A \to \prod_{\mathcal{V} \in S} A_\mathcal{V}$ lies in $\coprod_S^\nu A_\mathcal{V}$. *We claim that $\mathcal{U}$ is an open affine subscheme with $A_\mathcal{U} = \coprod_S^\nu A_\mathcal{V}$.* Indeed, it is clear that the canonical homomorphism $A \to A_\mathcal{U}$ induces a homeomorphism $\mathrm{Fspec}(A_\mathcal{U}) \xrightarrow{\sim} \mathcal{U}$. Let $\varphi : \mathcal{X}' = \mathrm{Fspec}(A') \to \mathcal{X}$ be a morphism of affine schemes whose image lies in $\mathcal{U}$. It follows from Proposition 4.2.3(iii) and (iv) that the family S' of non-empty open subsets $\mathcal{V}'$ of the form $\varphi^{-1}(\mathcal{V})$ with $\mathcal{V} \in S$ is an elementary family which is associated to a finite idempotent $\mathbf{F}_1$-subalgebra I'. In particular, there is an isomorphism of $\mathbf{F}_1$-algebras $A' \xrightarrow{\sim} \coprod_{S'}^\nu A'_{\mathcal{V}'}$. By the construction, the morphism φ gives rise to a morphism of disconnected sum data $\{S, A_\mathcal{V}, \nu_{\mathcal{V}\mathcal{W}}\} \to \{S', A'_{\mathcal{V}'}, \nu_{\mathcal{V}'\mathcal{W}'}\}$ which induces the required homomorphism $A_\mathcal{U} \to A'$.

The statement (ii) easily follows from Proposition 4.2.2(iii) and Definition 4.2.3. ∎

4.3.5. Corollary. *In the situation of Theorem 4.3.4, if $\mathcal{V}$ is an open affine subscheme of $\mathcal{U}$, then it is an open affine subscheme of $\mathcal{X}$.* ∎

4.3.6. Theorem. *Suppose that $\mathcal{X}$ is weakly decomposable. Then the following properties of an open subset $\mathcal{U} \subset \mathcal{X}$ are equivalent:*

(a) $\mathcal{U}$ is an open affine subscheme;

(b) $\mathcal{U}$ is a weak open affine subscheme;

(c) for every pair $\mathfrak{p}, \mathfrak{q} \in \mathcal{I}(\mathcal{U})$, one has $\inf(\mathfrak{p}, \mathfrak{q}) \in \mathcal{I}(\mathcal{U})$ and $\check{\mathcal{X}}_\mathfrak{p} \subset \mathcal{U}$.

Proof. The implications (a)$\Longrightarrow$(b) and (b)$\Longrightarrow$(c) follow from Theorem 4.3.4 and Lemma 4.1.2(iv), respectively, and so it remains to verify the implication (c)$\Longrightarrow$(a). Notice that there is a unique maximal Zariski prime ideal $\mathfrak{p}$ with $\check{\mathcal{X}}_\mathfrak{p} \subset \mathcal{U}$, and Proposition 3.1.3 implies that $A_\mathfrak{p} = A_f$

for some $f \notin \mathfrak{p}$.

Step 1. *If $\mathcal{X}$ is irreducible, $\mathcal{U}$ is a principal open subset.* Indeed, by the above remark it suffices to verify that $\mathcal{U} = \mathrm{Fspec}(A_{\mathfrak{p}})$, i.e., if $\mathfrak{q}$ is a Zariski prime ideal with $\mathfrak{p} \le \mathfrak{q}$ (i.e., $\mathfrak{q} \subset \mathfrak{p}$), then $\check{\mathcal{X}}_{\mathfrak{q}} \subset \mathcal{U}$. Irreducibility of $\mathcal{X}$ implies that, for every Zariski prime ideal $\mathfrak{q} \subset A$, one has $\mathcal{X}^{(\mathfrak{q})} = \mathcal{X}_{\mathfrak{q}}$. Thus, if $\mathfrak{q} \subset \mathfrak{p}$, then $\mathcal{X}_{\mathfrak{p}} \subset \mathcal{X}_{\mathfrak{q}}$ and, therefore, $\mathcal{X}^{(\mathfrak{p})} \subset \mathcal{X}^{(\mathfrak{q})}$. Since $\check{\mathcal{X}}_{\mathfrak{p}} \subset \mathcal{U}$ and the set $\mathcal{U}$ is open, it follows that the intersection $\check{\mathcal{X}}_{\mathfrak{q}} \cap \mathcal{U}$ is nonempty and, therefore, $\check{\mathcal{X}}_{\mathfrak{q}} \subset \mathcal{U}$.

Step 2. *$\mathcal{U}$ contains the minimal connected component of $D(f)$.* Indeed, replacing $\mathcal{X}$ by $D(f)$, we may assume that $\mathcal{X}_{\mathbf{m}} \subset \mathcal{U}$. Let $\mathcal{V}$ be the minimal connected component of $\mathcal{X}$. By Corollary 3.3.4, for any $\mathfrak{q} \in \mathcal{I}(\mathcal{V})$ there exists a chain of Zariski prime ideals $\mathfrak{p}_0 = \mathbf{m} \le \mathfrak{p}_1 \le \ldots \le \mathfrak{p}_n = \mathfrak{q}$ such that $\mathcal{X}^{(\mathfrak{p}_i)} \cap \mathcal{X}^{(\mathfrak{p}_{i+1})} \neq \emptyset$ for all $0 \le i \le n-1$. We have $\mathcal{X}^{(\mathfrak{p}_0)} \subset \mathcal{V}$. Suppose that $\mathcal{X}^{(\mathfrak{p}_i)} \subset \mathcal{U}$ for some $0 \le i \le n-1$. By Lemma 3.3.2(ii), one has $\mathcal{X}^{(\mathfrak{p}_i)} \cap \mathcal{X}^{(\mathfrak{p}_{i+1})} = \mathrm{Fspec}(A^{(\mathfrak{p}_{i+1})}/\mathbf{a})$, where $\mathbf{a}$ is the image of $\mathfrak{p}_i$ in $A^{(\mathfrak{p}_{i+1})}$. Since the Zariski ideal $\mathbf{a}$ is nontrivial, it is contained in the maximal Zariski ideal of $A^{(\mathfrak{p}_{i+1})}$ and, therefore, the above intersection contains $(\mathcal{X}^{(\mathfrak{p}_{i+1})})_{\mathbf{m}}$. It follows that the latter lies in $\mathcal{U}$ and, therefore, $\mathcal{X}^{(\mathfrak{p}_{i+1})} \subset \mathcal{U}$.

Step 3. *$\mathcal{U}$ is an open affine subscheme.* By Step 2, the claim is true if $\mathcal{U}$ is connected. Suppose that $\mathcal{U}$ is not connected and that the claim is true for open subsets with the property (c) and the number of connected components strictly less than that of $\mathcal{U}$. By Step 3, we may replace $\mathcal{X}$ by $D(f)$ and assume that the minimal connected component $\mathcal{V}$ of $\mathcal{X}$ lies in $\mathcal{U}$. Let $\mathcal{W}$ be a connected component of $\mathcal{U}$ different from $\mathcal{V}$. Then there exists a nontrivial idempotent $e \in I_A$ which equals to one at $\mathcal{W}$, and $\mathcal{X}$ is a disjoint union of the idempotent open subsets $\mathcal{X}' = \{x \in \mathcal{X} \big| e(x) = 0\}$ and $\mathcal{X}'' = \{x \in \mathcal{X} \big| e(x) = 1\}$. The intersections $\mathcal{U}' = \mathcal{U} \cap \mathcal{X}'$ and $\mathcal{U}'' = \mathcal{U} \cap \mathcal{X}''$ are unions of connected components of $\mathcal{U}$ and, by induction, they are open affine subschemes of $\mathcal{X}$. In particular, the connected components of $\mathcal{U}$ are elementary open subsets of $\mathcal{X}$, and there is a partial ordering of the set $\pi_0(\mathcal{U})$ of all of them. Finally, let $\mathcal{W}_1$ and $\mathcal{W}_2$ be connected components of $\mathcal{U}$ different from $\mathcal{V}$. If there exists a nontrivial idempotent $e \in I_A$ which is equal to zero at both $\mathcal{W}_1$ and $\mathcal{W}_2$, then applying the induction hypothesis to the idempotent open subset $\{x \in \mathcal{X} \big| e(x) = 0\}$, we get the required property of Definition 4.2.3 for the infimum of $\mathcal{W}_1$ and $\mathcal{W}_2$. Assume therefore that each nontrivial idempotent from I_A equals to one at least at one of them. We then claim that $\mathcal{V} = \inf(\mathcal{W}_1, \mathcal{W}_2)$. Indeed, if $\mathcal{W} \le \mathcal{W}_1$ and $\mathcal{W}_2 \le \mathcal{W}_2$ for some $\mathcal{W}$ different from $\mathcal{V}$, then there exists a nontrivial idempotent equal to zero at $\mathcal{W}$ and, therefore, at noth $\mathcal{W}_1$ and $\mathcal{W}_2$ which contradicts the assumption. Thus, $\mathcal{V} = \inf(\mathcal{W}_1, \mathcal{W}_2)$. It remains to notice that $\mathcal{V}$ is the only connected component of $\mathcal{X}$ at which all nontrivial idempotents equal to one. Together with the above assumption this implies that each of those idempotents equal to zero at $\mathcal{W}_1$ or at $\mathcal{W}_2$. ∎

4.3.7. Proposition. *Let Σ be a subset of an open scheme $\mathcal{X}$, and suppose that there exists a covering $\{\mathcal{U}_i\}_{i\in I}$ by open affine subschemes such that, for every $i \in I$, $\Sigma \cap \mathcal{U}_i$ is a strongly closed subset of $\mathcal{U}_i$. Then Σ is a strongly closed subset of $\mathcal{X}$.*

Proof. By Proposition 4.2.6, we may replace the covering by a refinement and assume that the covering is finite and all $\mathcal{U}_i$'s are pairwise disjoint elementary open subsets. In this case they are idempotent open subsets, and so every strongly closed subset of each $\mathcal{U}_i$ is a strongly closed subset of $\mathcal{X}$. This implies the required fact. ∎

4.4. Properties of open affine subschemes. Let $\mathcal{X} = \mathrm{Fspec}(A)$ be an affine scheme over $\mathbf{F}_1$, and let $\mathcal{U}$ be an open affine subscheme of $\mathcal{X}$.

4.4.1. Proposition. *If A is an integral domain (resp. reduced), then so is $A_{\mathcal{U}}$.*

Proof. If A is integral, then $\mathcal{U}$ is a principal open subset and, therefore, $A_{\mathcal{U}}$ is also integral. Suppose that A is reduced. Since $A_{\mathcal{U}}$ is a disconnected sum coproduct taken over elementary open subsets from an elementary family that covers $\mathcal{U}$, in order to show that $A_{\mathcal{U}}$ is reduced, we may assume that $\mathcal{U}$ is an elementary open subset. Replacing $\mathcal{X}$ by a principal open subset, we may assume that $A_{\mathcal{U}}$ is the quotient of A by a finitely generated Zariski ideal of I_A. It suffices to show that, if an $\mathbf{F}_1$-algebra A is reduced, then for any idempotent $e \in A$, the quotient A/Ae has no nilpotent elements. Suppose that the image of an element $f \in A$ in A/Ae is nilpotent, i.e., $f^n = ea$ for $f, a \in A$ and $n \geq 1$. Then $f^{n+i} = eaf^i = (fe)^{n+i}$ for all $i \geq 0$. This means that $(f, fe) \in \mathrm{z}\mathbf{n}(A)$. Since A is reduced, it follows that $f = fe$, i.e., the image of f in A/Ae is zero. ∎

4.4.2. Proposition. *Let $\varphi : \mathcal{Y} = \mathrm{Fspec}(B) \to \mathcal{X}$ be a morphism which is a homeomorphism between the underlying topological spaces. Then the correspondence $\mathcal{U} \mapsto \varphi^{-1}(\mathcal{U})$ gives rise to a bijection between the families of open affine subschemes of $\mathcal{X}$ and of $\mathcal{Y}$.*

Proof. Corollary 3.1.2(ii) the map $\mathrm{Zspec}(B) \to \mathrm{Zspec}(A)$ induced by the morphism φ is an isomorphism of posets. This implies that the correspondence considered gives rise to an injective of the family of principal open subsets of $\mathcal{X}$ to that of $\mathcal{Y}$. If $\mathfrak{q}$ is a Zariski prime ideal of B such that $B_{\mathfrak{q}} = B_g$ for some $g \in B\backslash\mathfrak{q}$, then the set $\mathrm{Fspec}(B_{\mathfrak{q}})$ is open in $\mathcal{Y}$. It follows that, if $\mathfrak{p}$ is the image of $\mathfrak{q}$ in $\mathrm{Zspec}(A)$ then $\mathrm{Fspec}(A_{\mathfrak{p}})$ which is the image of $\mathrm{Fspec}(B_{\mathfrak{q}})$ is open in $\mathcal{X}$, and Proposition 3.1.3 implies that $A_{\mathfrak{p}} = A_f$ for some $f \in A\backslash\mathfrak{p}$, i.e., the above injective map is a bijection. Furthermore, the morphism φ induces a homemorphism $\pi_0(\mathcal{Y}) \xrightarrow{\sim} \pi_0(\mathcal{X})$ and, by Theorem 3.3.1, it induces a homeomorphism $\mathrm{Fspec}(I_B) \xrightarrow{\sim} \mathrm{Fspec}(I_A)$. Proposition 3.1.4 then implies that φ induces a bijection between finitely generated Zariski ideals of I_A and I_B. This implies that the same is true for the principal open subsets of both affine schemes. Thus, φ induces a bijection between elementary open subsets of both affine schemes. This easily implies the required fact. ∎

4.4.3. Proposition. *The following properties of an open affine subscheme $\mathcal{U} \subset \mathcal{X}$ are equivalent:*

(a) the homomorphism $A \to A_{\mathcal{U}}$ is surjective;

(b) $\mathcal{U}$ is an idempotent open subset of $\mathcal{X}$.

Proof. The implication (b)$\Longrightarrow$(a) is trivial. Suppose (a) is true. Then the open affine subscheme $\mathcal{U}$ is a closed subset and, by Proposition 4.3.2(iv) it is the preimage of an open affine subscheme of $\mathrm{Fspec}(I_A)$. This implies that $\mathcal{U}$ is an idempotent open subset. ∎

For an open affine subscheme $\mathcal{U} \subset \mathcal{X} = \mathrm{Fspec}(A)$, let $A_{(\mathcal{U})}$ denote the localization of A with respect to the monoid of elements of A that do not vanish at any point of $\mathcal{U}$. It is clear that the canonical homomorphism $A \to A_{\mathcal{U}}$ goes through a homomorphism $A_{(\mathcal{U})} \to A_{\mathcal{U}}$.

4.4.4. Corollary. *The following properties of an open affine subscheme $\mathcal{U} \subset \mathcal{X}$ are equivalent:*

(a) the homomorphism $A_{(\mathcal{U})} \to A_{\mathcal{U}}$ is surjective;

(b) $\mathcal{U}$ is an idempotent open subset of a principal open subset of $\mathcal{X}$.

Proof. The implication (b)$\Longrightarrow$(a) is trivial. Suppose (a) is true, and let $D(f)$ be the minimal principal open subset of $\mathcal{X}$ that contains $\mathcal{U}$. Replacing $\mathcal{X}$ by $D(f)$, we may assume that $\mathcal{U}$ contains the minimal connected component of $\mathcal{X}$ and, in particular, $\mathcal{X}_{\mathbf{m}} \subset \mathcal{U}$. The latter implies that an element of A that does not vanish at any point of $\mathcal{U}$ is invertible, i.e., $A \xrightarrow{\sim} A_{(\mathcal{U})}$. Corollary 4.4.3 now implies that $\mathcal{U}$ is an idempotent open subset of $\mathcal{X}$. ∎

Notice that, if $\mathcal{X}$ is irreducible, every open affine subscheme is a principal open subset. There is a broader class of affine schemes which possess the latter property.

4.4.5. Definition. An affine scheme $\mathcal{X} = \mathrm{Spec}(A)$ (or the $\mathbf{F}_1$-algebra A) is said to be *quasi-irreducible* if, for every Zariski prime ideal $\mathfrak{p} \subset A$ such that $\mathrm{Fspec}(A/\Pi_{\mathfrak{p}})$ is an irreducible component of $\mathcal{X}$, $A/\mathfrak{p}$ is an integral domain (and, in particular, $A/\mathfrak{p} = A/\Pi_{\mathfrak{p}}$).

For example, if A is the quotient of an integral $\mathbf{F}_1$-algebra by a Zariski ideal, then $\mathcal{X}$ is quasi-irreducible. Notice that, if $\mathcal{X}$ is quasi-irreducible, then the quotient $A/\mathfrak{p}$ is an integral domain for all Zariski prime ideals $\mathfrak{p} \subset A$.

4.4.6. Proposition. *Suppose that $\mathcal{X}$ is quasi-irreducible. Then*

(i) every open affine subscheme of $\mathcal{X}$ is a connected principal open subset;

(ii) if B is a finitely generated A-algebra such that the homomorphism $A \to B$ is injective, then $A_{\mathfrak{p}} = A_f$ for $\mathfrak{p} = \mathbf{m}_B \cap A$ and some $f \in A \backslash \mathfrak{p}$ and the image of $\mathrm{Zspec}(B)$ *in* $\mathrm{Zspec}(A)$ *coincides with* $\mathrm{Zspec}(A_{\mathfrak{p}})$*;*

(iii) if, in addition to (ii), $\mathcal{Y} = \mathrm{Fspec}(B)$ is also quasi-irreducible, then the image of $\mathcal{Y}$ in $\mathcal{X}$ coincides with $D(f)$.

Proof. (i) It suffices to show that every nonempty principal open subset $D(f)$ is connected. Let $\mathfrak{p}$ be the maximal Zariski prime ideal that does not contain any powers of f. Then $D(f) \cap \mathcal{X}^{(\mathfrak{p})}$ is nonempty and connected. Let x be a point from $D(f)$ over a Zariski prime ideal $\mathfrak{q}$. Then $\mathfrak{q}$ does not contain any powers of f and, therefore, $\mathfrak{q} \subset \mathfrak{p}$. Since $A/\mathfrak{q}$ is an integral domain, then $\mathcal{X}^{(\mathfrak{p})} \subset \mathcal{X}^{(\mathfrak{q})}$ and $D(f) \cap \mathcal{X}^{(\mathfrak{q})}$ is connected and, therefore, the point x lies in the same connected component as $D(f) \cap \mathcal{X}^{(\mathfrak{p})}$. Thus, $D(f)$ is connected.

(ii) If $\mathfrak{p} = \mathbf{m}_B \cap A$, all elements from $A \backslash \mathfrak{p}$ are invertible in B. Since $A/\mathfrak{p}$ is an integral domain and the homomorphism $A \to B$ is injective, the induced homomorphisms of groups $A^* \to \kappa(\mathfrak{p})^* \to B^*$ are injective. It follows that the quotient group $\kappa(\mathfrak{p})^*/A^*$ is finitely generated. If $f_1, \dots, f_n$ are elements from $A/\backslash \mathfrak{p}$ whose images generate the group $\kappa(\mathfrak{p})^*/A^*$, then $A_{\mathfrak{p}} = A_f$ for $f = f_1 \cdot \ldots \cdot f_n$ and, in particular, $D(f) = \mathrm{Fspec}(A_{\mathfrak{p}})$. *We claim that, for every Zariski prime ideal of A that lies in $\mathfrak{p}$, there exists a Zariski prime $\mathfrak{q} \subset B$ with $\mathfrak{q} \cap A = \mathfrak{p}$.* Indeed, we can replace A by $A/\mathfrak{p}$ and B by $B/\mathfrak{p}B$ and assume that A is an integral domain and $\mathfrak{p} = 0$. If K is the fraction $\mathbf{F}_1$-field of A, B embeds in the finitely generated K-algebra $B \otimes_A K$. The latter has finitely many Zariski prime ideals. If $\mathfrak{q}_1, \dots, \mathfrak{q}_n$ are their preimages in B, the intersection $\bigcap_{i=1}^n \mathfrak{q}_i$ is the Zariski nilradical of B. It follows that $\bigcap_{i=1}^n (\mathfrak{q} \cap A) = 0$ and, therefore, $\mathfrak{q}_i \cap A = 0$ for some $1 \le i \le n$.

(iii) By (ii), it suffices to verify that, if $\mathfrak{q} \cap A = \mathfrak{p}$ for Zariski prime ideals $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$, then the image of $\check{\mathcal{Y}}_{\mathfrak{q}} = \mathrm{Fspec}(\kappa(\mathfrak{q})^*)$ in $\mathcal{X} = \mathrm{Fspec}(\kappa(\mathfrak{p})^*)$ coincides with $\check{\mathcal{X}}_{\mathfrak{p}}$. Since the quotients $A/\mathfrak{p}$ and $B/\mathfrak{q}$ are integral domains and the homomorphism $A/\mathfrak{p} \to B/\mathfrak{q}$ is injective, the induced homomorphism of groups $\kappa(\mathfrak{p})^* \to \kappa(\mathfrak{q})^*$ is injective. This implies the required fact. ∎

Let M be an A-module. For an open affine subscheme $\mathcal{U} \subset \mathcal{X}$, we set $M_{\mathcal{U}} = M \otimes_A A_{\mathcal{U}}$ and, for a covering of $\mathcal{X}$ by open affine subschemes $\mathfrak{U} = \{\mathcal{U}_i\}_{i \in I}$, we set

$$M_{\mathfrak{U}} = \mathrm{Ker}(\prod_{i \in I} M_{\mathcal{U}_i} \rightrightarrows \prod_{i,j \in I} M_{\mathcal{U}_i \cap \mathcal{U}_j}) \ .$$

Finally, let $\langle M \rangle$ denote the filtered inductive limit $\varinjlim M_{\mathfrak{U}}$ taken over all coverings $\mathfrak{U}$ of $\mathcal{X}$ by open affine subschemes.

4.4.7. Proposition. *For any A-module M and any open covering $\mathfrak{U}$ of $\mathcal{X}$ by open affine* *subschemes, the canonical homomorphism $M \to M_{\mathfrak{U}}$ is injective and, therefore, the canonical homomorphism $M \to \langle M \rangle$ is injective.*

Proof. By Proposition 4.2.6, we may assume that $\mathfrak{U}$ is an elementary family associated to a finite idempotent $\mathbf{F}_1$-subalgebra I of A. In this case, the sets from $\mathfrak{U}$ are pairwise disjoint, and they correspond to elements of $\check{I}$. For $e \in \check{I}$, one has $M^{(e)} = M/F_e$, where F_e is the I-submodule of M generated by the prime ideal Π_e. The required injectivity of the homomorphism $M \to \prod_{e \in \check{I}} M^{(e)}$ follows from Lemma 1.6.1. ∎

4.4.8. Proposition. *Let B be a commutative ring with unity. Given a homomorphism of $\mathbf{F}_1$-algebras $A \to B^{\cdot}$, the following is true:*

(i) the preimage of any open affine subscheme $\mathcal{U}$ of $\mathcal{X} = \mathrm{Fspec}(A)$ with respect to the induced map $\varphi : \mathcal{Y} = \mathrm{Spec}(B) \to \mathcal{X}$ is an open affine subscheme of $\mathcal{Y}$;

(ii) if $\varphi(\mathcal{Y}) \subset \mathcal{U}$, then the homomorphism $A \to B^{\cdot}$ goes through a unique homomorphism $A_{\mathcal{U}} \to B^{\cdot}$; in particular, the image of the map $\mathrm{Fspec}(B^{\cdot}) \to \mathcal{X}$ also lies in $\mathcal{U}$.

Proof. Both statements are trivial if $\mathcal{U}$ is a principal or idempotent open subset, and so they are true for elementary open subsets. In the general case, let S be an elementary family of open subsets that cover $\mathcal{U}$. Then for each $\mathcal{V} \in S$ its preimage is an open affine subscheme of $\mathcal{Y}$. Let T denote the subset of $\mathcal{V} \in S$ with $\varphi^{-1}(\mathcal{V}) \neq \emptyset$. Then the disjoint union $\mathcal{W} = \coprod_{\mathcal{V} \in T} \varphi^{-1}(\mathcal{V})$, which is the preimage of $\mathcal{U}$, is an open affine subscheme of $\mathcal{Y}$, i.e., (i) is true. Suppose that $\varphi(Y) \subset \mathcal{U}$. Then $\mathcal{W} = \mathcal{Y}$ and, therefore, $B \xrightarrow{\sim} \prod_{\mathcal{V} \in T} B_{\varphi^{-1}(\mathcal{V})}$. Every homomorphism $A \to B \to (B_{\varphi^{-1}(\mathcal{V})})^{\cdot}$ goes through a unique homomorphism $A_{\mathcal{V}} \to (B_{\varphi^{-1}(\mathcal{V})})^{\cdot}$. They induce a homomorphism $\prod_{\mathcal{V} \in S} A_{\mathcal{V}} \to B^{\cdot} = \prod_{\mathcal{V} \in T} (B_{\varphi^{-1}(\mathcal{V})})^{\cdot}$. The required homomorphism $A_{\mathcal{U}} \to B^{\cdot}$ is the composition of the latter with the canonical embedding $A_{\mathcal{U}} = \coprod_S^{\nu} A_{\mathcal{V}} \hookrightarrow \prod_{\mathcal{V} \in S} A_{\mathcal{V}}$. ∎

4.4.9. Corollary. *In the situation of Proposition 4.4.8, the homomorphism $A \to B^{\cdot}$ extends in a canonical way to a homomorphism $\langle A \rangle \to B^{\cdot}$.*

Proof. Let $\mathfrak{U} = \{\mathcal{U}_i\}_{i \in I}$ be a covering of $\mathcal{X}$ by open affine subschemes. By Proposition 4.4.8, it gives rise to a covering of $\mathcal{Y}$ by open affine subschemes $\mathfrak{V} = \{\mathcal{V}_i\}_{i \in I}$ with $\mathfrak{V}_i = \varphi^{-1}(\mathcal{U}_i)$. It follows also that the homomorphism $A \to B^{\cdot}$ extends in a canonical way to compatible homomorphisms $A_{\mathcal{U}_i} \to B^{\cdot}_{\mathcal{V}_i}$ and $A_{\mathcal{U}_i \cap \mathcal{U}_j} \to B^{\cdot}_{\mathcal{V}_i \cap \mathcal{V}_j}$. Since $B \xrightarrow{\sim} \mathrm{Ker}(\prod_{i \in I} B_{\mathcal{V}_i} \rightrightarrows \prod_{i,j \in I} B^{\cdot}_{\mathcal{V}_i \cap \mathcal{V}_j})$, it follows that the homomorphism $A \to B^{\cdot}$ extends in a canonical way to a homomorphism $A_{\mathfrak{U}} \to B^{\cdot}$. ∎

4.5. Open and closed immersions and finite morphisms. Let $\varphi : \mathcal{Y} = \mathrm{Fspec}(B) \to \mathcal{X} = \mathrm{Fspec}(A)$ be a morphism of affine schemes over $\mathbf{F}_1$.

4.5.1. Definition. φ is said to be an *open immersion* if it induces an isomorphism between $\mathcal{Y}$ and an open affine subscheme of $\mathcal{X}$.

4.5.2. Proposition. *The following properties of φ are equivalent:*

(a) φ is an open immersion;

(b) there is a covering of $\mathcal{X}$ by open affine subschemes $\{\mathcal{X}_i\}$ such that all of the induced morphisms $\varphi^{-1}(\mathcal{X}_i) \to \mathcal{X}_i$ are open immersions;

(c) φ is injective (as a map), and there is a covering of $\mathcal{Y}$ by open affine subschemes $\{\mathcal{Y}_i\}$ such that all of the induced morphisms $\mathcal{Y}_i \to \mathcal{X}$ are open immersions.

Proof. The implication (a)$\Longrightarrow$(b)$\Longrightarrow$(c) are trivial.

(c)$\Longrightarrow$(a). Let $\mathcal{U}_i$ be the open affine subscheme of $\mathcal{X}$ which is the image of $\mathcal{Y}_i$. It is covered by elementary open subsets of $\mathcal{X}$. It follows that we can replace the covering $\{\mathcal{Y}_i\}$ by a refinement that consists of elementary open subsets and assume that all $\mathcal{U}_i$'s are elementary open subsets of $\mathcal{X}$. Furthermore, by Proposition 4.2.6, we may assume that $\{\mathcal{Y}_i\}$ is an elementary family of open subsets of $\mathcal{Y}$. *We then claim that $\{\mathcal{U}_i\}$ is an elementary family of open subsets of $\mathcal{X}$.* Indeed, the assumptions on φ imply that the map $\mathrm{Zspec}(B) \to \mathrm{Zspec}(A)$. It follows that $\mathfrak{p}^{(\mathcal{U}_i)} \le \mathfrak{p}^{(\mathcal{U}_j)}$ if and only if $\mathfrak{p}^{(\mathcal{Y}_i)} \le \mathfrak{p}^{(\mathcal{Y}_j)}$ and, therefore, the partial orderings on the families $\{\mathcal{U}_i\}$ and $\{\mathcal{Y}_i\}$ coincide. Suppose that $\mathcal{U}_i = \inf(\mathcal{U}_j, \mathcal{U}_k)$ and $e \in \mathbf{a}_{\mathcal{U}_i}$. Then e vanishes at $\mathcal{Y}_i$ and, therefore, it vanishes either at $\mathcal{Y}_j$, or at $\mathcal{Y}_k$. It follows that e vanishes at $\mathcal{U}_j$, or at $\mathcal{U}_k$, and we get the claim. The claim implies that $\mathcal{U} = \bigcup_i \mathcal{U}_i$ is an open affine subscheme of $\mathcal{X}$ and $\mathcal{Y} \stackrel{\sim}{\to} \mathcal{U}$. ∎

We will denote by $\mathcal{A}sch^{oi}_{\mathbf{F}_1}$ the category in which the family of objects coincide with that of $\mathcal{A}sch_{\mathbf{F}_1}$ and morphisms are open immersions.

4.5.3. Definition. φ is said to be a *closed* (resp. *Zariski closed*) *immersion* if the homomorphism $A \to B$ is surjective (resp. and its kernel coincides with Zariski kernel).

For example, Proposition 4.4.3 implies that, for an open affine subscheme $\mathcal{U} \subset \mathcal{X}$, the canonical morphism $\mathcal{U} \to \mathcal{X}$ is a closed (resp. Zariski closed) immersion if and only if $\mathcal{U}$ is an idempotent open subset (resp. which is defined by the equations $e_1 = \ldots = e_n = 0$ for $e_1, \ldots, e_n \in I_A$).

4.5.4. Proposition. *Given a covering of $\mathcal{X}$ by open affine subschemes $\{\mathcal{X}_i\}$, the following properties of φ are equivalent:*

(a) φ is a closed immersion;

(b) all of the induced morphisms $\varphi^{-1}(\mathcal{X}_i) \to \mathcal{X}_i$ are closed immersions.

Proof. The implication (a)$\Longrightarrow$(b) is trivial. Suppose that (b) is true. By Proposition 4.2.6, we may assume that $\{\mathcal{X}_i\}$ is an elementary family associated to a finite idempotent $\mathbf{F}_1$-subalgebra I of A and, in particular, the set of indices is $\check{I}$. Let I' be the image of I in B. The surjection $I \to I'$ induces a map of posets $\check{I}' \to \check{I} : i' \mapsto i$ (see §3.4). By the assumption, for every $i' \in \check{I}'$ the homomorphism $A_i \to B_{i'}$ is surjective, where $A_i = A_{\mathcal{X}_i}$ and $B_{i'} = B_{\varphi^{-1}(\mathcal{X}_i)}$. Since $A = \coprod_{\check{I}}^{\nu} A_i$ and $B = \coprod_{\check{I}'}^{\nu} B_{i'}$, it follows that the homomorphism $A \to B$ is surjective. ∎

Notice that the implication (b)$\Longrightarrow$(a) in Proposition 4.5.4 does not hold in general for Zariski closed immersions.

4.5.5. Definition. (i) φ is said to be *finite* if B is a finite A-module;

(ii) φ is said to be *of finite type* if B is a finitely generated A-algebra.

For example, any closed immersion is a finite morphism, and any finite morphism is of finite type.

4.5.6. Proposition. *Given a covering of $\mathcal{X}$ by open affine subschemes $\{\mathcal{X}_i\}$, the following properties of φ are equivalent:*

(a) φ is a finite morphism (resp. of finite type);

(b) all of the induced morphisms $\varphi^{-1}(\mathcal{X}_i) \to \mathcal{X}_i$ are finite morphisms (resp. of finite type).

Proof. The implication (a)$\Longrightarrow$(b) is trivial. Suppose that (b) is true. As above, we may assume that $\{\mathcal{X}_i\}$ is an elementary family associated to a finite idempotent $\mathbf{F}_1$-subalgebra I of I_A. In the notation from the proof of Proposition 4.5.4, the assumption implies that, for every $i' \in \check{I}'$, $B_{i'}$ is a finite A_i-module (resp. a finitely generated A-algebra). Since the images of generators of all $B_{i'}$'s in B generate the A-module (resp. the A-algebra) B, the required fact follows. ∎

4.6. Piecewise affine schemes. Let $\mathcal{X} = \mathrm{Fspec}(A)$ and $\mathcal{Y} = \mathrm{Fspec}(B)$ be affine schemes. For a covering $\mathfrak{V} = \{\mathcal{V}_i\}_{i\in I}$ of $\mathcal{Y}$ by open affine subschemes, we set

$$\mathrm{Hom}_{\mathfrak{V}}(\mathcal{Y},\mathcal{X}) = \mathrm{Ker}(\prod_{i\in I}\mathrm{Hom}(\mathcal{V}_i,\mathcal{X}) \rightrightarrows \prod_{i,j\in I}\mathrm{Hom}(\mathcal{V}_i\cap\mathcal{V}_j,\mathcal{X}))\ .$$

One has $\mathrm{Hom}_{\mathfrak{V}}(\mathcal{Y},\mathcal{X}) = \mathrm{Hom}(A, B_{\mathfrak{V}})$. Furthermore, we set

$$\mathrm{Hom}^p(\mathcal{Y},\mathcal{X}) = \varinjlim \mathrm{Hom}_{\mathfrak{V}}(\mathcal{Y},\mathcal{X})\ ,$$

where the inductive limit is taken over coverings $\mathfrak{V}$ of $\mathcal{Y}$ by open affine subschemes. (By Proposition 4.4.7, all transition maps in this inductive limit are injective.) Recall that every covering has a finite refinement consisting of pairwise disjoint open affine subschemes. For such $\mathfrak{V} = \{\mathcal{V}_i\}_{i\in I}$, an element of $\mathrm{Hom}_{\mathfrak{Y}}(\mathcal{Y},\mathcal{X})$ is just a a system of morphisms of affine schemes $\mathcal{V}_i \to \mathcal{X}$. Of course, if $\mathcal{Y}$ is connected, then $\mathrm{Hom}(\mathcal{Y},\mathcal{X}) \xrightarrow{\sim} \mathrm{Hom}^p(\mathcal{Y},\mathcal{X})$.

Elements of $\mathrm{Hom}^p(\mathcal{Y},\mathcal{X})$ are said to be *p-morphisms* from $\mathcal{Y}$ to $\mathcal{X}$. A p-morphism from $\mathcal{Y}$ to $\mathcal{X}$ represented by a system of compatible morphisms $\varphi_i : \mathcal{V}_i \to \mathcal{X}$ defines a continuous map $\varphi : Y \to X$ and, for every point $y \in \mathcal{Y}$, there is a well defined embedding of $\mathbf{F}_1$-fields $\kappa(x) \to \kappa(y)$, where $x = \varphi(y)$. Furthermore, given a second p-morphism $\psi : \mathcal{Z} = \mathrm{Fspec}(C) \to \mathcal{Y}$ represented by a system of compatible morphisms $\psi_j : \mathcal{W}_j \to \mathcal{Y}$, there is a well defined composition p-morphism which is represented by the system of compatible morphisms $\varphi^{-1}(\mathcal{V}_i) \cap \mathcal{W}_j \xrightarrow{\psi_j} \mathcal{V}_i \xrightarrow{\varphi_i} \mathcal{X}$. This means that there is a well defined category $\mathcal{A}sch^p_{\mathbf{F}_1}$ whose family of objects coincides with that of $\mathcal{A}sch_{\mathbf{F}_1}$, and in which the set of morphisms from $\mathcal{Y}$ to $\mathcal{X}$ is the set of p-morphisms of affine schemes $\mathrm{Hom}^p(\mathcal{Y},\mathcal{X})$. The canonical functor $\mathcal{A}sch_{\mathbf{F}_1} \to \mathcal{A}sch^p_{\mathbf{F}_1}$ is faithful and even fully faithful on the full subcategory of connected affine schemes, but not fully faithful on the whole category. For example, if I and J are finite idempotent $\mathbf{F}_1$-algebras with the same number of elements, then the affine schemes $\mathrm{Fspec}(I)$ and $\mathrm{Fspec}(J)$ are isomorphic in $\mathcal{A}sch^p_{\mathbf{F}_1}$ although they are not necessarily isomorphic in $\mathcal{A}sch_{\mathbf{F}_1}$.

4.6.1. Proposition. *The category $\mathcal{A}sch^p_{\mathbf{F}_1}$ admits finite coproducts and fiber products.*

Proof. Let $\{\mathcal{X}_i = \mathrm{Fspec}(A_i)\}$ be a finite family of affine schemes over $\mathbf{F}_1$. Take an arbitrary idempotent $\mathbf{F}_1$-algebra I whose poset of nonzero elements $\check{I}$ can be identified with the set of indices of that family. For $i \le j$ in $\check{I}$, let ν_{ij} be the homomorphism $A_i \to A_j$ defined by $\nu_{ij}(a) = 1$, if $a \in A_i^*$, and $\nu_{ij}(a) = 0$, otherwise. Then the tuple $\{\check{I}, A_i, \nu_{ij}\}$ is a disconnected sum datum and, for $A = \coprod_I^\nu A_i$, the affine scheme $\mathcal{X} = \mathrm{Fspec}(A)$ is the coproduct of the family $\{\mathcal{X}_i\}$ in $\mathcal{A}sch^p_{\mathbf{F}_1}$.

Furthermore, suppose we are given p-morphisms $\mathcal{Y} \to \mathcal{X}$ and $\mathcal{X}' \to \mathcal{X}$. By Proposition 4.2.6, we may assume that these p-morphisms are defined by systems of morphisms $\mathcal{V}_i \to \mathcal{X}$ and $\mathcal{U}'_j \to \mathcal{X}$ for coverings $\{\mathcal{V}_i\}$ of $\mathcal{Y}$ and $\{\mathcal{U}'_j\}$ of $\mathcal{X}'$ by pairwise disjoint elementary open subsets. They define systems of morphisms from $\mathcal{V}'_{ij} = \mathcal{V}_i \times_{\mathcal{X}} \mathcal{U}'_j$ to $\mathcal{Y}$ and $\mathcal{X}'$. *We claim that the coproduct $\mathcal{Y}'$ of the affine schemes $\mathcal{V}'_{ij}$ is a fiber product of the p-morphisms we started from.* Indeed, let $f : \mathcal{T} \to \mathcal{Y}$ and $g : \mathcal{T} \to \mathcal{X}'$ be p-morphisms that induce the same p-morphism $\mathcal{T} \to \mathcal{X}$. Suppose first that f and g are morphisms. Then $\mathcal{W}_{ij} = f^{-1}(\mathcal{V}_j) \cap g^{-1}(\mathcal{U}'_i)$ are pairwise disjoint open affine subschemes of $\mathcal{T}$ that form its covering. Since the morphisms $\mathcal{W}_{ij} \to \mathcal{V}_i$ and $\mathcal{W}_{ij} \to \mathcal{U}'_j$ induce the same morphism $\mathcal{W}_{ij} \to \mathcal{X}$, they give rise a canonical morphism $\mathcal{W}_{ij} \to \mathcal{V}'_{ij} \subset \mathcal{Y}'$. All these morphisms define a p-morphism $\mathcal{T} \to \mathcal{Y}'$. In the general case, we can find a covering $\{\mathcal{T}_k\}$ of $\mathcal{T}$ by pairwise disjoint open affine subschemes such that the restrictions of f and g to every $\mathcal{T}_k$ are morphisms. By the previous case, there are p-morphisms $\mathcal{T}_k \to \mathcal{Y}$ which induce the required p-morphism $\mathcal{T} \to \mathcal{Y}'$. ∎

For an affine scheme $\mathcal{X} = \mathrm{Fspec}(A)$, the category of p-morphisms $\mathcal{Y} \to \mathcal{X}$ in which morphisms are p-morphisms commuting with those to $\mathcal{X}$ will be denoted by $\mathcal{A}sch_A^p$.

Let now $\mathcal{X} = \mathrm{Fspec}(A)$ be an affine scheme. For A-modules M and N, we set

$$\mathrm{Hom}_A^p(M, N) = \varinjlim \mathrm{Hom}_A(M, N_{\mathfrak{U}}) \ ,$$

where the limit is taken over coverings $\mathfrak{U}$ by open affine subschemes. If M is a finite A-module, then the latter limit coincides with $\mathrm{Hom}_A(M, \langle N \rangle)$. If $\mathcal{X}$ is connected, then $\mathrm{Hom}_A^p(M, N) = \mathrm{Hom}_A(M, N)$. Elements of $\mathrm{Hom}_A^p(M, N)$ are said to be *p-homomorphisms*. Notice that a p-homomorphism from M to N is just a family of homomorphisms of $A_{\mathcal{U}_i}$-modules $M_{\mathcal{U}_i} \to N_{\mathcal{U}_i}$ for a finite covering of $\mathcal{X}$ by pairwise disjoint open affine subschemes $\{\mathcal{U}_i\}_{i \in I}$. It follows that one can compose p-homomorphisms and, therefore, there is a well defined category $A\text{-Mod}^p$ whose family of objects coincides with that of A-Mod and in which the set of morphisms from M to N is the set of p-homomorphisms $\mathrm{Hom}^p(M, N)$.

If all of the A-modules in the above definitions are in fact A-algebras and all homomorphisms between them commute with multiplication, we get a category $A\text{-Alg}^p$. The correspondence $B \mapsto \mathrm{Fspec}(B)$ gives rise to a contravariant fully faithful functor $A\text{-Alg}^p \to \mathcal{A}sch_A^p$.

4.6.2. Definition. A p-morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ is said to be a *p-open immersion* if it is injective (as a map) and there is a covering of $\mathcal{Y}$ by open affine subschemes $\{\mathcal{Y}_i\}$ such that φ induces open immersions of affine schemes $\mathcal{Y}_i \to \mathcal{X}$.

It follows easily from the definition that, given a p-open immersion $\varphi : \mathcal{Y} \to \mathcal{X}$, any p-morphism $\psi : \mathcal{Z} \to \mathcal{X}$ with $\psi(\mathcal{Z}) \subset \varphi(\mathcal{Y})$ goes through a unique p-morphism $\mathcal{Z} \to \mathcal{Y}$. In particular, the set $\varphi(Y)$ defines the morphism φ uniquely up to a unique isomorphism in $\mathcal{A}sch_{\mathbf{F}_1}^p$. Such a subset of $\mathcal{X}$ is said to be an *open p-affine subscheme.*

4.6.3. Lemma. *Let $\mathcal{X} = \mathrm{Fspec}(A)$ be an affine scheme over $\mathbf{F}_1$. Then*

(i) a subset of $\mathcal{X}$ is an open p-affine subscheme if and only if it is a disjoint union of elementary open subsets;

(ii) the class of open p-affine subschemes of $\mathcal{X}$ is preserved under finite intersection;

(iii) given a p-morphism $\varphi : \mathcal{Y} \to \mathcal{X}$, the preimage of an open p-affine subscheme of $\mathcal{X}$ is a open p-affine subscheme of $\mathcal{Y}$.

Proof. (i) Suppose that a p-open immersion $\varphi : \mathcal{Y} \to \mathcal{X}$ is represented by a compatible system of morphisms $\mathcal{V}_i \to \mathcal{X}$ for elementary open subsets $\mathcal{V}_i \subset \mathcal{X}$. By Proposition 4.2.6, we may assume that $\{\mathcal{V}_i\}$ is a finite system of pairwise disjoint elementary open subsets of $\mathcal{Y}$. This implies that $\varphi(\mathcal{Y})$ is a disjoint union of elementary open subsets of $\mathcal{X}$. The converse implication follows from Proposition 4.6.1.

The statements (ii) and (iii) follow from (i). ∎

Notice that, by Proposition 4.5.2, any morphism of affine schemes which is a p-open immersion is an open immersion and, in particular, the functor $\mathcal{A}sch_{\mathbf{F}_1} \to \mathcal{A}sch^p_{\mathbf{F}_1}$ is conservative (i.e., any morphism in the first category, which becomes an isomorphism in the second one, is an isomorphism). Furthermore, two affine schemes $\mathcal{X}$ and $\mathcal{Y}$ are isomorphic in $\mathcal{A}sch^p_{\mathbf{F}_1}$ if and only if there exist finite coverings $\{\mathcal{U}_i\}_{1\le i\le n}$ of $\mathcal{X}$ and $\{\mathcal{V}_i\}_{1\le i\le n}$ by pairwise disjoint open affine subschemes such that each $\mathcal{U}_i$ is isomorphic to $\mathcal{V}_i$ in $\mathcal{A}sch_{\mathbf{F}_1}$. It is also easy to see that the class of p-open immersions is preserved by compositions, and so there is a category $\mathcal{A}sch^{poi}_{\mathbf{F}_1}$ the category whose objects are affine schemes and morphisms are p-open immersions.

4.6.4. Definition. A p-morphism of affine schemes $\varphi : \mathcal{Y} \to \mathcal{X}$ is said to be a *p-closed* (resp. *Zariski p-closed*) *immersion* if it is an injective map and there exists a covering $\{\mathcal{U}_i\}_{i\in I}$ of $\mathcal{X}$ by open affine subschemes such that, for every $i \in I$, $\varphi^{-1}(\mathcal{U}_i)$ is a finite disjoint union of its open affine subschemes $\{\mathcal{V}_{ij}\}_{j\in J_i}$ and, for every $j \in J_i$, $\mathcal{V}_{ij} \to \mathcal{U}_i$ is a closed (resp. Zariski closed) immersion of affine schemes.

Notice that, since the intersection of nonempty subsets of the form $V(\mathbf{a})$ in $\mathcal{X}$ is nonempty, it follows that, for a Zariski closed immersion as in Definition 4.6.4, all $\varphi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i$ are Zariski closed immersions of affine schemes.

4.6.5. Lemma. *A morphism of affine schemes $\varphi : \mathcal{Y} \to \mathcal{X}$ is a p-closed immersion if and only if it is a closed immersion.*

Proof. The converse implication is trivial. Suppose that φ is a p-closed immersion. By Proposition 4.5.4, we may assume that $\mathcal{Y} = \mathrm{Fspec}(B)$ is a disjoint union of open affine subschemes $\{\mathcal{V}_i\}_{i\in I}$ such that each $\mathcal{V}_i \to \mathcal{X} = \mathrm{Fspec}(A)$ is a closed immersion of affine schemes. Furthermore, by Proposition 4.2.6, we may assume that the above covering of $\mathcal{Y}$ is an elementary family associated to a finite idempotent $\mathbf{F}_1$-subalgebra of B. Then B is a disconnected sum $\coprod_I^\nu B_{\mathcal{V}_i}$. Since all of the homomorphisms $A \to B_{\mathcal{V}_i}$ are surjective, it follows that the homomorphism $A \to B$ is surjective. ∎

4.6.6. Definition. A p-morphism of affine schemes $\varphi : \mathcal{Y} \to \mathcal{X}$ is said to be *p-finite* (resp. *of p-finite type*) if there exists a covering $\{\mathcal{U}_i\}_{i\in I}$ of $\mathcal{X}$ by open affine subschemes such that, for every $i \in I$, $\varphi^{-1}(\mathcal{U}_i)$ is a (finite) disjoint union of its open affine subschemes $\{\mathcal{V}_{ij}\}_{j\in J_i}$ and, for every

$j \in J_i$, $\mathcal{V}_{ij} \to \mathcal{U}_i$ is a morphism of affine schemes which is finite (resp. of finite type).

4.6.7. Lemma. *A morphism of affine schemes $\varphi : \mathcal{Y} \to \mathcal{X}$ is p-finite (resp. of p-finite type) if and only if it is a finite morphism (resp. of finite type).*

Proof. The converse implication is trivial. Suppose that the morphism φ is p-finite (resp. of p-finite type). By Proposition 4.5.6, we may assume that $\mathcal{Y} = \mathrm{Fspec}(B)$ is a finite disjoint union of open affine subschemes $\{\mathcal{V}_i\}$ such that all of the morphism $\mathcal{V}_i \to \mathcal{X}$ are finite (resp. of finite type). Furthermore, by Proposition 4.2.6, we can find a refinement $\{\mathcal{W}_j\}$ of the above covering which is an elementary family associated to a finite idempotent $\mathbf{F}_1$-subalgebra of A. Since every $\mathcal{W}_j$ lies in some $\mathcal{V}_i$ and is an open-closed subset of $\mathcal{V}_i$, it follows that the canonical homomorphism $A_{\mathcal{V}_i} \to \mathcal{A}_{\mathcal{W}_j}$ is surjective. This reduces the situation to the case that $\{\mathcal{V}_i\}$ is an elementary family associated to a finite idempotent $\mathbf{F}_1$-subalgebra of A. In this case, the required property of the homomorphism $A \to B$ follows from Proposition 3.4.8. ∎

4.6.8. Lemma. *Given a covering of $\mathcal{Y}$ by open p-affine subschemes $\{\mathcal{V}_i\}_{i \in I}$, the following sequence of maps of sets is exact*

$$\mathrm{Hom}^p(\mathcal{Y}, \mathcal{X}) \to \prod_{i \in I} \mathrm{Hom}^p(\mathcal{Y}_i, \mathcal{X}) \rightrightarrows \prod_{i,j \in I} \mathrm{Hom}^p(\mathcal{Y}_i \cap \mathcal{Y}_j, \mathcal{X}) \ . \qquad ∎$$

§5. Schemes over $\mathbf{F}_1$

5.1. The category of schemes $\mathcal{S}ch_{\mathbf{F}_1}$. Let X be a topological space, and let τ be a collection of subsets. (All subsets are provided with the induced topology.) Recall (see [Ber1, §1.1]) that τ is said to be a *quasinet* on X if, for each point $x \in X$, there exist $V_1, \dots, V_n \in \tau$ such that $x \in V_1 \cap \ldots \cap V_n$ and the set $V_1 \cup \ldots \cup V_n$ is a neighborhood of x. If τ is a quasinet, then a subset $\mathcal{U}$ is open in X if and only if for each $V \in \tau$ the intersection $\mathcal{U} \cap V$ is open in V (see [Ber1, Lemma 1.1.1(i)]). The collection τ is said to be a *net* if it is quasinet and, for every pair $U, V \in \tau$, $\tau\big|_{U \cap V}$ is a quasinet on $U \cap V$. Notice that, if all sets from τ are open, then the property to be a quasinet means that τ is an open covering, and the property to be a net means that τ is a base of a topology (which is weaker than or coincides with the topology on X). In what follows we consider a quasinet (or net) τ as a category (its objects are sets from τ and morphisms are inclusion maps), and we denote by $\mathcal{T}$ the canonical functor $\tau \to \mathcal{T}op$ to the category of topological spaces $\mathcal{T}op$.

Let $\mathcal{T}^a$ denote the forgetful functor $\mathcal{A}sch^{poi}_{\mathbf{F}_1} \to \mathcal{T}op$ that takes an affine scheme to the underlying topological space.

5.1.1. Definition. A *scheme over* $\mathbf{F}_1$ is a triple $(\mathcal{X}, A, \tau)$, where $\mathcal{X}$ is a topological space, τ is a net of open subsets of $\mathcal{X}$, and A is a *p-affine atlas on $\mathcal{X}$ with the net* τ, i.e., a pair consisting of a functor $A : \tau \to \mathcal{A}sch^{poi}_{\mathbf{F}_1}$ and an isomorphism of functors $\mathcal{T}^a \circ A \stackrel{\sim}{\to} \mathcal{T}$.

Let $(\mathcal{X}, A, \tau)$ be a K-analytic space. The functor A takes each $\mathcal{U} \in \tau$ to an affine scheme $\mathrm{Fspec}(A_{\mathcal{U}})$, and the isomorphism of functors provides a homeomorphism $\mathrm{Fspec}(A_{\mathcal{U}}) \stackrel{\sim}{\to} \mathcal{U}$. We consider such $\mathcal{U}$ as an object of the category $\mathcal{A}sch^p_{\mathbf{F}_1}$.

5.1.2. Definition. A *strong morphism* $\varphi : (\mathcal{X}, A, \tau) \to (\mathcal{X}', A', \tau')$ is a pair consisting of a continuous map $\varphi : \mathcal{X} \to \mathcal{X}'$ such that for every $\mathcal{U} \in \tau$ there exists $\mathcal{U}' \in \tau'$ with $\varphi(\mathcal{U}) \subset \mathcal{U}'$, and a compatible system of p-morphisms of affines schemes $\varphi_{\mathcal{U}/\mathcal{U}'} : \mathcal{U} \to \mathcal{U}'$ with $\varphi_{\mathcal{U}/\mathcal{U}'} = \varphi\big|_{\mathcal{U}}$ (as maps) for all pairs $\mathcal{U} \in \tau$ and $\mathcal{U}' \in \tau'$ with $\varphi(\mathcal{U}) \subset \mathcal{U}'$.

5.1.3. Lemma. *For any pair of strong morphisms $\varphi : (\mathcal{X}, A, \tau) \to (\mathcal{X}', A', \tau')$ and $\psi : (\mathcal{X}', A', \tau') \to (\mathcal{X}'', A'', \tau'')$, there is a unique morphism $\chi : (\mathcal{X}, A, \tau) \to (\mathcal{X}'', A'', \tau'')$ such that, for every triple $\mathcal{U} \in \tau$, $\mathcal{U}' \in \tau'$ and $\mathcal{U}'' \in \tau''$ with $\varphi(\mathcal{U}) \subset \mathcal{U}'$ and $\psi(\mathcal{U}') \subset \mathcal{U}''$, one has $\chi_{\mathcal{U}/\mathcal{U}''} = \psi_{\mathcal{U}'/\mathcal{U}''} \circ \varphi_{\mathcal{U}/\mathcal{U}''}$.*

Proof. Let χ be the composition map $\psi\varphi : \mathcal{X} \to \mathcal{X}''$. We have to construct, for every pair $\mathcal{U} \in \tau$ and $\mathcal{U}'' \in \tau''$ with $\chi(\mathcal{U}) \subset \mathcal{U}''$, a p-morphism of affine schemes $\chi_{\mathcal{U}/\mathcal{U}''} : \mathcal{U} \to \mathcal{U}''$. For this we take $\mathcal{U}' \in \tau'$ and $\mathcal{V}'' \in \tau''$ with $\varphi(\mathcal{U}) \subset \mathcal{U}'$ and $\psi(\mathcal{U}') \subset \mathcal{V}''$. Then $\chi(\mathcal{U}) \subset \mathcal{U}'' \cap \mathcal{V}''$. Since τ'' is a net and $\mathcal{U}$ is quasi-compact, one has $\chi(\mathcal{U}) \subset \mathcal{W}_1 \cup \ldots \cup \mathcal{W}_n$ for some $\mathcal{W}_1, \ldots, \mathcal{W}_n \in \tau''\big|_{\mathcal{U}'' \cap \mathcal{V}''}$. Then $\mathcal{U}'_i = \psi^{-1}_{\mathcal{U}'/\mathcal{V}''}(\mathcal{W}_i)$ and $\mathcal{U}_i = \varphi^{-1}_{\mathcal{U}/\mathcal{U}'}(\mathcal{U}'_i)$ are open p-affine subschemes of $\mathcal{U}'$ and $\mathcal{U}$, respectively. The morphisms $\varphi_{\mathcal{U}/\mathcal{U}'}$ and $\psi_{\mathcal{U}'/\mathcal{V}''}$ induce p-morphisms of affine schemes $\mathcal{U}_i \to \mathcal{U}'_i$ and $\mathcal{U}'_i \to \mathcal{W}_i$ and, therefore, the composition p-morphisms $\mathcal{U}_i \to \mathcal{W}_i \to \mathcal{U}''$. Since they are compatible on intersections, they give rise to the required p-morphism $\chi_{\mathcal{U}/\mathcal{U}''}$. ∎

Lemma 5.1.3 implies that the family of schemes with strong morphisms between them forms a category which is denoted by $\widetilde{\mathcal{S}ch}_{\mathbf{F}_1}$.

Definition 5.1.4. (i) A strong morphism $\varphi : (\mathcal{X}, A, \tau) \to (\mathcal{X}', A', \tau')$ is said to be a *quasi-isomorphism* φ induces a homeomorphism of topological spaces $\mathcal{X} \stackrel{\sim}{\to} \mathcal{X}'$ and, for every pair $\mathcal{U} \in \tau$ and $\mathcal{U}' \in \tau'$ with $\varphi(\mathcal{U}) \subset \mathcal{U}'$, $\varphi_{\mathcal{U}/\mathcal{U}'}$ is a p-open immersion of affine schemes.

(ii) The *category $\mathcal{S}ch_{\mathbf{F}_1}$ of schemes over* $\mathbf{F}_1$ is the category of fractions of $\widetilde{\mathcal{S}ch}_{\mathbf{F}_1}$ with respect to the system of quasi-isomorphisms.

We are going to describe morphisms in the category $\mathcal{S}ch_{\mathbf{F}_1}$.

5.1.5. Lemma. *Let $(\mathcal{X}, A, \tau)$ be a scheme over $\mathbf{F}_1$. then* *(i) if $\mathcal{W}$ is an open p-affine subscheme in some $\mathcal{U} \in \tau$, it is an open p-affine subscheme in any $\mathcal{V} \in \tau$ that contains $\mathcal{W}$;*

(ii) the family $\overline{\tau}$ consisting of all $\mathcal{W}$ as above is a net on $\mathcal{X}$, and there exists a unique (up to a canonical isomorphism) p-affine atlas $\overline{A}$ on $\mathcal{X}$ with the net $\overline{\tau}$ which extends A.

Proof. (i) Since τ is a net and $\mathcal{W}$ is quasi-compact, one has $\mathcal{W} \subset \mathcal{U}_1 \cup \ldots \cup \mathcal{U}_n$ for some $\mathcal{U}_1, \ldots, \mathcal{U}_n \in \tau\big|_{\mathcal{U}\cap\mathcal{V}}$. Furthermore, since $\mathcal{W}$ and $\mathcal{U}_i$ are open p-affine subschemes of $\mathcal{U}$, then $\mathcal{W}_i = \mathcal{W} \cap \mathcal{U}_i$ is an open p-affine subscheme of $\mathcal{W}$ and $\mathcal{U}_i$. It follows that each $\mathcal{W}_i$ is an open p-affine subscheme of $\mathcal{V}$ and, therefore, the p-open immersions $\mathcal{W}_i \to \mathcal{V}$ give rise to a p-open immersion $\mathcal{W} \to \mathcal{V}$.

(ii) For $\mathcal{U}, \mathcal{V} \in \overline{\tau}$, take $\mathcal{U}', \mathcal{V}' \in \tau$ with $\mathcal{U} \subset \mathcal{U}'$ and $\mathcal{V} \subset \mathcal{V}'$. Every point $x \in \mathcal{U} \cap \mathcal{V}$ lies in some $\mathcal{W} \in \tau\big|_{\mathcal{U}'\cap\mathcal{V}'}$. Since $\mathcal{U} \cap \mathcal{W}$ and $\mathcal{V} \cap \mathcal{W}$ are open p-affine subschemes of $\mathcal{W}$, it follows that $\mathcal{U} \cap \mathcal{V} \cap \mathcal{W}$ is an open p-affine subscheme of $\mathcal{W}$ and, therefore, $\overline{\tau}$ is a net. Furthermore, for each $\mathcal{U} \in \overline{\tau}$ we fix $\mathcal{U}' \in \tau$ and provide $\mathcal{U}$ is the structure of an affine scheme for which the canonical embedding $\mathcal{U} \to \mathcal{U}'$ is a p-open immersion. The reasoning from the proof of Lemma 5.1.3 that, for any pair $\mathcal{U} \subset \mathcal{V}$ in $\overline{\tau}$, there is a canonical p-open immersion of affine schemes $\mathcal{U} \to \mathcal{V}$, and the required fact follows. ∎

Notice that the canonical strong morphism $(\mathcal{X}, \overline{A}, \overline{\tau}) \to (\mathcal{X}, A, \tau)$ is a quasi-isomorphism, and that any strong morphism $(\mathcal{X}, A, \tau) \to (\mathcal{X}', A', \tau')$ extends in a unique way to a strong morphism $(\mathcal{X}, \overline{A}, \overline{\tau}) \to (\mathcal{X}', \overline{A}', \overline{\tau}')$. Lemma 5.1.5 easily implies that the system of quasi-isomorphisms in $\widetilde{\mathcal{S}ch}_{\mathbf{F}_1}$ admits calculus of right fractions.

Let now $(\mathcal{X}, A, \tau)$ be a scheme over $\mathbf{F}_1$. If σ is a net on $\mathcal{X}$, we write $\sigma \prec \tau$ if $\sigma \subset \overline{\tau}$. The the affine atlas $\overline{A}$ defines an affine atlas A_σ with the net σ, and there is a canonical quasi-isomorphism $(\mathcal{X}, A_\sigma, \sigma) \to (\mathcal{X}, A, \tau)$. The system of nets $\sigma \prec \tau$ is filtered and, for any scheme $(\mathcal{X}', A', \tau')$ over $\mathbf{F}_1$, one has

$$\mathrm{Hom}((\mathcal{X}, A, \tau), (\mathcal{X}', A', \tau')) = \varinjlim_{\sigma \prec \tau} \mathrm{Hom}_{\widetilde{\mathcal{S}ch}}((\mathcal{X}, A_\sigma, \sigma), (\mathcal{X}', A', \tau')) \ .$$

For example, let $\mathcal{X} = \mathrm{Fspec}(A)$ be an affine scheme over $\mathbf{F}_1$. Then $\{\mathcal{X}\}$ is a net on $\mathcal{X}$ and the identity correspondence $\mathcal{X} \mapsto \mathcal{X}$ is an affine atlas. In this way we get a scheme over $\mathbf{F}_1$ denoted by $\mathcal{X}$ and a functor $\mathcal{A}sch_{\mathbf{F}_1} \to \mathcal{S}ch_{\mathbf{F}_1}$. The following statement follows straightforwardly from the above description of morphisms in the category of schemes.

5.1.6. Lemma. *In the above situation, for any scheme $(\mathcal{Y}, B, \sigma)$ morphisms $(\mathcal{Y}, B, \sigma) \to \mathcal{X}$ can be identified with compatible families of p-morphisms of affine schemes $\mathcal{V} \to \mathcal{X}$ with $\mathcal{V} \in \sigma$. In particular, the above functor gives rise to a fully faithful functor $\mathcal{A}sch^p_{\mathbf{F}_1} \to \mathcal{S}ch_{\mathbf{F}_1}$.* ∎

The set of morphisms from $(\mathcal{Y}, B, \sigma)$ to $\mathcal{X}$ to $\mathrm{Fspec}(\mathbf{F}_1[T])$ (the *affine line over* $\mathbf{F}_1$) is denoted by $\mathcal{O}(\mathcal{Y})$. It is an $\mathbf{F}_1$-algebra, and Lemma 5.1.6 means that there is a canonical bijection $\mathrm{Hom}((\mathcal{Y}, B, \sigma), \mathcal{X}) \xrightarrow{\sim} \mathrm{Hom}(A, \mathcal{O}(\mathcal{Y}))$.

5.1.7. Lemma. *A strong morphism $\varphi : (\mathcal{X}, A, \tau) \to (\mathcal{X}', A', \tau')$ becomes an isomorphism in the category $\mathcal{S}ch_{\mathbf{F}_1}$ if and only if it is a quasi-isomorphism.*

Proof. The converse implication is trivial. Suppose that φ is an isomorphism in $\mathcal{S}ch_{\mathbf{F}_1}$. It is clear that φ is a homeomorphism. The assumption implies that one can find nets $\sigma \prec \tau$ and $\sigma' \prec \tau'$ and strong morphisms $\psi : (\mathcal{X}', A'_{\sigma'}, \sigma') \to (\mathcal{X}, A, \tau)$ and $\varphi' : (\mathcal{X}, A_\sigma, \sigma) \to (\mathcal{X}', A'_{\sigma'}, \sigma')$ such that the following diagram is commutative

$$\begin{array}{ccc} (\mathcal{X}, A, \tau) & \xrightarrow{\varphi} & (\mathcal{X}', A', \tau') \\ \uparrow & \nwarrow \psi & \uparrow \\ (\mathcal{X}, A_\sigma, \sigma) & \xrightarrow{\varphi'} & (\mathcal{X}', A'_{\sigma'}, \sigma') \end{array}$$

where the vertical arrows are the canonical quasi-isomorphisms. Let $\mathcal{U} \in \sigma$. We can find $\mathcal{U}' \in \sigma'$, $\mathcal{V} \in \tau$ and $\mathcal{V}' \in \tau'$ with $\varphi'(\mathcal{U}) \subset \mathcal{U}'$, $\psi(\mathcal{U}') \subset \mathcal{V}$ and $\varphi(\mathcal{V}) \subset \mathcal{V}'$. Since $\mathcal{U}$ is an open p-affine subscheme of $\mathcal{V}$, its preimage $\mathcal{U}'' = \psi^{-1}_{\mathcal{U}'/\mathcal{V}}(U)$ is an open p-affine subscheme of $\mathcal{U}'$. The commutativity of the lower triangle implies that the composition of the p-morphisms $\varphi'_{\mathcal{U}/\mathcal{U}'} : \mathcal{U} \to \mathcal{U}''$ and $\psi_{\mathcal{U}''/\mathcal{U}} : \mathcal{U}'' \to \mathcal{U}$ is the identity p-morphism on $\mathcal{U}$. The commutativity of the higher triangle implies that the composition of the morphisms $\psi_{\mathcal{U}''/\mathcal{U}} : \mathcal{U}'' \to \mathcal{U}$ and $\varphi_{\mathcal{U}/\mathcal{U}'} : \mathcal{U} \to \mathcal{U}''$ is the identity p-morphism on $\mathcal{U}''$. Thus, $\mathcal{U} \xrightarrow{\sim} \mathcal{U}''$. The required fact follows. ∎

In what follows, we do not make difference between a scheme (X, A, τ) and the schemes isomorphic to it, and denote it simply by X. We call any net τ that defines the scheme structure on X a *net of definition*.

5.2. Open subschemes and the schematic and Zariski topologies. Let $\mathcal{X}$ be a scheme over $\mathbf{F}_1$. We fix a triple $(\mathcal{X}, A, \tau)$ that represents it.

5.2.1. Definition. An open subset $\mathcal{Y} \subset \mathcal{X}$ is said to be an *open subscheme* if, for every point $x \in \mathcal{Y}$, there exists $\mathcal{U} \in \overline{\tau}$ with $x \in \mathcal{U} \subset \mathcal{Y}$. (Notice that this property does not depend on the choice of τ.)

If $\mathcal{Y}$ is an open subscheme, then the restriction of the affine atlas $\overline{A}$ to the net $\overline{\tau}\big|_Y$ defines a scheme $(\mathcal{Y}, \overline{A}, \overline{\tau}\big|_{\mathcal{Y}})$. (If $\sigma \prec \tau$, then $\overline{\sigma}\big|_{\mathcal{Y}} \prec \overline{\tau}\big|_{\mathcal{Y}}$.) The scheme $(\mathcal{Y}, \overline{A}, \overline{\tau}\big|_{\mathcal{Y}})$, which will be denoted by $\mathcal{Y}$, possesses the following property: any morphism of schemes $\varphi : \mathcal{X}' \to \mathcal{X}$ with $\varphi(\mathcal{X}') \subset \mathcal{Y}$ goes through a unique morphism $\mathcal{X}' \to \mathcal{Y}$. Notice that the intersection of two open subschemes is an open subscheme, and the preimage of an open subscheme with respect to a morphism of schemes is an open subscheme.

5.2.2. Definition. An *open p-affine subscheme* of $\mathcal{X}$ is an open subscheme $\mathcal{U}$ isomorphic to an affine scheme over $\mathbf{F}_1$. If $\mathcal{U}$ is connected, it will be called an *open affine subscheme.*

5.2.3. Proposition. *(i) A subset is an open p-affine subscheme if and only if it is a finite disjoint union of sets from $\overline{\tau}$;*

(ii) the family $\widehat{\tau}$ of open p-affine subschemes is a net on $\mathcal{X}$, and there is a unique (up to a canonical isomorphism) p-affine atlas $\widehat{A}$ on $\mathcal{X}$ with the net $\widehat{\tau}$ that extends A;

(iii) the strong morphism $\varphi : (\mathcal{X}, \overline{A}, \overline{\tau}) \to (\mathcal{X}, \widehat{A}, \widehat{\tau})$ is a quasi-isomorphism.

Proof. (i) The direct implication follows from Proposition 4.2.6, and the converse one follows from the fact that the category $\mathcal{A}sch^p$ admits finite coproducts.

(ii) That $\widehat{\tau}$ is a net is trivial. We fix an affine scheme structure on every $\mathcal{W} \in \widehat{\tau}$, and our purpose is to construct, for every pair $\mathcal{W} \subset \mathcal{W}'$ in $\widehat{\tau}$ a canonical p-open immersion $\mathcal{W} \to \mathcal{W}'$. Let $\{\mathcal{U}_i\}$ and $\{\mathcal{U}'_k\}$ be finite coverings of $\mathcal{W}$ and $\mathcal{W}'$ by sets from $\overline{\tau}$. Since each $\mathcal{U}_i$ is quasi-compact, it is covered by a finite number of sets from some $\overline{\tau}_{\mathcal{U}_i \cap \mathcal{U}'_k}$. Replacing all $\mathcal{U}_i$'s by them, we may assume that each $\mathcal{U}_i$ lies in some $\mathcal{U}'_k$ and, in particular, every $\mathcal{U}_i$ is an open p-affine subscheme of some U'_k. It follows that there are canonical p-open immersions of affine schemes $\mathcal{U}_i \to \mathcal{W}'$. It is easy to see that they are compatible on intersections and, therefore, they give rise to a p-open immersion $\mathcal{W} \to \mathcal{W}'$.

The statement (iii) is trivial. ∎

5.2.4. Proposition. *Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism of schemes over $\mathbf{F}_1$. Then for every connected open affine subscheme $\mathcal{V} \subset \mathcal{Y}$ there exists an open p-affine subscheme $\mathcal{U} \subset \mathcal{X}$ with $\varphi(\mathcal{V}) \subset \mathcal{U}$. If every point of $\mathcal{X}$ lies in a connected open affine subscheme, then such $\mathcal{U}$ can be found to be connected.*

Proof. Take a point $y \in \mathcal{V}_{\mathbf{m}}$ and an open p-affine subscheme $\mathcal{U}$ of $\mathcal{X}$ that contains the point $\varphi(y)$. *We claim that $\varphi(\mathcal{V}) \subset \mathcal{U}$.* Indeed, the intersection $\mathcal{V} \cap \varphi^{-1}(\mathcal{U})$ is covered by open affine subschemes of $\mathcal{V}$. Since $\mathcal{V}$ is connected, every open affine subscheme of $\mathcal{V}$ that contains a point from $\mathcal{V}_{\mathbf{m}}$ coincides with $\mathcal{V}$. This implies that $\mathcal{V} \cap \varphi^{-1}(\mathcal{U}) = \mathcal{V}$, i.e., $\varphi(\mathcal{V}) \subset \mathcal{U}$. ∎

We say that $\mathcal{X}$ is *locally connected* if, for each point $x \in \mathcal{X}$, every open subscheme of $\mathcal{X}$ which is a neighborhood of x contains a connected open affine subscheme which is also a neighborhood of x. For example, if every point of $\mathcal{X}$ has an open p-affine neighborhood with finitely many irreducible components, then $\mathcal{X}$ is locally connected. If $\mathcal{X}$ is locally connected, then the family τ_c of connected open affine subschemes is a net on $\mathcal{X}$ and the p-affine atlas on $\mathcal{X}$ induces an *affine atlas* with the net τ_c, i.e., a functor $\tau_c \to \mathcal{A}sch^{oi}_{\mathbf{F}_1}$.

The *schematic topology* on a scheme $\mathcal{X}$ is the topology in which open sets are open subschemes. This topology is weaker than the canonical topology on $\mathcal{X}$, and it is denoted by $\mathcal{X}_{\mathcal{S}\mathrm{ch}}$.

5.2.5. Lemma. *Any representable presheaf is a sheaf on* $\mathcal{X}_{\mathcal{S}ch}$.

Proof. Let $\{\mathcal{X}_i\}_{i\in I}$ be a covering of $\mathcal{X}$ by open subschemes. We have to verify that, for every scheme $\mathcal{Y}$, the following sequence of maps is exact

$$\mathrm{Hom}(\mathcal{X},\mathcal{Y}) \to \prod_i \mathrm{Hom}(\mathcal{X}_i,\mathcal{Y}) \rightrightarrows \prod_{i,j} \mathrm{Hom}(\mathcal{X}_i\cap\mathcal{X}_j,\mathcal{Y}) \ .$$

Let $\varphi_i : \mathcal{X}_i \to \mathcal{Y}$ be a family of morphisms such that, for every pair $i,j \in I$, $\varphi_i\big|_{\mathcal{X}_i\cap\mathcal{X}_j} = \varphi_j\big|_{\mathcal{X}_i\cap\mathcal{X}_j}$. It obviously defines a continuous map $\varphi : \mathcal{X} \to \mathcal{Y}$. Every point $x \in \mathcal{X}$ lies in some $\mathcal{X}_i$ and, therefore, we can find an open p-affine subscheme $\mathcal{U} \subset \mathcal{X}_i$ which contains x and whose image in $\mathcal{Y}$ lies in an open p-affine subscheme $\mathcal{U}'$ of $\mathcal{Y}$. The morphism φ defines a p-morphism of affine schemes $\mathcal{U} \to \mathcal{V}$ which does not depend on the choice of i, by the assumption. The statement follows. ∎

If we apply Lemma 5.2.1 to the presheaf representable by $\mathrm{Fspec}(\mathbf{F}_1[T])$, we get the *structural sheaf* $\mathcal{O}_{\mathcal{X}}$ *on* $\mathcal{X}_{\mathcal{S}ch}$, which is a sheaf of $\mathbf{F}_1$-algebras. Its value on $\mathcal{X}$ is the $\mathbf{F}_1$-algebra $\mathcal{O}(\mathcal{X})$ introduced in the previous subsection. If $\mathcal{X} = \mathrm{Fspec}(A)$ is an affine scheme, then $\mathcal{O}(\mathcal{X}) = \langle A\rangle$. Notice that every morphism of schemes $\varphi : \mathcal{Y} \to \mathcal{X}$ gives rise to a homomorphism of sheaves of $\mathbf{F}_1$-algebras $\mathcal{O}_{\mathcal{X}} \to \varphi_*\mathcal{O}_{\mathcal{Y}}$ in the schematic topology of $\mathcal{X}$.

5.2.6. Corollary. *If* $\mathcal{X} = \mathrm{Fspec}(A)$ *is affine, then* $\mathrm{Hom}(\mathcal{Y},\mathcal{X}) = \mathrm{Hom}(A, \mathcal{O}(\mathcal{Y}))$. ∎

For a triple $(\mathcal{X}, A, \tau)$ that represents $\mathcal{X}$, let $\mathrm{Qcoh}(\mathcal{X}, A, \tau)$ (resp. $\mathrm{Qcoh}_a(\mathcal{X}, A, \tau)$) denote the following category. Its objects M are pairs consisting of a map that takes each $\mathcal{U} \in \tau$ to an $A_{\mathcal{U}}$-module (resp. $A_{\mathcal{U}}$-algebra) $M_{\mathcal{U}}$ and a system of p-isomorphisms of $A_{\mathcal{V}}$-modules in $A_{\mathcal{V}}$-Mod^p (resp. $A_{\mathcal{V}}$-algebras in $A_{\mathcal{V}}$-Alg^p) $\gamma_{\mathcal{U}/\mathcal{V}} : M_{\mathcal{U}} \otimes_{A_{\mathcal{U}}} A_{\mathcal{V}} \stackrel{\sim}{\to} M_{\mathcal{V}}$ for all pairs $\mathcal{U} \supset \mathcal{V}$ in τ such that, for every triple $\mathcal{U} \supset \mathcal{V} \supset \mathcal{W}$ in τ, one has $\gamma_{\mathcal{V}/\mathcal{W}} \circ (\gamma_{\mathcal{U}/\mathcal{V}} \otimes_{A_{\mathcal{U}}} A_{\mathcal{V}}) = \gamma_{\mathcal{U}/\mathcal{W}}$. Morphisms between such objects are defined in the evident way. If $\sigma \prec \tau$, there is an evident faithful functor $\mathrm{Qcoh}(\mathcal{X}, A, \tau) \to \mathrm{Qcoh}(\mathcal{X}, A_\sigma, \sigma)$ (resp. $\mathrm{Qcoh}_a(\mathcal{X}, A, \tau) \to \mathrm{Qcoh}_a(\mathcal{X}, A_\sigma, \sigma)$). Of course, if all sets from τ are connected (which is possible, for example, if $\mathcal{X}$ is locally connected), the latter is an equivalence of categories.

5.2.7. Definition. The *category of quasi-coherent* $\mathcal{O}_{\mathcal{X}}$*-modules* $\mathrm{Qcoh}(\mathcal{X})$ (resp. $\mathcal{O}_{\mathcal{X}}$*-algebras*

$\mathrm{Qcoh}_a(\mathcal{X})$) is the inductive limit of categories

$$\varinjlim_{\sigma \prec \tau} \mathrm{Qcoh}(\mathcal{X}, A_\sigma, \sigma) \ \ (\text{resp. } \varinjlim_{\sigma \prec \tau} \mathrm{Qcoh}_a(\mathcal{X}, A_\sigma, \sigma)) \ .$$

The object of $\mathrm{Qcoh}(\mathcal{X})$ (resp. $\mathrm{Qcoh}_a(\mathcal{X})$) that corresponds to the above object M will be denoted by $\widetilde{M}$. It can be viewed as a sheaf of $\mathcal{O}_\mathcal{X}$-modules (resp. $\mathcal{O}_\mathcal{X}$-algebras), i.e., there is a faithful functor from $\mathrm{Qcoh}(\mathcal{X})$ (resp. $\mathrm{Qcoh}_a(\mathcal{X})$) to that of sheaves of $\mathcal{O}_\mathcal{X}$-modules (resp. $\mathcal{O}_\mathcal{X}$-algebras). This functor is fully faithful only if every point of $\mathcal{X}$ admits a connected open affine neighborhood. Of course, any quasi-coherent $\mathcal{O}_\mathcal{X}$-algebra can be viewed as a quasi-coherent $\mathcal{O}_\mathcal{X}$-module, i.e., there is a faithful functor $\mathrm{Qcoh}_a(\mathcal{X}) \to \mathrm{Qcoh}(\mathcal{X})$. Notice that both categories admit direct sums and tensor product (defined in the evident way) which commute with the latter functor.

For example, suppose that $\mathcal{X} = \mathrm{Fspec}(A)$ is affine. Then $\mathrm{Qcoh}(\mathcal{X}, \{\mathcal{X}\}, A)$ is just the category of A-modules A-Mod. For an A-module, one has $\Gamma(\mathcal{X}, \widetilde{M}) = \langle M \rangle$. An arbitrary object of $\mathrm{Qcoh}(\mathcal{X})$ can be represented as a system of $A_{\mathcal{U}_i}$-modules M_i for a finite covering of $\mathcal{X}$ by pairwise disjoint open affine subschemes $\{\mathcal{U}_i\}_{i \in I}$, and there is a fully faithful functor $A\text{-Mod}^p \to \mathrm{Qcoh}(\mathcal{X})$. The similar description holds for the category $\mathrm{Qcoh}_a(\mathcal{X})$. Of course, if $\mathcal{X}$ is connected, then there are equivalences of categories $A\text{-Mod} \xrightarrow{\sim} \mathrm{Qcoh}(\mathcal{X})$ and $A\text{-Alg} \xrightarrow{\sim} \mathrm{Qcoh}_a(\mathcal{X})$.

A quasi-coherent $\mathcal{O}_\mathcal{X}$-module is said to be *of finite type* (resp. *coherent*; resp. *locally free*) if it comes from M as above such that, for every $\mathcal{U} \in \tau$, the $A_\mathcal{U}$-module $M_\mathcal{U}$ is finitely generated (resp. finitely presented; resp. free). (Notice that, if $\mathcal{X}$ is affine, any locally free $\mathcal{O}_\mathcal{X}$-module of constant rank is free.) Locally free $\mathcal{O}_\mathcal{X}$-modules of rank one are said to be *invertible*. The isomorphism classes of invertible $\mathcal{O}_\mathcal{X}$-modules form an abelian group with respect to tensor product. This group is canonically isomorphic to the first Čech cohomology group (in the schematic topology) $\check{H}^1(\mathcal{X}, \mathcal{O}^*_\mathcal{X})$.

Recall that, for an ideal E of an $\mathbf{F}_1$-algebra A, we set $V(E) = \{x \in \mathrm{Fspec}(A) \big| f(x) = g(x)$ for all $(f,g) \in E\}$. It is a closed subset of $\mathrm{Fspec}(A)$, and it coincides with $\mathrm{Fspec}(A/E)$. If E is associated to a Zariski ideal $\mathbf{a} \subset A$, the set $V(E)$ is denoted by $V(\mathbf{a})$. Notice that the intersection of any family of nonempty Zariski closed subsets is nonempty (since it contains $V(\mathbf{m}_A)$).

5.2.8. Definition. (i) A subset $\Sigma \subset \mathcal{X}$ is said to be *strongly closed* if every point of $\mathcal{X}$ has an p-affine neighborhood $\mathcal{U}$ such that the intersection $\Sigma \cap \mathcal{U}$ is of the form $V(E_1) \cup \ldots \cup V(E_n)$ for ideals $E_1, \ldots, E_n$ of $A_\mathcal{U}$.

(ii) A strongly closed subset $\Sigma \subset \mathcal{X}$ is said to be *schematically closed* if $\mathcal{U}$ can be found in such a way that the sets $V(E_i)$ from (i) are in addition pairwise disjoint.

(iii) A subset $\Sigma \subset \mathcal{X}$ is said to be *Zariski closed* if every point of $\mathcal{X}$ has an p-affine neighborhood $\mathcal{U}$ such that the intersection $\Sigma \cap \mathcal{U}$ is of the form $V(\mathbf{a})$ for a Zariski ideal $\mathbf{a} \subset A_{\mathcal{U}}$. The complement of a Zariski closed subset is said to be *Zariski open*.

Foe example, the closure $\overline{\{x\}}$ of any point $x \in \mathcal{X}$ is a schematically closed subset (see Corollary 3.1.2(i)). In particular, all irreducible components of $\mathcal{X}$ are schematically closed. Notice that the intersection of a finite family of strongly (resp. schematically) closed subsets is strongly (resp. schematically) closed. We also notice that a schematically closed subset is not necessarily closed in the schematic topology.

5.2.9. Proposition. *(i) If Σ is strongly (resp. schematically) closed, then the intersection $\Sigma \cap \mathcal{U}$ with every open p-affine subset of $\mathcal{X}$ is of the form $V(E_1) \cup \ldots \cup V(E_n)$ as in (i) (resp. (ii));*

(ii) every Zariski open set is an open subscheme.

Proof. We may assume that $\mathcal{X} = \mathrm{Fspec}(A)$ is affine.

(i) The assumption implies that there exists a covering $\{\mathcal{U}_i\}_{i \in I}$ by open affine subschemes such that, for every $i \in I$, $\Sigma \cap \mathcal{U}_i$ is strongly (resp. schematically) closed subset of $\mathcal{U}_i$. By Proposition 4.2.6, we may replace the covering by a refinement and assume that the covering is finite and all $\mathcal{U}_i$'s are pairwise disjoint elementary open subsets. In this case they are idempotent open subsets, and so every strongly closed subset of each $\mathcal{U}_i$ is a strongly closed subset of $\mathcal{X}$. This easily implies the required fact.

(ii) We may in addition assume that a Zariski open subset $\mathcal{U}$ is the complement of the set $V(\mathbf{a})$ for some Zariski ideal $\mathbf{a} \subset A$. If $x \in \mathcal{U}$, then there exists an element $f \in \mathbf{a}$ with $f(x) \neq 0$ and, therefore, $D(f) \subset \mathcal{U}$. It follows that $\mathcal{U}$ is a union of principal open subsets of $\mathcal{X}$ and, therefore, it is an open subscheme. ■

In general the union of an infinite number of Zariski open subsets is not necessarily a Zariski open subset (see Remark 5.2.12(i)). This means that the family of Zariski open subsets does not form a usual topology. It forms a Grothendieck topology denoted by $\mathcal{X}_{Zar}$. By the above remark, there is a morphism of sites $\mathcal{X}_{\mathcal{S}ch} \to \mathcal{X}_{Zar}$. If every point of $\mathcal{X}$ has a connected affine neighborhood (e.g., if $\mathcal{X}$ is locally connected), then the family of Zariski open subsets forms a usual topology. Indeed, if $\mathcal{U}$ is an open affine neighborhood of a point $x \in \mathcal{X}$, then $\mathcal{U}$ is the minimal (connected) affine neighborhood of any point $y \in \mathcal{U}_{\mathbf{m}}$. It follows that every Zariski closed subset of $\mathcal{U}$ is of the form $V(\mathbf{a})$ and, therefore, the intersection of any families of them is of the same form. In general the Zariski topology is weaker than the schematic topology (see Remark 5.2.12(ii)).

We now consider a process of gluing schemes over $\mathbf{F}_1$.

5.2.10. Definition. A morphism of schemes $\varphi : \mathcal{Y} \to \mathcal{X}$ is said to be an *open* (resp. *Zariski open*) *immersion* if it induces an isomorphism of $\mathcal{Y}$ with an open (resp. Zariski open) subscheme of $\mathcal{X}$.

Let $\{\mathcal{X}_i\}_{i\in I}$ be a family of schemes over $\mathbf{F}_1$, and suppose that, for each pair $i, j \in I$, we are given an open subscheme $\mathcal{X}_{ij} \subset \mathcal{X}_i$ and an isomorphism of schemes $\nu_{ij} : \mathcal{X}_{ij} \stackrel{\sim}{\to} \mathcal{X}_{ji}$ so that $\mathcal{X}_{ii} = \mathcal{X}_i$, $\nu_{ij}(\mathcal{X}_{ij} \cap \mathcal{X}_{ik}) = \mathcal{X}_{ji} \cap \mathcal{X}_{jk}$, and $\nu_{ik} = \nu_{jk} \circ \nu_{ij}$ on $\mathcal{X}_{ij} \cap \mathcal{X}_{ik}$. We are looking for a scheme $\mathcal{X}$ with a family of morphisms $\mu_i : \mathcal{X}_i \to \mathcal{X}$ such that:

(1) μ_i is an open immersion;

(2) $\{\mu_i(\mathcal{X}_i)\}_{i\in I}$ is a covering of $\mathcal{X}$;

(3) $\mu_i(\mathcal{X}_{ij}) = \mu_i(\mathcal{X}_i) \cap \mu_j(\mathcal{X}_j)$;

(4) $\mu_i = \mu_j \circ \nu_{ij}$ on $\mathcal{X}_{ij}$.

If such $\mathcal{X}$ exists we say that it is obtained by *gluing $\mathcal{X}_i$'s along $\mathcal{X}_{ij}$'s.*

5.2.11. Lemma. *A scheme $\mathcal{X}$ obtained by gluing of $\mathcal{X}_i$ along $\mathcal{X}_{ij}$ exists and is unique (up to a canonical isomorphism).*

Proof. Let X be the disjoint union $\coprod_i \mathcal{X}_i$. The system $\{\nu_{ij}\}$ defines an equivalence relation R on X. We denote by $\mathcal{X}$ the quotient space X/R and by μ_i the induced maps $\mathcal{X}_i \to \mathcal{X}$. Then the equivalence relation R is open (see [Bou], Ch. I, §9, n° 6), and therefore all $\mu_i(\mathcal{X}_i)$ are open in $\mathcal{X}$, and each μ_i induces a homeomorphism $\mathcal{X}_i \stackrel{\sim}{\to} \mu_i(\mathcal{X}_i)$. Furthermore, let τ denote the collection of all open subsets $\mathcal{U} \subset \mathcal{X}$ for which there exists $i \in I$ such that $\mathcal{U} \subset \mu_i(\mathcal{X}_i)$ and $\mu_i^{-1}(\mathcal{U})$ is an open p-affine subscheme of $\mathcal{X}_i$. It is easy to see that τ is a net on $\mathcal{X}$, and there is an evident p-affine atlas A with the net τ. In this way we get a scheme $(\mathcal{X}, A, \tau)$ that possesses the properties (1)-(4). That $\mathcal{X}$ is unique up to a canonical isomorphism is trivial. ∎

5.2.12. Remark. (i) Let A be the idempotent $\mathbf{F}_1$-algebra from Remark 3.4.8, i.e., $A = \{0, e_{-\infty}, \dots, e_{-2}, e_{-1}\}$ with $e_{-i}e_{-j} = e_{-\min(i,j)}$. Every Zariski ideal of A is prime, and it is either $\mathfrak{p}_n = e_n A = \{0, e_{-n}, \dots, e_{-1}\}$ for $1 \leq n < \infty$, or $\mathbf{m} = A\backslash\{1\}$. Every open neighborhood of the point $\Pi_{\mathbf{m}}$ contains almost all points $\Pi_{\mathfrak{p}_n}$, and the topology on the open subset $\mathrm{Fspec}(A)\backslash\{\Pi_{\mathbf{m}}\}$ is discrete. For $n \geq 1$, the set $\mathcal{U}_n = \{\Pi_{\mathfrak{p}_i} \big| i \leq n \text{ and } i \text{ is even}\}$ is Zariski open, but the union $\bigcup_{n=1}^{\infty} \mathcal{U}_n$ consists of the points $\Pi_{\mathfrak{p}_i}$ with even i and, therefore, it is not Zariski open.

(ii) Let $A = \mathbf{F}_1[T_1, T_2, T_3]/E$, where E is the ideal generated by the pair (T_1T_2, T_2^2), and let t_i be the image of T_i in A. The affine scheme $\mathcal{X} = \mathrm{Fspec}(A)$ is a union of two irreducible components $\mathcal{X}_1$ and $\mathcal{X}_2$ defined by the equations $t_2 = 0$ and $t_1 = t_2$, respectively, whose intersection is the "line" defined by the equations $t_1 = t_2 = 0$. Let $\mathcal{U}_1 = D_{\mathcal{X}_1}(t_1)$ and $\mathcal{U}_2 = D_{\mathcal{X}_2}(t_1t_3)$. Then $\mathcal{U} = \mathcal{U}_1 \coprod \mathcal{U}_2$

is an open affine subscheme of $\mathcal{X}$. The complement $\Sigma = \mathcal{X}\backslash\mathcal{U}$ is a union of two "lines" defined by the equations $t_1 = t_2$ and $t_3 = 0$ ($\mathcal{L}_1$) and $t_1 = t_2 = 0$ ($\mathcal{L}_2$), which intersect at the point x with $t_i(x) = t_2(x) = t_3(x) = 0$. *We claim the Σ is not Zariski closed in $\mathcal{X}$.* Indeed, assume that Σ is Zariski closed, and let $B = A/\mathbf{b}$, where $\mathbf{b}$ is the Zariski ideal of A generated by the element t_3. Then B is isomorphic to the quotient of $\mathbf{F}_1[T_1, T_2]$ by the ideal generated by the pair (T_1T_2, T_2^2). The affine scheme $\mathcal{Y} = \mathrm{Fspec}(B)$ is a union of the above line $\mathcal{L}_1$ and the line $\mathcal{L}_3$ defined by the equation $t_2 = 0$. The assumption on Σ implies that the line $\mathcal{L}_1 = \Sigma \cap \mathcal{Y}$ is Zariski closed in $\mathcal{Y}$. This is impossible because any element of B, which vanishes at a point from $\mathcal{L}_1\backslash\{x\}$, is zero.

5.3. Fiber products and classes of morphisms of schemes.

5.3.1. Proposition. *The category $\mathcal{S}ch_{\mathbf{F}_1}$ admits coproducts and finite fiber products.*

Proof. Given a family of schemes $\{(\mathcal{X}_i, A_i, \tau_i)\}_{i\in I}$, let $\mathcal{X}$ be the disjoint union $\coprod_{i\in I} \mathcal{X}_i$, τ is the net on $\mathcal{X}$ with $\tau\big|_{\mathcal{X}_i} = \tau_i$ for all $i \in I$, and A be the p-affine atlas with the net τ whose restriction on each τ_i is A_i. Then the triple $(\mathcal{X}, A, \tau)$ is a scheme over $\mathbf{F}_1$, and it is the coproduct of the above family.

Let now $\varphi : \mathcal{Y} \to \mathcal{X}$ and $f : \mathcal{X}' \to \mathcal{X}$ be morphisms of schemes over $\mathbf{F}_1$. Suppose first that the scheme $\mathcal{X}$ is affine. If the other two schemes are also affine and φ and f are morphisms in $\mathcal{A}sch_{\mathbf{F}_1}$, Corollary 5.2.6 implies that their fiber product in $\mathcal{A}sch_{\mathbf{F}_1}$ is also a fiber product in $\mathcal{S}ch_{\mathbf{F}_1}$. If φ and f are p-morphisms of affine schemes, the reasoning from the proof of Proposition 4.6.1 shows that the fiber product in $\mathcal{A}sch^p_{\mathbf{F}_1}$ is also a fiber product in $\mathcal{S}ch_{\mathbf{F}_1}$. Moreover, in this case, if $\mathcal{Z} \subset \mathcal{Y}$ and $\mathcal{X}'' \subset \mathcal{X}'$ are open subschemes, then the preimage of $\mathcal{Z} \times \mathcal{X}''$ with respect to the canonical map $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}' \to \mathcal{Y} \times \mathcal{X}'$ is a fiber product $\mathcal{Z} \times_{\mathcal{X}} \mathcal{X}''$. Furthermore, if $\mathcal{Y}$ and $\mathcal{X}'$ are arbitrary, we take coverings $\{\mathcal{Y}_i\}$ of $\mathcal{Y}$ and $\{\mathcal{X}'_k\}$ of $\mathcal{X}'$ by open p-affine subschemes. Then a fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ is the scheme $\mathcal{Y}'$ obtained by gluing all $\mathcal{Y}_i \times_{\mathcal{X}} \mathcal{X}'_k$ along $(\mathcal{Y}_i \cap \mathcal{Y}_j) \times_{\mathcal{X}} (\mathcal{X}'_k \cap \mathcal{X}'_l)$. Finally, suppose that $\mathcal{X}$ is an arbitrary scheme over $\mathbf{F}_1$. If the morphisms φ and f go through a morphisms to an open p-affine subscheme $\mathcal{U}$, then $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}' = \mathcal{Y} \times_{\mathcal{U}} \mathcal{X}'$. In the general case, we take a covering $\{\mathcal{U}_i\}$ of $\mathcal{X}$ by open p-affine subschemes. Then the scheme $\mathcal{Y}'$ obtained by gluing all $\varphi^{-1}(\mathcal{U}_i) \times_{\mathcal{X}} f^{-1}(\mathcal{U}_i)$ along $\varphi^{-1}(\mathcal{U}_i \cap \mathcal{U}_j) \times_{\mathcal{X}} f^{-1}(\mathcal{U}_i \cap \mathcal{U}_j)$ is a fiber product of $\mathcal{Y}$ and $\mathcal{X}'$ over $\mathcal{X}$. ∎

5.3.2. Definition. A morphism of schemes over $\mathbf{F}_1$, $\varphi : \mathcal{Y} \to \mathcal{X}$, is said to be a *finite morphism* (resp. a *closed immersion*; resp. a *Zariski closed immersion*) if there exists a covering of $\mathcal{X}$ by open p-affine subschemes $\{\mathcal{U}_i\}_{i\in I}$ such that, for every $i \in I$, $\varphi^{-1}(\mathcal{U}_i) \to \mathcal{U}_i$ is a p-finite morphism (resp. p-closed immersion; resp. a Zariski p-closed immersion) of affine schemes.

Notice that, if both schemes $\mathcal{X}$ and $\mathcal{Y}$ are affine, this definition is consistent with those in §4.6.

5.3.3. Proposition. *Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a finite morphism (resp. a closed immersion) of schemes over $\mathbf{F}_1$. Then for any open p-affine subscheme $\mathcal{U} \subset \mathcal{X}$, $\varphi^{-1}(\mathcal{U})$ is an open p-affine subscheme of $\mathcal{Y}$ and, in particular, $\varphi^{-1}(\mathcal{U}) \to \mathcal{U}$ is a p-finite morphism (resp. a p-closed immersion) of affine schemes.*

Proof. Suppose that the morphism φ is p-finite. It follows from Definition 5.3.2 that we can find a covering $\{\mathcal{U}_i\}$ of $\mathcal{U}$ by open affine subschemes such that, for every $i \in I$, $\varphi^{-1}(\mathcal{U}_i)$ is an open p-affine subscheme and there is a covering $\{\mathcal{V}_{ij}\}_{j \in J_i}$ of it by pairwise disjoint open affine subschemes such that all of the induced morphisms $\mathcal{V}_{ij} \to \mathcal{U}_i$ are finite morphisms of affine schemes. By Proposition 4.2.6, we may assume that all of $\mathcal{U}_i$'s are pairwise disjoint. In this case all $\mathcal{V}_{ij}$ are pairwise disjoint open p-affine subschemes of $\mathcal{X}$ and, therefore, there union, which coincides with $\varphi^{-1}(\mathcal{U})$, is an open p-affine subscheme. ∎

5.3.4. Corollary. *The classes of finite morphisms and of closed and Zariski closed immersions are preserved by composition and any base change.* ∎

Notice that the image of a close (resp. Zariski closed) immersion $\varphi : \mathcal{Y} \to \mathcal{X}$ is a schematically (resp. Zariski) closed subset of $\mathcal{X}$.

5.3.5. Proposition. *Given a schematically (resp. Zariski) closed subset $\mathcal{Y} \subset \mathcal{X}$, there exists a closed (resp. Zariski closed) immersion $\varphi : \mathcal{Y}' \to \mathcal{X}$ such that $\mathcal{Y}'$ is reduced (resp. Zariski reduced), φ induces a homeomorphism $\mathcal{Y}' \overset{\sim}{\to} \mathcal{Y}$, and any morphism $\psi : \mathcal{Z} \to \mathcal{X}$ from a reduced (resp. Zariski reduced) scheme $\mathcal{Z}$ with $\psi(\mathcal{Z}) \subset \mathcal{Y}$ goes through a unique morphism $\mathcal{Z} \to \mathcal{Y}'$.*

Proof. By the definition, the family τ of open affine subschemes $\mathcal{U} \subset \mathcal{X}$ such that $\mathcal{Y} \cap \mathcal{U}$ is of the form $V(E_1) \coprod \dots \coprod V(E_n)$ for ideals $E_1, \dots E_n$ (resp. $V(\mathbf{a})$ for a Zariski ideal $\mathbf{a}$) of $A_{\mathcal{U}}$ is a net on $\mathcal{X}$. Then the family of intersections $\mathcal{V} = \mathcal{U} \cap \mathcal{Y}$ with $\mathcal{U} \in \tau$ is a net on $\mathcal{Y}$. Given $\mathcal{V}$ with $\mathcal{U}$ as above, let $\mathcal{V}'$ be the reduced (resp. Zariski reduced) affine scheme $\mathrm{Fspec}(A_{\mathcal{U}}/\mathbf{r}(E_1)) \coprod \dots \coprod \mathrm{Fspec}(A_{\mathcal{U}}/\mathbf{r}(E_n))$ (resp. $\mathrm{Fspec}(A/\mathrm{z}\mathbf{r}(\mathbf{a}))$). Then the canonical morphism $\mathcal{V}' \to \mathcal{U}$ is a closed (resp. Zariski closed) immersion which possesses the property from the formulation with $\mathcal{U}$ and $\mathcal{V}$ instead of $\mathcal{X}$ and $\mathcal{Y}$, respectively. It follows easily the family of $\mathcal{V}$'s defines the structure of a reduced (resp. Zariski reduced) scheme on $\mathcal{Y}$ with the required property. ∎

Proposition 5.3.5 implies that we can associate with each scheme $\mathcal{X}$ over $\mathbf{F}_1$ its *reduction* $\mathcal{X}^{\mathrm{r}}$ (resp. *Zariski reduction* $\mathcal{X}^{\mathrm{zr}}$). Of course, there is a canonical closed immersion $\mathcal{X}^{\mathrm{r}} \to \mathcal{X}^{\mathrm{rd}}$. In what follows, if we mention a schematically closed subset $\mathcal{Y}$ of $\mathcal{X}$, we consider it by default as a reduced scheme.

A scheme $\mathcal{X}$ is said to be *integral* if the $\mathbf{F}_1$-algebra $A_{\mathcal{U}}$ of every open p-affine subscheme $\mathcal{U}$ is an integral domain or, equivalently, if $\mathcal{X}$ is irreducible and reduced. A scheme $\mathcal{X}$ is said to be *normal* if it is integral and the $\mathbf{F}_1$-algebra $A_{\mathcal{U}}$ of every open affine subscheme $\mathcal{U}$ is integrally closed in its fraction $\mathbf{F}_1$-field. For every integral scheme $\mathcal{X}$ one can construct in the evident way its *normalization* $\mathcal{X}^{\mathrm{nor}}$, i.e., a morphism $\varphi : \mathcal{X}^{\mathrm{nor}} \to \mathcal{X}$ from a normal scheme such that, for every open affine subscheme $\mathcal{U} \subset \mathcal{X}$, $\varphi^{-1}(\mathcal{U})$ is an affine scheme which is the spectrum of the integral closure of $A_{\mathcal{U}}$ in its fraction $\mathbf{F}_1$-field.

The $\mathbf{F}_1$*-field of rational functions* on an irreducible scheme $\mathcal{X}$ is the $\mathbf{F}_1$-field $\kappa(x)$ of the generic point x of $\mathcal{X}$. It is denoted by $\kappa(\mathcal{X})$ and, if $\mathcal{X}$ is defined over an $\mathbf{F}_1$-field K, it is also denoted by $K(\mathcal{X})$. If $\mathcal{X}$ is integral, then $\kappa(\mathcal{X})$ is the fraction $\mathbf{F}_1$-field of the $\mathbf{F}_1$-algebra $A_{\mathcal{U}}$ of any nonempty open affine subscheme $\mathcal{U}$ of $\mathcal{X}$.

A scheme $\mathcal{X}$ is said to be *Zariski integral* if each open p-affine subscheme $\mathcal{U}$ is connected and its the $\mathbf{F}_1$-algebra $A_{\mathcal{U}}$ has no zero divisors.

5.3.6. Proposition. *Every Zariski closed immersion $\varphi : \mathcal{Y} \to \mathcal{X}$ from a Zariski integral scheme $\mathcal{Y}$ to a scheme $\mathcal{X}$ has a canonical section $\psi : \mathcal{X} \to \mathcal{Y}$ (i.e., $\psi \circ \varphi = 1_{\mathcal{Y}}$).*

Proof. For every open p-affine subscheme $\mathcal{U} = \mathrm{Fspec}(A)$, one has $\varphi^{-1}(\mathcal{U}) = \mathrm{Fspec}(A/\mathfrak{p})$, where $\mathfrak{p}$ is a Zariski prime ideal of A. The canonical homomorphism $A/\mathfrak{p} \to A$ defines a section $\mathcal{U} \to \varphi^{-1}(\mathcal{U})$ of φ restricted to $\varphi^{-1}(\mathcal{U})$. All these sections are compatible on intersections and induce the required morphism. ∎

5.3.7. Definition. (i) A morphism of schemes over $\mathbf{F}_1$, $\varphi : \mathcal{Y} \to \mathcal{X}$, is said to be an *immersion* if it is a composition of a closed immersion $i : \mathcal{Y} \to \mathcal{X}'$ with an open immersion $j : \mathcal{X}' \to \mathcal{X}$.

(ii) A *subscheme* of $\mathcal{X}$ is the isomorphism class of an immersion $\mathcal{Y} \to \mathcal{X}$.

Notice that an immersion $\varphi : \mathcal{Y} \to \mathcal{X}$ is a closed immersion if and only if its image $\varphi(\mathcal{X})$ is a closed subset of $\mathcal{X}$, and that immersions are preserved by composition and any base change.

An example of an immersion is the diagonal morphism $\Delta_\varphi : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ for an arbitrary morphism $\varphi : \mathcal{Y} \to \mathcal{X}$. Indeed, each point from the image of Δ_φ has an open p-affine neighborhood of the form $\mathcal{V} \times_{\mathcal{U}} \mathcal{V}$ for open p-affine subschemes $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{V} \subset \mathcal{Y}$ with $\varphi(\mathcal{V}) \subset \mathcal{U}$, and the base change of Δ_φ with respect to the canonical open immersion $\mathcal{V} \times_{\mathcal{U}} \mathcal{V} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is the diagonal morphism $\mathcal{V} \to \mathcal{V} \times_{\mathcal{U}} \mathcal{V}$, which is a p-closed immersion of affine schemes. Thus, if $\mathcal{W}$ is the union of such subschemes $\mathcal{V} \times_{\mathcal{U}} \mathcal{V}$, then the morphism Δ_φ goes through a closed immersion $\mathcal{Y} \to \mathcal{W}$.

5.3.8. Definition. Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism of schemes over $\mathbf{F}_1$.

(i) φ is said to be *quasi-compact* if, for any open quasi-compact subscheme $\mathcal{U} \subset \mathcal{X}$, the open subscheme $\varphi^{-1}(\mathcal{U})$ is quasi-compact.

(ii) φ is said to be *of finite type* if there exist a covering of $\mathcal{X}$ by open p-affine subschemes $\{\mathcal{U}_i\}_{i\in I}$ and, for every $i \in I$, a finite covering of $\varphi^{-1}(\mathcal{U}_i)$ by open p-affine subschemes $\{\mathcal{V}_{ij}\}_{j\in J_i}$ such that all of the induced p-morphisms $\mathcal{V}_{ij} \to \mathcal{U}_i$ are morphisms of affine schemes of finite type.

For example, the diagonal morphism $\Delta_\varphi : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ as above is quasi-compact if and only if, for any pair $\mathcal{V}', \mathcal{V}''$ of open p-affine subschemes of $\mathcal{Y}$ with $\varphi(\mathcal{V}')$ and $\varphi(\mathcal{V}'')$ lying in an open p-affine subscheme of $\mathcal{X}$, the intersection $\mathcal{V}' \cap \mathcal{V}''$ is quasi-compact.

It is clear that any morphism of finite type is quasi-compact. By Lemma 4.6.7, Definition 5.3.8(ii) is consistent with that for morphisms of affine schemes. It is easy to see that the classes of quasi-compact morphisms and of morphisms of finite type are preserved by composition and any base change.

5.3.9. Definition. Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism of schemes over $\mathbf{F}_1$.

(i) φ is said to be *quasi-separated* (resp. *separated*) if the diagonal morphism $\Delta_\varphi : \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ is quasi-compact (resp. a closed immersion).

(ii) φ is said to be *closed* if it takes closed sets to closed sets. It is said to be *universally closed* if any base change of φ is closed.

(iii) φ is said to be *proper* if it is of finite type, separated and universally closed.

It is also easy to see that the above classes of morphisms are preserved by composition and any base change. By the remark above, a morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ is separated if and only if the image $\Delta_\varphi(\mathcal{Y})$ of the diagonal morphism is closed in $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$.

5.3.10. Proposition. *Let $\mathcal{X}$ be a quasi-separated scheme over $\mathbf{F}_1$. Then for any strongly closed subset Σ of a quasi-compact open subscheme $\mathcal{U}$, the set $\Sigma \cup (\mathcal{X}\backslash\mathcal{U})$ is strongly closed in $\mathcal{X}$.*

Proof. It suffices to verify that the intersection $\mathcal{V} \cap (\Sigma \cup (\mathcal{X}\backslash\mathcal{U})) = (\Sigma \cap \mathcal{V}) \cup (\mathcal{V}\backslash\mathcal{U})$ with every open p-affine subscheme $\mathcal{V}$ of $\mathcal{X}$ is strongly closed in $\mathcal{V}$. The quasi-separatedness assumption implies that the open subscheme $\mathcal{U} \cap \mathcal{V}$ of $\mathcal{V}$ is quasi-compact. We can therefore replace $\mathcal{X}$ by $\mathcal{V}$ and assume that $\mathcal{X} = \mathrm{Fspec}(A)$ is affine. In this case, $\mathcal{U}$ is a finite union $\bigcup_{i=1}^n \mathcal{U}_i$ of elementary open subsets of $\mathcal{X}$. Since $\Sigma \cup (\mathcal{X}\backslash\mathcal{U}) = \bigcap_{i=1}^n (\Sigma_i \cup (\mathcal{X}\backslash\mathcal{U}_i))$, where $\Sigma_i = \Sigma \cap \mathcal{U}_i$, the situation is reduced to the case when $\mathcal{U}$ is an elementary open subset, i.e., $\mathcal{U} = \{x \in \mathcal{X} \big| f(x) \neq 0,\ e_1(x) = \ldots = e_n(x) = 0\}$ for some $f \in A$ and $e_1, \ldots, e_n \in I_{A_f}$ and $A_{\mathcal{U}}$ is the quotient of A_f by the Zariski ideal of A_f generated by the idempotents $e_1, \ldots, e_n$. We may also assume that $\Sigma = V(E)$ for an ideal E of $A_{\mathcal{U}}$. One has $e_i = \frac{g_i}{f^m}$ for some $g_1, \ldots, g_n \in A$ and $m \geq 0$, and we have $\mathcal{U} = \{x \in \mathcal{X} \big| f(x) \neq 0,\ g_1(x) \neq$

$f^m(x), \dots, g_n(x) \neq f^m(x)\}$. If E_0 and E_i for $1 \le i \le n$ are the ideals of A generated by the pairs $(f, 0)$ and (g_i, f^m), respectively, then $\mathcal{X}\backslash\mathcal{U} = \bigcup_{i=0}^n V(E_i)$. Finally, each element of E is of the form $(\frac{a}{f^k}, \frac{b}{f^k})$ for some $a, b \in A$ and $k \ge 0$. Let F be the ideal of A generated by the pairs (a, b). Then $\Sigma = V(F) \cap \mathcal{U}$ and, therefore, $\Sigma \cup (\mathcal{X}\backslash\mathcal{U}) = V(F) \cup (\mathcal{X}\backslash\mathcal{U})$. The required fact follows. ∎

5.3.11. Corollary. *Every closed subset of a quasi-separated scheme over* $\mathbf{F}_1$ *is the intersection of a filtered family of strongly closed subsets.* ∎

5.4. Valuative criterions of separateness and properness.

5.4.1. Theorem. *A morphism of schemes over* $\mathbf{F}_1$, $\varphi : \mathcal{Y} \to \mathcal{X}$, *is separated if and only if it is quasi-separated and, for any morphism* $\psi : \mathrm{Fspec}(K^\circ) \to \mathcal{X}$ *from the spectrum of a valuation* $\mathbf{F}_1$*-algebra* K°*, the following map is injective*

$$\mathrm{Hom}_{\mathcal{X}}(\mathrm{Fspec}(K^\circ), \mathcal{Y}) \to \mathrm{Hom}_{\mathcal{X}}(\mathrm{Fspec}(K), \mathcal{Y}) \ .$$

We introduce a partial ordering on points of a scheme $\mathcal{X}$ over $\mathbf{F}_1$ as follows. Given $x, y \in \mathcal{X}$, we write $y \preceq x$ or $x \succeq y$, and say that y is a *specialization* of x or that x is a *generization* of y if $y \in \overline{\{x\}}$ and the following condition is satisfied. Let $\mathcal{U} = \mathrm{Fspec}(A)$ be an open p-affine neighborhood of y. Then $x \in \mathcal{U}$ and $\overline{\{x\}} \cap \mathcal{U} = \mathrm{Fspec}(A^{(x)})$, where $A^{(x)} = A/\Pi_x$. Let $\mathfrak{q}$ be the Zariski prime ideal of $A^{(x)}$ that corresponds to the point y. The condition is that y is the image of the point x under the morphism $\mathrm{Fspec}(A^{(x)}) \to \mathrm{Fspec}(A^{(x)}/\mathfrak{q})$ induced by the canonical injective homomorphism $A^{(x)}/\mathfrak{q} \hookrightarrow A^{(x)}$. This means that the prime ideal of $A^{(x)}$ that corresponds to the point y coincides with the prime ideal $\Pi_{\mathfrak{q}}$ of $A^{(x)}$ that corresponds to $\mathfrak{q}$. In particular, if $x \succeq y$, there is a canonical injective embedding of $\mathbf{F}_1$-fields $\kappa(y) \hookrightarrow \kappa(x)$. For example, if $\mathcal{X} = \mathrm{Fspec}(K^\circ)$ is the spectrum of a valuation $\mathbf{F}_1$-algebra K°, then $x \succeq y$, where x and y are the generic points of $\mathrm{Fspec}(K)$ and $\mathrm{Fspec}(\widetilde{K})$, respectively. We also notice that the above partial ordering is compatible with morphisms of schemes. Furthermore, suppose $\mathcal{X} = \mathrm{Fspec}(A)$ is affine, and let $\mathfrak{p}$ be a Zariski prime ideal of A. Then for every point $y \in \mathcal{X}^{(\mathfrak{p})}$, there exists a point $x \in \check{\mathcal{X}}_{\mathfrak{p}}$ with $x \succeq y$. Indeed, let $\mathcal{Y} = \mathcal{X}^{(\mathfrak{p})}$ and $\mathfrak{q}$ the Zariski prime ideal of $A^{(\mathfrak{p})} = A/\Pi_{\mathfrak{p}}$ with $y \in \check{\mathcal{Y}}_{\mathfrak{q}}$. Then the induced homomorphism of $\mathbf{F}_1$-fields $\kappa(\mathfrak{q}) \to \kappa(\mathfrak{p})$ is injective. The point y corresponds to a subgroup $H \subset \kappa(\mathfrak{q})^*$. If G is an arbitrary subgroup of $\kappa(\mathfrak{p})^*$ whose intersection with $\kappa(\mathfrak{q})^*$ coincides with H (e.g., H itself), then for the corresponding point $x \in \check{\mathcal{X}}_{\mathfrak{p}}$ one has $x \succeq y$.

5.4.2. Lemma. *Given points* $x, y \in \mathcal{X}$ *with* $x \succeq y$ *and an* $\mathbf{F}_1$*-field, for any any morphism* $\mathrm{Fspec}(K) \to \mathcal{X}$ *that takes the generic point of* $\mathrm{Fspec}(K)$ *to* x *there exist a valuation* $\mathbf{F}_1$*-subalgebra* K° *of* K *and a morphism* $\mathrm{Fspec}(K^\circ) \to \mathcal{X}$ *that takes the generic point of* $\mathrm{Fspec}(\widetilde{K})$ *to* y*.*

Proof. First of all, we may assume that $\mathcal{X} = \mathrm{Fspec}(A)$ is affine. Furthermore, replacing A by A/Π_x, we may assume that A is an integral domain and x is the generic point of $\mathcal{X}$. By Proposition 2.7.2, there exists a valuation $\mathbf{F}_1$-subalgebra K° of K with $K^{\circ\circ} \cap A = \mathfrak{p}_y$. The homomorphism $A \to K^\circ$ induces a morphism $\mathrm{Fspec}(K^\circ) \to \mathcal{X}$ that possesses the required property. ∎

5.4.3. Lemma. *The following properties of a quasi-compact morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ are equivalent:*

(a) φ takes strongly closed sets to strongly closed sets;

(b) for every point $y \in \mathcal{Y}$, one has $\varphi(\overline{\{y\}}) = \overline{\{\varphi(y)\}}$;

(c) for every point $y \in \mathcal{Y}$ and a specialization x' of the point $x = \varphi(y)$, there exists a specialization y' of y such that $\varphi(y') = y$.

Proof. The implication (a)$\Longrightarrow$(b) is trivial.

(b)$\Longrightarrow$(c). Replacing $\mathcal{X}$ by by an open p-affine neighborhood of the point x' and $\mathcal{Y}$ by the preimage of that neighborhood in $\mathcal{Y}$, we may assume that $\mathcal{X} = \mathrm{Fspec}(A)$ is affine. We can also replace $\mathcal{X}$ by $\mathrm{Fspec}(A/\Pi_x)$ and $\mathcal{Y}$ by the preimage of the later in $\mathcal{Y}$, and so we may assume that $\mathcal{X}$ is integral and x is its generic point. One has $\Pi_{x'} = \Pi_{\mathfrak{p}}$, where $\mathfrak{p} = \mathfrak{p}_{x'}$. Furthermore, let y'' be a point from $\overline{\{y\}}$ with $\varphi(y'') = x'$. Replacing $\mathcal{Y}$ by an open p-affine neighborhood of y'', we may assume that φ is a morphism of affine schemes $\mathcal{Y} = \mathrm{Fspec}(B) \to \mathcal{X} = \mathrm{Fspec}(A)$. We can also replace $\mathcal{Y}$ by $\mathrm{Fspec}(B/\Pi_y)$, and so we may assume that $\mathcal{Y}$ is integral and y is its generic point. Since the canonical homomorphism $\kappa(x) \to \kappa(y)$ is injective, the homomorphism $A \to B$ is also injective. If $\mathfrak{q} = \mathfrak{p}_{y''}$, then $\mathfrak{q} \cap A = \mathfrak{p}$, and so the point y' of $\mathcal{Y}$ that corresponds to the prime ideal $\Pi_{\mathfrak{q}}$ is a specialization of the point y and one has $\varphi(y') = x'$.

(c)$\Longrightarrow$(a). Let Σ be a strongly closed subset of $\mathcal{Y}$. We may assume that $\mathcal{X} = \mathrm{Fspec}(A)$ is affine and reduced. Since φ is quasi-compact, $\mathcal{Y}$ is a finite union of open p-affine subschemes $\mathcal{Y}_i = \mathrm{Fspec}(B_i)$, $1 \le i \le m$, such that each of the morphisms $\mathcal{Y}_i \to \mathcal{X}$ is induced by a homomorphism of $\mathbf{F}_1$-algebras $A \to B_i$ and $\Sigma \cap \mathcal{Y}_i = \bigcup_{j=1}^{n_i} \mathcal{Y}_{ij}$, where $\mathcal{Y}_{ij} = \mathrm{Fspec}(B_i/F_{ij})$ for ideals F_{ij} of B_i. Let E_{ij} denote the kernel of the induced homomorphism $A \to B_i/F_{ij}$. Then $\varphi(\mathcal{Y}_{ij}) \subset \mathcal{X}_{ij} = \mathrm{Fspec}(A/E_{ij})$. *We claim that* $\varphi(\Sigma) = \bigcup_{i,j} \mathcal{X}_{ij}$ (and, in particular, $\varphi(\Sigma)$ is strongly closed in $\mathcal{X}$). Indeed, let x' be a point from the set on the right hand side, i.e., $x' \in \mathcal{X}_{ij}$ for some $1 \le i \le m$ and $1 \le j \le n_i$. Let $\mathfrak{p}$ be a Zariski ideal of A/E_{ij} such that $\mathcal{X}_{ij}^{(\mathfrak{p})}$ is an irreducible component of $\mathcal{X}_{ij}$ that contains the point x'. Then there exists a generization x of x' in $\check{\mathcal{X}}_{ij}^{(\mathfrak{p})}$. By Theorem 3.2.2(i), we can find an irreducible component $\mathcal{Y}_{ij}^{(\mathfrak{q})}$ of $\mathcal{Y}_{ij}$ with $\varphi(\check{\mathcal{Y}}_{ij,\mathfrak{q}}) = \check{\mathcal{X}}_{ij,\mathfrak{p}}$. Let y be a point in $\check{\mathcal{Y}}_{ij,\mathfrak{q}}$ with $\varphi(y) = x$. By the assumption (c), there exists a specialization y' of y in $\mathcal{Y}$ with $\varphi(y') = x'$. Since $y \in \Sigma$, then

$y' \in \overline{\{y\}} \subset \Sigma$ and, therefore, $x' \in \varphi(\Sigma)$. ■

Proof of Theorem 5.4.1. Suppose first that φ is separated. Then it is evidently quasi-separated. Any pair of morphisms $\chi_1, \chi_2 : \mathrm{Fspec}(K^\circ) \to \mathcal{Y}$ over $\mathcal{X}$ gives rise to a morphism $\chi : \mathrm{Fspec}(K^\circ) \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$. If x and y are the images of the generic points of $\mathrm{Fspec}(K)$ and $\mathrm{Fspec}(\widetilde{K})$, then $x \succeq y$. Assume that the restrictions of χ_1, χ_2 to $\mathrm{Fspec}(K)$ coincide, then $\chi(x) \in \Delta_\varphi(\mathcal{Y})$. Since the latter set is closed, it follows that $y \in \Delta_\varphi(\mathcal{Y})$ and, therefore, $\chi_1 = \chi_2$. Conversely, suppose that the morphism φ is quasi-separated and the map considered is injective. To show that $\Delta_\varphi(\mathcal{Y})$ is closed in $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$, we apply the criterion of Lemma 5.4.3. It suffices to verify that, for every point $y \in \mathcal{Y}$, each specialization z' of $z = \Delta_\varphi(y)$ in $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ lies in $\Delta_\varphi(\mathcal{Y})$. By Lemma 5.4.2, there exists a morphism $\chi : \mathrm{Fspec}(K^\circ) \to \mathcal{Y} \times_{\mathcal{X}} \mathcal{Y}$ from the spectrum of a valuation $\mathbf{F}_1$-algebra K° such that the images of the generic points of $\mathrm{Fspec}(K)$ and $\mathrm{Fspec}(\widetilde{K})$ are z and z', respectively. We get two morphisms $\chi_1, \chi_2 : \mathrm{Fspec}(K^\circ) \to \mathcal{Y}$ which are compositions of the above morphism with the canonical projections $\mathcal{Y} \times_{\mathcal{X}} \mathcal{Y} \to \mathcal{Y}$. Since $z \in \Delta_\varphi(\mathcal{Y})$, the restrictions of χ_1 and χ_2 to $\mathrm{Fspec}(K)$ coincide, the assumption implies that $\chi_1 = \chi_2$ and, in particular, $z' \in \Delta_\varphi(\mathcal{Y})$. ■

5.4.4. Theorem. *A separated morphism of finite type $\varphi : \mathcal{Y} \to \mathcal{X}$ is proper if and only if, for any morphism $\psi : \mathrm{Fspec}(K^\circ) \to \mathcal{X}$ from the spectrum of a valuation $\mathbf{F}_1$-algebra K°, the following map is bijective*

$$\mathrm{Hom}_{\mathcal{X}}(\mathrm{Fspec}(K^\circ), \mathcal{Y}) \to \mathrm{Hom}_{\mathcal{X}}(\mathrm{Fspec}(K), \mathcal{Y}) \ .$$

5.4.5. Lemma. *Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a quasi-compact morphism, and assume that the scheme $\mathcal{Y}$ is quasi-separated. Then φ is closed if and only if it possesses the equivalent properties of Lemma 5.4.3.*

Proof. If φ is closed, it possesses the property (b) of Lemma 5.4.3. Conversely, suppose φ possesses the property (a), i.e., it takes strongly closed sets to strongly closed sets. To prove the required fact, we may assume that $\mathcal{X}$ is affine. In this case $\mathcal{Y}$ is quasi-compact, i.e., $\mathcal{Y}$ is a finite union of open p-affine subschemes $\mathcal{Y}_i = \mathrm{Fspec}(B_i)$, $1 \leq i \leq m$, such that each of the morphisms $\varphi_i : \mathcal{Y}_i \to \mathcal{X}$ is induced by a homomorphism of $\mathbf{F}_1$-algebras $A \to B_i$. Let Σ be a closed subset of $\mathcal{Y}$. We have to show that every point $x \in \mathcal{X} \backslash \varphi(\Sigma)$ has an open neighborhood that does not intersect $\varphi(\Sigma)$. Every point $y \in \varphi_i^{-1}(x)$ has an open neighborhood $\mathcal{V}_y$ of the form $\bigcap_{j=1}^k D(a_j, b_j)$ for some $a_j, b_j \in B_i$ and $k \geq 1$. The set $\Sigma'_y = \mathcal{Y}_i \backslash \mathcal{V}_y$ is strongly closed in $\mathcal{Y}_i$ and contains the set $\Sigma \cap \mathcal{Y}_i$. Since the set $\varphi_i^{-1}(x)\mathrm{Fspec}(B_i \otimes_A \kappa(x)$ is quasi-compact, we can cover it by a finite number of sets of the above form $\mathcal{V}_y$. If Σ'_i is the finite intersection of the corresponding strongly closed sets Σ_y then, by Proposition 5.3.10, the set $\Sigma_i = \Sigma'_i \cup (\mathcal{Y} \backslash \mathcal{Y}_i)$ is strongly closed in $\mathcal{Y}$. Thus, we get a strongly closed set $\Sigma' = \bigcap_{i=1}^m \Sigma_i$ that contains Σ and has empty intersection with $\varphi^{-1}(x)$. The assumption on φ implies that $\varphi(\Sigma')$ is a strongly closed subset of $\mathcal{X}$. Since it contains $\varphi(\Sigma)$ and does not contain the point x, the required fact follows. ∎

5.4.6. Corollary. *Finite morphisms are proper.*

Proof. By Definitions 5.3.2 and 4.6.6, it suffices to consider the case of a finite morphism of affine schemes $\varphi : \mathcal{Y} = \mathrm{Fspec}(B) \to \mathcal{X} = \mathrm{Fspec}(A)$ and, by Lemma 5.4.5, it suffices to show that the image $\varphi(\mathcal{Y})$ is strongly closed in $\mathcal{X}$. We can replace A by A/E, where E is the kernel of the homomorphism $A \to B$, and so we may assume that the latter homomorphism is injective. In this case Proposition 2.6.5(ii) implies that $\varphi(\mathcal{Y}) = \mathcal{X}$. ∎

Proof of Theorem 5.4.4. Suppose fist that φ is proper. Then the map considered is injective, by Theorem 5.4.1. To establish its bijectivity, we can replace $\mathcal{X}$ by $\mathrm{Fspec}(K^\circ)$ and $\mathcal{Y}$ by the base change $\mathcal{Y} \times_{\mathcal{X}} \mathrm{Fspec}(K^\circ)$ with respect to the morphism ψ, and we have to show that any morphism $\sigma : \mathrm{Fspec}(K) \to \mathcal{Y}$ over $\mathcal{X} = \mathrm{Fspec}(K^\circ)$ extends to a morphism $\mathcal{X} \to \mathcal{Y}$. Let x and x' be the generic points of $\mathrm{Fspec}(K)$ and $\mathrm{Fspec}(\widetilde{K})$, respectively, and set $y = \sigma(x)$. By Lemma 5.4.3, there exists a specialization y' of y with $\varphi(y') = x'$. Let $\mathcal{V} = \mathrm{Fspec}(B)$ be an open p-affine neighborhood of the point y'. Then $\overline{\{y\}} \cap \mathcal{U} = \mathrm{Fspec}(B^{(y)})$, where $B^{(y)} = B/\Pi_y$. The morphism φ induces a homomorphism $K^\circ \to B$. Since $\varphi(y) = x$, the composition of the latter with the canonical surjection $B \to B^{(y)}$ is an injective homomorphism $\alpha : K^\circ \to B^{(y)}$. The latter identifies K with the fraction $\mathbf{F}_1$-field of $B^{(y)}$ because the morphism σ is a section of the restriction of φ to $\mathrm{Fspec}(K)$. Let $\mathfrak{q}$ be the Zariski prime ideal $B^{(y)}$ that corresponds to the point y'. Since $\varphi(y') = x'$, it follows that $\alpha^{-1}(\mathfrak{q}) = K^{\circ\circ}$ and, since K° is valuation $\mathbf{F}_1$-algebra, it follows that α is an isomorphism. The inverse isomorphism $B^{(x)} \xrightarrow{\sim} K^\circ$ provides the required extension of the morphism σ.

Conversely, suppose that the map considered is bijective. Since both of these properties are preserved under any base change, it suffices to verify that φ is closed. Let y be a point from $\mathcal{Y}$, and let x' be a specialization of the point $x = \varphi(y)$. The morphism φ defines an embedding of $\mathbf{F}_1$-fields $\kappa(x) \hookrightarrow \kappa(y)$. By Lemma 5.4.2, there exists a valuation $\mathbf{F}_1$-subalgebra $\kappa(y)^\circ$ of $\kappa(y)$ such that the morphism $\mathrm{Fspec}(\kappa(y)) \to \mathcal{X}$ extends to a morphism $\mathrm{Fspec}(\kappa(y)^\circ) \to \mathcal{X}$ which takes the generic point of $\mathrm{Fspec}(\widetilde{\kappa(y)})$ to x'. By the bijectivity, the latter morphism comes from a morphism $\mathrm{Fspec}(\kappa(y)^\circ) \to \mathcal{Y}$ which extends the canonical morphism $\mathrm{Fspec}(\kappa(y)) \to \mathcal{Y}$. The image y' of the generic point of $\mathrm{Fspec}(\widetilde{\kappa(y)})$ in $\mathcal{Y}$ is a specialization of the point y and, by the construction, one has $\varphi(y') = x'$. Lemma 5.4.5 implies that the morphism φ is closed. ∎

5.5. The projective spectrum Proj(A).

5.5.1. Definition. (i) A *graded* $\mathbf{F}_1$*-algebra* is an $\mathbf{F}_1$-algebra A provided with a $\mathbf{Z}_+$*-gradation*, i.e., a direct sum representation by Zariski $\mathbf{F}_1$-submodules $A = \oplus_{i\geq 0} A_i$ such that $A_i \cdot A_j \subset A_{i+j}$ for all $i, j \leq 0$.

(ii) A *graded module* over a graded $\mathbf{F}_1$-algebra A is an A-module M provided with a $\mathbf{Z}$*-gradation*, i.e., a direct sum representation by Zariski $\mathbf{F}_1$-submodules $M = \oplus_{i\in\mathbf{Z}} M_i$ such that $A_i \cdot M_j \subset M_{i+j}$ for all $i, j \in \mathbf{Z}$.

(iii) For M as above and $n \in \mathbf{Z}$, we denote by $M(n)$ the graded A-module with $M(n)_i = M_{i+n}$ for all $i \in \mathbf{Z}$.

For a graded $\mathbf{F}_1$-algebra A, any Zariski ideal $\mathbf{a}$ is homogeneous in the sense that $\mathbf{a} = \oplus_{n\geq 0}\mathbf{a}_n$, where $\mathbf{a}_n = \mathbf{a} \cap A_n$ and, therefore, the quotient $A/\mathbf{a}$ is a graded $\mathbf{F}_1$-algebra, and the canonical homomorphism $A \to A/\mathbf{a}$ is a homomorphism of graded $\mathbf{F}_1$-algebras. The localization $S^{-1}A$ of A with respect to any submonoid S is also a graded $\mathbf{F}_1$-algebra. In particular, the $\mathbf{F}_1$-field $\kappa(\mathfrak{p})$ of any Zariski prime ideal $\mathfrak{p}$ is graded, and the canonical homomorphism $A \to \kappa(\mathfrak{p})$ is a homomorphism of graded $\mathbf{F}_1$-algebras.

5.5.2. Definition. A pair $(a, b) \in A \times A$ is said to be *homogeneous* if $a, b \in A_n$ for some $n \geq 0$. An ideal E of A is said to be *homogeneous* if it consists of homogeneous pairs.

For a homogeneous ideal E, the quotient A/E is a graded $\mathbf{F}_1$-algebra, and the homomorphism $A \to A/E$ is a homomorphism of graded $\mathbf{F}_1$-algebras.

Let A be a graded $\mathbf{F}_1$-algebra, A_+ the Zariski ideal $\oplus_{n\geq 1} A_n$, and $\mathcal{X}$ the set of all homogeneous prime ideals Π of A with $\mathbf{a}_\Pi \not\supset A_+$. We provide $\mathcal{X}$ with the topology whose basis consists of sets of the form $\bigcap_{i=1}^n D_+(a_i, b_i)$ where, for a homogeneous pair $(a, b) \in A \times A$, $D_+(a, b) = \{\Pi \in \mathcal{X} \big| (a, b) \notin \Pi\}$. We notice that $\mathcal{X}$ is covered by open sets of the form $D_+(f) = D_+(f, 0)$ for $f \in A_+$. Notice also that $D_+(fg) = D_+(f) \cap D_+(g)$ for all $f, g \in A_+$. For $f \in A_+$, the localization A_f is provided with the evident $\mathbf{Z}$-gradation. Let $A_{(f)}$ denote the $\mathbf{F}_1$-subalgebra of A_f consisting of elements of degree zero. Notice that, if $f \in A_d$ and $g \in A_e$, there are canonical isomorphisms $(A_{(f)})_{\frac{g^d}{f^e}} \xrightarrow{\sim} A_{(fg)}$ and $(A_{(g)})_{\frac{f^e}{g^d}} \xrightarrow{\sim} A_{(fg)}$.

5.5.3. Lemma. *There is a system of compatible homeomorphisms* $D_+(f) \xrightarrow{\sim} \mathrm{Fspec}(A_{(f)})$.

Proof. We may assume that $f \neq 0$, and let $k = \deg(f)$. If $\Pi \in D_+(f)$, then $\Pi_{(f)} = \{(\frac{a}{f^m}, \frac{b}{f^n}) \in A_{(f)} \times A_{(f)} \big| (af^n, bf^m) \in \Pi\}$ is a prime ideal of $D_+(f)$. Conversely, if Π is a prime ideal of $A_{(f)}$, we define a homogeneous prime ideal $\Pi^{(f)}$ of A as follows. First of all, $\mathbf{a}_{\Pi^{(f)}} = \{a \in A \big| \frac{a^m}{f^n} \in \mathbf{a}_\Pi$ for some $m, n \geq 1\}$ (it is a Zariski prime ideal of A). If (a, b) is a homogeneous pair with both a and b outside $\mathbf{a}_{\Pi^{(f)}}$, then $(a, b) \in \Pi^{(f)}$ if $(\frac{a^k}{f^m}, \frac{a^{k-1}b}{f^m}) \in \Pi$, where $m = \deg(a) = \deg(b)$.

(It is a symmetric equivalence relation since the latter inclusion implies that $(\frac{b^k}{f^m}\cdot\frac{a^k}{f^m},\frac{b^{k-1}a}{f^m}\cdot\frac{a^k}{f^m})=(\frac{a^{k-1}b}{f^m}\cdot\frac{ab^{k-1}}{f^m},\frac{a^k}{f^m}\cdot\frac{ab^{k-1}}{f^m})\in\Pi$ and, therefore, $(\frac{b^k}{f^m},\frac{b^{k-1}a}{f^m})\in\Pi$, i.e., $(b,a)\in\Pi^{(f)}$.) It is easy to see that $\Pi^{(f)}$ is really a homogeneous prime ideal and that the both maps are inverse one to another and continuous. ∎

The family of open subsets $\tau=\{D_+(f)\}_{f\in A_+}$ is a net on $\mathcal{X}$, and Lemma 5.5.3 implies that the correspondence $D_+(f)\mapsto \mathrm{Fspec}(A_{(f)})$ defines a functor $A:\tau\to\mathcal{A}sch^{oi}_{\mathbf{F}_1}$ and an isomorphism of functors $\mathcal{T}^a\circ A\xrightarrow{\sim}\mathcal{T}$, i.e., an affine atlas A with the net τ. Since τ is preserved by intersection, we get a scheme which is called the *projective spectrum of* A and denoted by $\mathrm{Proj}(A)$. Notice that this scheme is separated since, for any pair $f,g\in A_+$, the canonical homomorphism $A_{(f)}\otimes A_{(g)}\to A_{(fg)}$ is surjective. If the Zariski ideal A_+ is generated by elements from A_1 (or equivalently A is generated by A_1 as an A_0-algebra), then $\mathcal{X}=\bigcup_{f\in A_1}D_+(f)$.

Furthermore, given a graded A-module M, the localization M_f with respect to an element $f\in A$ is also provided with the evident $\mathbf{Z}$-gradation. Then the set $M_{(f)}$ of elements of degree zero is an $A_{(f)}$-module, and so it defines a quasi-coherent sheaf of modules $\widetilde{M_{(f)}}$ on the affine scheme $D_+(f)=\mathrm{Fspec}(A_{(f)})$. If $f\in A_d$ and $g\in A_e$, there are canonical isomorphisms $(M_{(f)})_{\frac{g^d}{f^e}}\xrightarrow{\sim}M_{(fg)}$ and $(M_{(g)})_{\frac{f^e}{g^d}}\xrightarrow{\sim}M_{(fg)}$. This means that the restrictions of the sheaves $\widetilde{M_{(f)}}$ on $D_+(f)$ and $\widetilde{M_{(g)}}$ on $D_+(g)$ are canonically isomorphic on the intersection $D_+(fg)$, and there is a well defined quasi-coherent $\mathcal{O}_\mathcal{X}$-module $\widetilde{M}$ on $\mathcal{X}$ whose restriction to each $D_+(f)$ coincides with $\widetilde{M_{(f)}}$. The above isomorphisms also give rise to a canonical injective map $M_{(f)}\to\Gamma(\mathcal{X},\widetilde{M})$ which is a bijection if $\mathcal{X}$ is connected.

For example, one has $\widetilde{A}=\mathcal{O}_\mathcal{X}$. For $n\in\mathbf{Z}$, the $\mathcal{O}_\mathcal{X}$-module $\widetilde{A(n)}$ is denoted by $\mathcal{O}_\mathcal{X}(n)$. Notice that, if $f\in A_d$, the multiplication by f^n gives rise to an isomorphism of $A_{(f)}$-modules $A_{(f)}\xrightarrow{\sim}A(n)_{(f)}$and, therefore, it defines an isomorphism $\mathcal{O}_\mathcal{X}(n)\big|_{D_+(f)}\xrightarrow{\sim}\mathcal{O}_\mathcal{X}\big|_{D_+(f)}$. In particular, if the Zariski ideal A_+ is generated by A_1, the $\mathcal{O}_\mathcal{X}$-module $\mathcal{O}_\mathcal{X}(n)$ is invertible.

For example, for an $\mathbf{F}_1$-algebra S and $n\geq 0$, the $\mathbf{F}_1$-algebra $S[T_0,T_1,\dots,T_n]$ is provided with the evident gradation. The projective spectrum $\mathrm{Proj}(S[T_0,T_1,\dots,T_n])$ is said to be the *projective space over* S and denoted by P^n_S. If A is a graded S-algebra such that the Zariski ideal A_+ is generated by a finite set of elements of A_1 over S, then there is a surjective homomorphism of graded S-algebras $S[T_0,T_1,\dots,T_n]\to A$, $m\geq 0$, which gives rise to a closed immersion $\mathcal{X}=\mathrm{Proj}(A)\to\mathrm{P}^n_S$, and all of the $\mathcal{O}_\mathcal{X}$-modules $\mathcal{O}_\mathcal{X}(m)$ are invertible. In this case, for all $m,n\in\mathbf{Z}$ there are also canonical isomorphisms $\mathcal{O}_\mathcal{X}(m)\otimes_{\mathcal{O}_\mathcal{X}}\mathcal{O}_\mathcal{X}(n)\xrightarrow{\sim}\mathcal{O}_\mathcal{X}(m+n)$ and $\widetilde{M(n)}\xrightarrow{\sim}\widetilde{M}(n)$, where M is a graded A-module and $\widetilde{M}(n)=\widetilde{M}\otimes_{\mathcal{O}_\mathcal{X}}\mathcal{O}_\mathcal{X}(n)$.

§6. The category of schemes $\mathcal{S}ch$

In this section we introduce a category $\mathcal{S}ch$ whose family of objects is a disjoint union of those of the categories $\mathcal{S}ch_{\mathbf{Z}}$ of *schemes over* $\mathbf{Z}$ (i.e., classical schemes) and $\mathcal{S}ch_{\mathbf{F}_1}$ of schemes over $\mathbf{F}_1$. The category $\mathcal{S}ch$ in fact contains $\mathcal{S}ch_{\mathbf{Z}}$ and $\mathcal{S}ch_{\mathbf{F}_1}$ as full subcategories. If $\mathcal{X}$ and $\mathcal{Y}$ are schemes over $\mathbf{F}_1$ and $\mathbf{Z}$, respectively, then the set $\mathrm{Hom}(\mathcal{X},\mathcal{Y})$ of morphisms in $\mathcal{S}ch$ is always empty, but the set $\mathrm{Hom}(\mathcal{Y},\mathcal{X})$ is not necessarily empty, e.g., $\mathrm{Fspec}(\mathbf{F}_1)$ is the final object of $\mathcal{S}ch$. The main feature of the category $\mathcal{S}ch$ is that it admits fiber products.

6.1. Definition of the category $\mathcal{S}ch$. The family of objects of the category of schemes $\mathcal{S}ch$ is defined as the disjoint union of the families of objects of the category $\mathcal{S}ch_{\mathbf{Z}}$ of schemes over $\mathbf{Z}$ and that of the category $\mathcal{S}ch^p_{\mathbf{F}_1}$ of schemes over $\mathbf{F}_1$. The sets of morphisms between two objects of $\mathcal{S}ch_{\mathbf{Z}}$ or of $\mathcal{S}ch^p_{\mathbf{F}_1}$ are defined as the corresponding sets in their categories. Furthermore, let $\mathcal{X}$ and $\mathcal{Y}$ be schemes over $\mathbf{F}_1$ and $\mathbf{Z}$, respectively. We set $\mathrm{Hom}(\mathcal{X},\mathcal{Y})=\emptyset$. A morphism from $\mathcal{Y}$ to $\mathcal{X}$ is a pair consisting of a continuous map $\varphi:\mathcal{Y}\to\mathcal{X}$ and a homomorphism $\nu_\varphi:\mathcal{O}_{\mathcal{X}}\to(\varphi_*\mathcal{O}_{\mathcal{Y}})^{\cdot}$ of sheaves of $\mathbf{F}_1$-algebras (in the schematic topology of $\mathcal{X}$) with the following property: for every point $y\in\mathcal{Y}$, there exist an open affine neighborhood $\mathcal{V}$ of y and an open p-affine neighborhood $\mathcal{U}$ of $\varphi(y)$ such that $\varphi(\mathcal{V})\subset\mathcal{U}$ and the map $\varphi:\mathcal{V}\to\mathcal{U}$ coincides with that induced by the homomorphism of $\mathbf{F}_1$-algebras $A_{\mathcal{U}}\to B^{\cdot}_{\mathcal{V}}$ (which is in its turn induced by ν_φ).

It follows from the definition that the above property holds for every pair consisting of an open affine subscheme $\mathcal{V}\subset\mathcal{Y}$ and an open p-affine subscheme $\mathcal{U}\subset\mathcal{X}$ with $\varphi(\mathcal{V})\subset\mathcal{U}$. It follows also that for any pair of morphisms $\psi:\mathcal{Y}'\to\mathcal{Y}$ and $\chi:\mathcal{X}\to\mathcal{X}'$ there is a well defined composition morphism $\chi\varphi\psi:\mathcal{Y}'\to\mathcal{X}'$. Thus, $\mathcal{S}ch$ is really a category.

6.1.1. Lemma. *The correspondence $\mathcal{Y}'\mapsto\mathrm{Hom}(\mathcal{Y}',\mathcal{X})$ is a sheaf on $\mathcal{Y}$.*

Proof. Let $\{\mathcal{Y}_i\}_{i\in I}$ be a covering of $\mathcal{Y}$ by open subschemes, and suppose we are given a compatible system of morphisms $\varphi_i:\mathcal{Y}_i\to\mathcal{X}$. It is clear that they induce a continuous map $\varphi:\mathcal{Y}\to\mathcal{X}$. Let $\mathcal{V}$ be an open affine subscheme of $\mathcal{Y}$ and $\mathcal{U}$ an open p-affine subscheme of $\mathcal{X}$, and suppose $\varphi(\mathcal{V})\subset\mathcal{U}$. For every $i\in I$, we take a covering $\{\mathcal{V}_{ij}\}_{j\in J_i}$ of $\mathcal{V}\cap\mathcal{Y}_i$ by open affine subschemes. Then we get a compatible system of homomorphisms of $\mathbf{F}_1$-algebras $A_{\mathcal{U}}\to B^{\cdot}_{\mathcal{V}_{ij}}$. Since $B_{\mathcal{V}}\stackrel{\sim}{\to}\mathrm{Ker}(\prod B_{\mathcal{V}_{ij}}\rightrightarrows\prod B_{\mathcal{V}_{ij}\cap\mathcal{V}_{kl}})$, that system is induced by a unique homomorphism $A_{\mathcal{U}}\to B^{\cdot}_{\mathcal{V}}$. In this way we get a homomorphism of sheaves of $\mathbf{F}_1$-algebras $\nu_\varphi:\mathcal{O}_{\mathcal{X}}\to(\varphi_*\mathcal{O}_{\mathcal{Y}})^{\cdot}$. That it satisfies the required property is trivial. It follows that the morphisms φ_i's are induced by a unique morphism $\varphi:\mathcal{Y}\to\mathcal{X}$. ∎

6.1.2. Lemma. *If* $\mathcal{X} = \mathrm{Fspec}(A)$ *is affine, then* $\mathrm{Hom}(\mathcal{Y}, \mathcal{X}) = \mathrm{Hom}(A, \mathcal{O}(\mathcal{Y})^{\cdot})$.

Proof. Lemma 6.1.1 reduces the situation to the case when $\mathcal{Y} = \mathrm{Spec}(B)$ is also affine. By Proposition 4.4.8, any homomorphism of $\mathbf{F}_1$-algebras $A \to B^{\cdot}$ extends in a unique way to a compatible system of homomorphisms $A_{\mathcal{U}} \to B^{\cdot}_{\mathcal{V}}$ for all pairs of open affine subschemes $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{V} \subset \mathcal{Y}$ with $\varphi(\mathcal{V}) \subset \mathcal{U}$. We have to extend the homomorphisms $A_{\mathcal{U}} \to B^{\cdot}_{\mathcal{V}}$ to similar pairs in which $\mathcal{U}$ is an open p-affine subscheme. For this we take a covering $\{\mathcal{U}_i\}_{i \in I}$ of $\mathcal{U}$ by pairwise disjoint open affine subschemes. By Proposition 4.4.8, each $\mathcal{V}_i = \varphi^{-1}(\mathcal{U}_i)$ is an open affine subscheme of $\mathcal{Y}$, and they form a finite covering of $\mathcal{V}$. One therefore has $B_{\mathcal{V}} \xrightarrow{\sim} \prod_{i \in I} B_{\mathcal{V}_i}$. This gives a homomorphism of $\mathbf{F}_1$-algebras $A_{\mathcal{U}} \to \prod_{i \in I} A_{\mathcal{U}_i} \to B^{\cdot}$ which induces a continuous map $\mathcal{V} \to \mathcal{U}$ that coincides with the map $\varphi\big|_{\mathcal{V}}$. ∎

6.1.3. Proposition. *The category* $\mathcal{S}ch$ *admits fiber products.*

Proof. First of all, it is trivial that the canonical fully faithful functor $\mathcal{S}ch_{\mathbf{Z}} \to \mathcal{S}ch$ commutes with fiber products. Furthermore, Lemma 6.1.2 implies that the canonical functor $\mathcal{A}sch_{\mathbf{F}_1} \to \mathcal{S}ch$ commutes with fiber products. One deduces from this using the reasoning from the proof of Proposition 5.3.1 that the canonical fully faithful functor $\mathcal{S}ch_{\mathbf{F}_1} \to \mathcal{S}ch$ commutes with fiber products. Finally, suppose we are given a morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ from a scheme $\mathcal{Y}$ over $\mathbf{Z}$ and a morphism $f : \mathcal{X}' \to \mathcal{X}$ of schemes over $\mathbf{F}_1$. Construction of the fiber product $\mathcal{Y}' = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ is done in several steps.

Step 1. Suppose that f is a morphism of affine schemes $\mathcal{X}' = \mathrm{Fspec}(A') \to \mathcal{X} = \mathrm{Fspec}(A)$ and φ is a morphism $\mathcal{Y} = \mathrm{Spec}(B) \to \mathcal{X}$. The latter is defined by a homomorphism of $\mathbf{F}_1$-algebras $\varphi^* : A \to B^{\cdot}$ and enables one to view the $\mathbf{F}_1$-algebra $C^{\cdot}$ of every B-algebra C as an A-algebra. It is easy to see that the quotient $B_{\varphi}[A']$ of the B-algebra of polynomials $B[T_{a'}]_{a' \in A'}$ by the ideal generated by the elements $T_{a'_1 a'_2} - T_{a'_1} T_{a'_2}$ with $a'_1, a'_2 \in A'$ and $T_{f^*(a)} - \varphi^*(a)$ with $a \in A$ represents the covariant functor $C \mapsto \mathrm{Hom}_A(A', C^{\cdot})$. Lemma 6.1.2 implies that $\mathrm{Fspec}(B_{\varphi}[A'])$ is a fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ in $\mathcal{S}ch$.

Step 2. Suppose that φ is the same as in Step 1, but f is a p-morphism of affine schemes as in Step 1. It is defined by morphisms $f_i : \mathcal{U}'_i \to \mathcal{X}$ for a finite covering $\{\mathcal{U}'_i\}_{i \in I}$ of $\mathcal{X}'$ by pairwise disjoint open affine subschemes. *We claim that the affine scheme* $\mathcal{Y}'$ *which is a finite disjoint union* $\mathcal{Y}'$ *of the affine schemes* $\mathcal{Y}_i = \mathcal{Y} \times_{\mathcal{X}} \mathcal{U}_i$ *is a fiber product* $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ *in* $\mathcal{S}ch$. Indeed, given morphisms $g : \mathcal{Z} \to \mathcal{Y}$ and $\psi : \mathcal{Z} \to \mathcal{X}'$ with $\varphi g = f \psi$, we set $\mathcal{Z}_i = \psi^{-1}(\mathcal{U}'_i)$. By Step 2, there are canonical morphisms $\mathcal{Z}_i \to \mathcal{Y}_i$ which induce a canonical morphism $\mathcal{Z} \to \mathcal{Y}'$ whose composition with the projections to $\mathcal{Y}$ and $\mathcal{X}'$ coincide with g and ψ, respectively. The claim follows.

Notice that in this case, given open subschemes $\mathcal{V} \subset \mathcal{Y}$ and $\mathcal{U}' \subset \mathcal{X}$, the preimage of $\mathcal{V} \times \mathcal{U}'$ in $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ is a fiber product $\mathcal{V} \times_{\mathcal{X}} \mathcal{U}$.

Step 3. *A fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ exists if $\mathcal{X}$ is affine.* Indeed, take coverings $\{\mathcal{V}_i\}$ of $\mathcal{Y}$ and $\{\mathcal{U}'_k\}$ of $\mathcal{X}'$ by open affine and p-affine subschemes, respectively. Lemma 6.1.1 easily implies that the scheme $\mathcal{Y}'$ obtained by gluing all $\mathcal{V}_i \times_{\mathcal{X}} \mathcal{U}'_k$ along $(\mathcal{V}_i \cap \mathcal{V}_j) \times_{\mathcal{X}} (\mathcal{U}'_k \cap \mathcal{U}'_l)$ is a fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$.

Step 4. *A fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ exists in the general case.* Indeed, if the morphisms φ and f go through a morphisms to an open p-affine subscheme $\mathcal{U}$, then $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}' = \mathcal{Y} \times_{\mathcal{U}} \mathcal{X}'$. In the general case, we take a covering $\{\mathcal{U}_i\}$ of $\mathcal{X}$ by open p-affine subschemes. Then the scheme $\mathcal{Y}'$ obtained by gluing all $\varphi^{-1}(\mathcal{U}_i) \times_{\mathcal{X}} f^{-1}(\mathcal{U}_i)$ along $\varphi^{-1}(\mathcal{U}_i \cap \mathcal{U}_j) \times_{\mathcal{X}} f^{-1}(\mathcal{U}_i \cap \mathcal{U}_j)$ is a fiber product of $\mathcal{Y}$ and $\mathcal{X}'$ over $\mathcal{X}$. ∎

Given morphisms $f : \mathcal{X}' \to \mathcal{X}$ of schemes over $\mathbf{F}_1$ and $\varphi : \mathcal{Y} \to \mathcal{X}$ from a scheme $\mathcal{Y}$ over $\mathbf{Z}$, if $\mathcal{X} = \mathrm{Fspec}(A)$ and $\mathcal{Y} = \mathrm{Spec}(B)$ are affine, the fiber product $\mathcal{X}' \times_{\mathcal{X}} \mathcal{Y}$ will be denoted by $\mathcal{X}' \otimes_A B$. For example, given a scheme $\mathcal{X}$ over $\mathbf{F}_1$, any morphism $\mathcal{Y} \to \mathcal{X}$ from a scheme over $\mathbf{Z}$ goes through a unique morphism $\mathcal{Y} \to \mathcal{X} \otimes_{\mathbf{F}_1} \mathbf{Z}$.

6.2. Lifting of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules. Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism from a scheme over $\mathbf{Z}$ to a scheme over $\mathbf{F}_1$. For an $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{G}$, the direct image $\varphi_*\mathcal{G}$ considered as a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules (in the schematic topology of $\mathcal{X}$) will be denoted by $(\varphi_*\mathcal{G})^{\cdot}$. Given a sheaf of $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, consider the covariant functor on the category of $\mathcal{O}_{\mathcal{Y}}$-modules that takes $\mathcal{G}$ to the set of homomorphisms of sheaves of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{F} \to (\varphi_*\mathcal{G})^{\cdot}$.

6.2.1. Proposition. *Suppose that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module. Then*

(i) the above functor is representable by a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-module denoted by $\varphi^*\mathcal{F}$;

(ii) if $\mathcal{F}$ is of finite type or coherent, then so is $\varphi^*\mathcal{F}$;

(iii) the correspondence $\mathcal{F} \mapsto \varphi^*\mathcal{F}$ is a functor which commutes with direct sums and tensor products;

(iv) if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra, then $\varphi^*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-algebra, and it represents the covariant functor that takes an $\mathcal{O}_{\mathcal{Y}}$-algebra $\mathcal{G}$ to the set of homomorphism of $\mathcal{O}_{\mathcal{X}}$-algebras $\mathcal{F} \to (\varphi_*\mathcal{G})^{\cdot}$.

6.2.2. Lemma. *If $\mathcal{X} = \mathrm{Fspec}(A)$ is affine and $\mathcal{F} = \mathcal{O}_{\mathcal{X}}(M)$ for an A-module M, then* $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, (\varphi_*\mathcal{G})^{\cdot}) \xrightarrow{\sim} \mathrm{Hom}_A(M, \mathcal{G}(\mathcal{Y})^{\cdot})$.

Proof. That the map considered is injective is easy. Suppose we are given a homomorphism of A-modules $M \to \mathcal{G}(\mathcal{Y})^{\cdot}$. It induces a system of compatible homomorphisms of $A_{\mathcal{U}}$-modules $M \otimes_A A_{\mathcal{U}} \to \mathcal{G}(\varphi^{-1}(\mathcal{U}))^{\cdot}$ for open affine subschemes $\mathcal{U} \subset \mathcal{X}$. Given a covering of $\mathcal{U}$ by open affine subschemes $\mathfrak{U} = \{\mathcal{U}_i\}_{i \in I}$, one has $\mathcal{G}(\varphi^{-1}(\mathcal{U})) \xrightarrow{\sim} \mathrm{Ker}(\prod_{i \in I} \mathcal{G}(\mathcal{U}_i) \rightrightarrows \prod_{i,j \in I} \mathcal{G}(\varphi^{-1}(\mathcal{U}_i \cap \mathcal{U}_j)))$ and, therefore, a homomorphism $M_{\mathfrak{U}} \to \mathcal{G}(\varphi^{-1}(\mathcal{U}))^{\cdot}$. In their turn the latter induce a system of compatible homomorphisms $\mathcal{F}(\mathcal{U}) = \langle M_{\mathcal{U}} \rangle \to \mathcal{G}(\varphi^{-1}(\mathcal{U}))^{\cdot}$ and, therefore, a homomorphism of sheaves of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{F} \to (\varphi_* \mathcal{G})^{\cdot}$ which gives rise to the homomorphism we started from. ∎

Notice that if, in the situation of Lemma 6.2.2, M is in fact an A-algebra, the same is true for the sets of homomorphisms of $\mathbf{F}_1$-algebras instead of homomorphisms of modules.

6.2.3. Lemma. *Suppose that both $\mathcal{X} = \mathrm{Fspec}(A)$ and $\mathcal{Y} = \mathrm{Spec}(B)$ are affine. Then*

(i) for any A-module M, the covariant functor $N \to \mathrm{Hom}_A(M, N^{\cdot})$ on the category of B-modules is representable by a B-module denoted by $B \otimes_A M$;

(ii) the correspondence $M \mapsto B \otimes_A M$ is a functor that commutes with direct sums and tensor products;

(iii) if M is an A-algebra, then $B \otimes_A M$ is a B-algebra that represents the covariant functor that takes a B-algebra N to the set of homomorphisms of A-algebras $M \to N^{\cdot}$.

Proof. Let f denote the homomorphism $A \to B^{\cdot}$ that induces φ. The functor considered is representable by the quotient of the free B-module $\oplus_{m \in M} BT_m$ by the B-submodule generated by the elements $T_{am} - f(a)T_m$ with $m \in M$ and $a \in A$, i.e., the statement (i) is true. Notice that, if M is a quotient of a free A-module $A^{(I)}$ by an A-submodule $E \subset M \times M$, then $B \otimes_A M$ is also the quotient of the free B-module $\oplus_{i \in I} BT_i$ by the B-submodule generated by the elements $a'T_i - a''T_j$ with $(a't_i, a''t_j) \in E$, where t_i is the image of the canonical i-th generator of $A^{(I)}$. In particular, if M is finite or finitely presented, then so is $B \otimes_A M$. The statements (ii) and (iii) easily follow from (i). ∎

Proof of Proposition 6.2.1. The situation is easily reduced to the case when both $\mathcal{X} = \mathrm{Fspec}(A)$ and $\mathcal{Y} = \mathrm{Spec}(B)$ are affine.

Step 1. *For any open affine subscheme $\mathcal{U} \subset \mathcal{X}$, one has $(B \otimes_A M)_{\varphi^{-1}(\mathcal{U})} \xrightarrow{\sim} B_{\varphi^{-1}(\mathcal{U})} \otimes_{A_{\mathcal{U}}} M_{\mathcal{U}}$.* Indeed, this is trivial if $\mathcal{U}$ is a principal open subset or defined by vanishing a finite number of idempotents and, therefore, this is true if $\mathcal{U}$ is an elementary open subset. If $\mathcal{U}$ is arbitrary, we take an elementary family $\{\mathcal{U}_i\}_{i \in \check{I}}$ that covers $\mathcal{U}$. Then $B_{\varphi^{-1}(\mathcal{U})} \xrightarrow{\sim} \prod_{i \in \check{I}} B_{\varphi^{-1}(\mathcal{U}_i)}$ and, therefore, $(B \otimes_A M)_{\varphi^{-1}(\mathcal{U})} \xrightarrow{\sim} \prod_{i \in \check{I}} B_{\varphi^{-1}(\mathcal{U}_i)} \otimes_{A_{\mathcal{U}_i}} M_{\mathcal{U}_i}$. For the same reason, the right hand side coincides with $B_{\varphi^{-1}(\mathcal{U})} \otimes_{A_{\mathcal{U}}} M_{\mathcal{U}}$, and the claim follows.

Step 2. *The functor $M \mapsto \mathcal{O}_{\mathcal{Y}}(B \otimes_A M)$ is extended to a functor* $\mathrm{Qcoh}(\mathcal{X}) \to \mathrm{Qcoh}(\mathcal{Y})$

that takes $\mathcal{F}$ to a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-module $\varphi^*\mathcal{F}$. Recall that, for A-modules M and P, one has $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}(M), \mathcal{O}_{\mathcal{X}}(P)) = \varinjlim \mathrm{Hom}_A(M, P_{\mathfrak{U}})$, where the inductive limit is taken over finite coverings $\mathfrak{U}$ of $\mathcal{X}$ by open affine subschemes. Step 1 and acyclicity of quasi-coherent modules on affine schemes over $\mathbf{Z}$ imply that the canonical homomorphism $P \to (B \otimes_A P)^{\cdot}$ extends in a canonical way to a homomorphism $P_{\mathfrak{U}} \to (B \otimes_A P)^{\cdot}$. This gives the required extension of the functor considered to the essential image of the category of A-modules in $\mathrm{Qcoh}(\mathcal{X})$. Suppose now that $\mathcal{O}_{\mathcal{X}}(M)$ is an arbitrary object of the category $\mathrm{Qcoh}(\mathcal{X})$. We may assume that there is a finite covering of $\mathcal{X}$ by pairwise disjoint open affine subschemes $\{\mathcal{U}_i\}_{i \in I}$ such that $\mathcal{O}_{\mathcal{X}}(M)$ is associated with a system of $A_{\mathcal{U}_i}$-modules $M_{\mathcal{U}_i}$. By the previous case, each $M_{\mathcal{U}_i}$ gives rise to a quasi-coherent $\mathcal{O}_{\varphi^{-1}(\mathcal{U}_i)}$-module on $\varphi^{-1}(\mathcal{U}_i)$, and all of them define the required quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-module.

Step 3. *The coherent $\mathcal{O}_{\mathcal{Y}}$-module $\varphi^*\mathcal{F}$ possesses all of the required properties.* Indeed, it suffices to verify the claim in the case $\mathcal{F} = \mathcal{O}_{\mathcal{X}}(M)$ for an A-module M. In this case $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, (\varphi_*\mathcal{G})^{\cdot}) \overset{\sim}{\to} \mathrm{Hom}_A(M, \mathcal{G}(\mathcal{Y})^{\cdot})$, by Lemma 6.2.2. By Lemma 6.2.3, the latter coincides with $\mathrm{Hom}_B(B \otimes_A M, \mathcal{G}(\mathcal{Y}))$ and, by quasi-coherence, with $\mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}}(\mathcal{O}_{\mathcal{Y}}(B \otimes_A M), \mathcal{G})$, i.e., the property (i) holds. The other properties follow from Lemma 6.2.3. ∎

6.3. The image of the map $\mathcal{Y} \to \mathcal{X}$. Recall that an abelian group is said to be locally cyclic if every subgroup of it generated by a finite number of elements is cyclic. For example, the torsion subgroup of the multiplicative group k^* of any field k is locally cyclic. It follows that, given a morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ from a scheme over $\mathbf{Z}$ to a scheme over $\mathbf{F}_1$, the torsion subgroup of $\kappa(x)^*$ of every point x from the image of φ is locally cyclic.

6.3.1. Proposition. *Let $\mathcal{X}$ be a scheme over $\mathbf{F}_1$, and let k be a field of characteristic zero (resp. $p > 0$). Then the image of the map $\mathcal{X} \otimes_{\mathbf{F}_1} k \to \mathcal{X}$ is the set of points $x \in \mathcal{X}$ with the property that the torsion subgroup of $\kappa(x)^*$ is locally cyclic (resp. and has no elements of order p).*

Proof. Let x be a point of $\mathcal{X}$ with that property, and set $K = \kappa(x)$. We notice that it suffices to show that there exists an embedding $K^* \hookrightarrow k'^*$ for an extension k' of k. Indeed, such an embedding gives rise to a morphism $\mathrm{Spec}(k') \to \mathcal{X}$ whose image is the point x and which goes through a morphism $\mathrm{Spec}(k') \to \mathcal{X} \otimes_{\mathbf{F}_1} k$. If x' is the image of the latter morphism, then the induced homomorphism $K^* \to \kappa(x')^*$ is injective. The required fact is a version of a result of Cohn [Cohn], and here is an easy proof of it.

We may assume that K is infinite, and we can increase the field k and assume that it is algebraically closed and its cardinality is greater than that of K. Then there is an emdedding $K^*_{tors} \hookrightarrow k^*$. Let S be the set of pairs (G, α), where G of a subgroup of K^* that contains K^*_{tors}

and α is an embedding $G \hookrightarrow k^*$. We provide S with a partial ordering as follows: $(G, \alpha) \leq (G', \alpha')$ if $G \subset G'$ and $\alpha'\big|_G = \alpha$. The poset S satisfies the condition of Zorn's Lemma and, therefore, it contains a maximal element (G, α). It suffices to show that $G = K^*$. Suppose this is not true. We then can find an element $\lambda \in K^* \backslash G$. If $\lambda^n \notin G$ for all $n \geq 1$, then we take an arbitrary element $x \in k^*$ transcendental over the subfield of k generated by $\alpha(G)$. (It exists since the cardinality of k is greater than that of K.) If G' is the subgroup of K^* generated by G and λ and α' is the homomorphism $G' \to k^*$ that coincides with α on G and takes λ to x, then the pair (G', α') is an element of S strictly bigger that (G, α), which is a contradiction. Suppose now that $\lambda^n \in G$ for some $n > 1$. We may assume that n is minimal with this property and, therefore, each element of the subgroup G' of K^* generated by G and λ has a unique representation in the form $\lambda^i g$ with $0 \leq i \leq n-1$ and $g \in G$. Let x be an element of k with $x^n = \alpha(\lambda^n)$. If α' is the homomorphism of $G' \to k^*$ that coincides with α on G and takes λ to x, then the pair (G', α') is an element of S strictly bigger that (G, α), which is a contradiction. ∎

Suppose now we are given an $\mathbf{F}_1$-field K, a commutative ring with unity k, and a homomorphism of $\mathbf{F}_1$-algebras $K \to k^\cdot$, i.e., a morphism $\mathrm{Spec}(k) \to \mathrm{Fspec}(K)$. Then for any K-algebra A there are an induced morphism $\mathrm{Spec}(k \otimes_K A) \to \mathrm{Fspec}(A)$ and an induced map $\mathrm{Spec}(k \otimes_K A) \to \mathrm{Zspec}(A)$.

6.3.2. Lemma. *The following properties of a Zariski ideal $\mathfrak{p} \subset A$ are equivalent:*

(a) $\mathfrak{p}$ lies in the image of the map $\mathrm{Spec}(k \otimes_K A) \to \mathrm{Zspec}(A)$;

(b) $k \otimes_K \kappa(\mathfrak{p}) \neq 0$;

(c) the stabilizer of every element $f \in A \backslash \mathfrak{p}$ in K^* lies in $\mathrm{Ker}(K^* \to k^*)$.

Proof. We set $B = k \otimes_K A$. A Zariski ideal $\mathfrak{p} \subset A$ lies in the image of $\mathrm{Spec}(B)$ if and only if there exists an ideal $\mathfrak{q} \subset B$ with $\mathfrak{p} = \mathrm{Zker}(A \to (B/\mathfrak{q})^\cdot)$, The latter is equivalent to the property $B \otimes_A \kappa(\mathfrak{p}) \neq 0$. Since $B \otimes_A \kappa(\mathfrak{p}) = k \otimes_K \kappa(\mathfrak{p})$, the equivalence (a)$\Longleftrightarrow$(b) follows. The equivalence (b)$\Longleftrightarrow$(c) is trivial. ∎

6.3.3. Corollary. *The following properties of a K-algebra A are equivalent:*

(a) the map $\mathrm{Spec}(k \otimes_K A) \to \mathrm{Zspec}(A)$ is surjective;

(b) the stabilizer of every non-nilpotent element $f \in A$ in K^* is contained in $\mathrm{Ker}(K^* \to k^*)$. ∎

6.4. Schemes over k with a topologized prelogarithmic K-structure. Suppose now we are given an $\mathbf{F}_1$-algebra K, a commutative ring with unity k, and a homomorphism of $\mathbf{F}_1$-algebras $\alpha : K \to k^\cdot$, i.e., a morphism $\mathrm{Spec}(k) \to \mathrm{Fspec}(K)$. For a scheme $\mathcal{X}$ over K, we set $\mathcal{X}^{(\alpha)} = \mathcal{X} \otimes_K k$, and denote by π the morphism $\mathcal{X}^{(\alpha)} \to \mathcal{X}$.

6.4.1. Definition. (i) A scheme $\mathcal{X}$ over K is said to be α-*nontrivial* if every point of $\mathcal{X}$ has an open p-affine neighborhood $\mathcal{U}$ for which the map $\mathcal{U}^{(\alpha)} \to \mathrm{Zspec}(A_{\mathcal{U}})$ is surjective.

(ii) The full subcategory of $\mathcal{S}ch_K$ of schemes over K consisting of locally connected α-nontrivial schemes is denoted by $\mathcal{S}ch_K^{[\alpha]}$.

For example, if K is an $\mathbf{F}_1$-field, Corollary 6.3.3 implies that an affine scheme $\mathcal{X} = \mathrm{Fspec}(A)$ over K is k/K-nontrivial if and only if the stabilizer of every non-nilpotent element $f \in A$ in K^* is contained in $\mathrm{Ker}(K^* \to k^*)$.

6.4.2. Lemma. *Suppose that $\mathcal{X}$ is α-nontrivial. If $\pi^{-1}(\mathcal{U}) \subset \pi^{-1}(\mathcal{V})$ for open subschemes of $\mathcal{X}$, then $\mathcal{U} \subset \mathcal{V}$. In particular, if $\pi^{-1}(\mathcal{U}) = \pi^{-1}(\mathcal{V})$, then $\mathcal{U} = \mathcal{V}$.*

Proof. The situation is easily reduced to the case when $\mathcal{X} = \mathrm{Fspec}(A)$ is affine. Since the map $\mathcal{X} \otimes_K k \to \mathrm{Zspec}(A)$ is surjective, it follows that the images of the sets $\pi^{-1}(\mathcal{U})$ and $\pi^{-1}(\mathcal{V})$ coincide with the sets of Zariski prime ideals $\mathfrak{p} \subset A$ with $\mathcal{U} \cap \mathcal{X}^{(\mathfrak{p})} \neq \emptyset$ and $\mathcal{V} \cap \mathcal{X}^{(\mathfrak{p})} \neq \emptyset$, respectively. It remains to notice that, if $\mathcal{V} \cap \mathcal{X}^{(\mathfrak{p})} \neq \emptyset$, then $\mathcal{X}^{(\mathfrak{p})} \subset \mathcal{V}$. ∎

6.4.3. Definition. (i) A scheme over k with a *topologized prelogarithmic K-structure* is a quadruple $(\mathcal{Y}, \sigma, \mathcal{A}, \nu)$ consisting of a scheme $\mathcal{Y}$ over k, a topology σ on $\mathcal{Y}$ formed by open subschemes, a σ-sheaf of K-algebras $\mathcal{A}$, and a homomorphism of σ-sheaves of K-algebras $\nu : \mathcal{A} \to \mathcal{O}_{\mathcal{Y}}\big|_{\sigma}$ which is compatible with the homomorphism α.

(ii) A morphism $(\mathcal{Y}, \sigma, \mathcal{A}, \nu) \to (\mathcal{Y}', \sigma', \mathcal{A}', \nu')$ is a pair consisting of a morphism of schemes over k, $\varphi : \mathcal{Y} \to \mathcal{Y}'$, which is σ-continuous (i.e., $\varphi^{-1}(\mathcal{V}')$ is σ-open for all σ'-open subschemes $\mathcal{V}' \subset \mathcal{Y}'$) and a homomorphism of σ-sheaves of K-algebras $\mathcal{A}' \to \varphi_* \mathcal{A}$ which is compatible with the homomorphism $\mathcal{O}_{\mathcal{Y}'} \to \varphi_* \mathcal{O}_{\mathcal{Y}}$.

(iii) The category of schemes over k with a topologized prelogarithmic K-structure is denoted by $\mathcal{S}ch_k^{[\alpha]}$.

If $(\mathcal{Y}, \sigma, \mathcal{A}, \nu)$ is an object of $\mathcal{S}ch_k^{[\alpha]}$, every σ-open subscheme $\mathcal{V} \subset \mathcal{Y}$ gives rise to an object of $\mathcal{S}ch_k^{[\alpha]}$, namely, $(\mathcal{V}, \sigma\big|_{\mathcal{V}}, \mathcal{A}\big|_{\mathcal{V}}, \nu\big|_{\mathcal{V}})$. (In the formulation of Theorem 6.4.4(ii), $\mathcal{V}$ denotes the latter tuple.) A morphism $(\mathcal{Y}', \sigma', \mathcal{A}', \nu') \to (\mathcal{Y}, \sigma, \mathcal{A}, \nu)$ in $\mathcal{S}ch_k^{[\alpha]}$ is said to be an *open immersion* if it gives rise to an isomorphism of the first tuple with the object of $\mathcal{S}ch_k^{[\alpha]}$ induced by a σ-open subscheme of $\mathcal{Y}$.

6.4.4. Theorem *(i) The correspondence $\mathcal{X} \mapsto \mathcal{X}^{(\alpha)}$ gives rise to a fully faithful functor*

$$\mathcal{S}ch_K^{[\alpha]} \to \mathcal{S}ch_k^{[\alpha]} \ ;$$

(ii) an object $(\mathcal{Y}, \sigma, \mathcal{A}, \nu)$ of $\mathcal{S}ch_k^{[\alpha]}$ lies in the essential image of the above functor if and only if the σ-topology admits a base b_σ with the following properties:

(1) for every $\mathcal{V} \in b_\sigma$, $\mathcal{A}(\mathcal{V})$ is a K-algebra with connected and locally connected spectrum, and the canonical morphism $\mathcal{V} \to \mathrm{Spec}(\mathcal{A}(\mathcal{V})] \otimes_K k)$ is an isomorphism in $\mathcal{S}ch_k^{[\alpha]}$;

(2) for every pair $\mathcal{V} \subset \mathcal{W}$ in b_σ, the canonical homomorphism $\mathcal{A}(\mathcal{W}) \to \mathcal{A}(\mathcal{V})$ induces an open immersion of affine schemes $\mathrm{Fspec}(\mathcal{A}(\mathcal{V})) \to \mathrm{Fspec}(\mathcal{A}(\mathcal{W}))$ which is compatible with the open immersion $\mathcal{V} \to \mathcal{W}$.

Proof. (i) Let $\mathcal{X}$ be locally connected α-nontrivial scheme over K. The scheme $\mathcal{X}^{(\alpha)}$ is provided with the topology σ whose open sets are the preimages of open subschemes of $\mathcal{X}$ with respect to the map $\pi : \mathcal{X}^{(\alpha)} \to \mathcal{X}$. If π_σ denotes the map from $\mathcal{X}^{(\alpha)}$, provided with the σ-topology, to $\mathcal{X}$, provided with the schematic topology, the homomorphism $\mathcal{O}_\mathcal{X} \to \pi_* \mathcal{O}^{\cdot}_{\mathcal{X}^{(\alpha)}}$ induces a homomorphism $\nu : \pi_\sigma^* \mathcal{O}_\mathcal{X} \to \mathcal{O}^{\cdot}_{\mathcal{X}^{(\alpha)}}\big|_\sigma$. The tuple $(\mathcal{X}^{(\alpha)}, \sigma, \pi_\sigma^* \mathcal{O}_\mathcal{X}, \nu)$ is an object of the category $\mathcal{S}ch_k^{[\alpha]}$. That the correspondence $\mathcal{X} \mapsto (\mathcal{X}^{(\alpha)}, \sigma, \pi_\sigma^* \mathcal{O}_\mathcal{X}, \nu)$ is a faithful functor is easy. To show that it is fully faithful (and to prove (ii)), we need the following simple fact.

6.4.5. Lemma. *One has $\mathcal{O}_\mathcal{X} \stackrel{\sim}{\to} \pi_{\sigma *} \pi_\sigma^* \mathcal{O}_\mathcal{X}$.*

Proof. Since $\mathcal{X}$ is locally connected, it suffices to verify that, for a connected open affine subscheme $\mathcal{U}$ of $\mathcal{X}$, one has $(\pi_\sigma^* \mathcal{O}_\mathcal{X})(\mathcal{U}^{(\alpha)}) = A_\mathcal{U}$. The σ-open affine subscheme $\mathcal{U}^{(\alpha)} = \pi^{-1}(\mathcal{U})$ contains a point x with $\pi(x) \in \mathcal{U}_\mathbf{m}$. Since $\mathcal{U}$ is the minimal open subscheme of $\mathcal{X}$ that contains a point from $\mathcal{U}_\mathbf{m}$, Lemma 6.4.2 implies that $\mathcal{U}^{(\alpha)}$ is the minimal σ-open subscheme of $\mathcal{X}^{(\alpha)}$ that contains the point x and, therefore, any σ-open covering $\{\mathcal{V}_i\}_{i \in I}$ of $\mathcal{U}^{(\alpha)}$ one has $\mathcal{U}^{(\alpha)} \subset \mathcal{V}_i$ for some $i \in I$. The required fact follows. ∎

Let $\varphi : (\mathcal{X}^{(\alpha)}, \sigma, \pi^* \mathcal{O}_\mathcal{X}, \nu) \to (\mathcal{X}'^{(\alpha)}, \sigma', \pi'^\alpha \mathcal{O}_{\mathcal{X}'}, \nu')$ be a morphism in $\mathcal{S}ch_k^{[\alpha]}$. Given a connected open affine subscheme $\mathcal{U} \subset \mathcal{X}$, take a point $x \in \mathcal{U}^{(\alpha)}$ with $p(x) \in \mathcal{U}_\mathbf{m}$ and a connected open affine subscheme $\mathcal{U}' \subset \mathcal{X}'$ with $\varphi(x) \in \mathcal{U}'^{(\alpha)}$. *We claim that $\varphi(\mathcal{U}^{(\alpha)}) \subset \mathcal{U}'^{(\alpha)}$.* Indeed, since the map φ is σ-continuous, $\varphi^{-1}(\mathcal{U}'^{(\phi)}) = \pi^{-1}(\mathcal{V})$ for some open subscheme $\mathcal{V} \subset \mathcal{X}$ and, since $\mathcal{U}$ is the minimal open subscheme of $\mathcal{X}$ that contains the point $\pi(x)$, it follows that $\mathcal{U} \subset \mathcal{V}$ and, therefore, $\mathcal{U}^{(\alpha)} \subset \varphi^{-1}(\mathcal{U}'^{(\alpha)})$.

Furthermore, suppose we are given connected open affine subschemes $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{U}' \subset \mathcal{X}'$ with $\varphi(\mathcal{U}^{(\alpha)}) \subset \mathcal{U}'^{(\alpha)}$. By Lemma 6.4.5, one has $(\pi_\sigma^* \mathcal{O}_\mathcal{X})(\mathcal{U}^{(\alpha)}) = A_\mathcal{U}$ and $(\pi'^*_{\sigma'} \mathcal{O}_{\mathcal{X}'}) = A'_{\mathcal{U}'}$ and, therefore, the homomorphism $\pi'^*_{\sigma'} \mathcal{O}_{\mathcal{X}'} \to \varphi_*(\pi_\sigma^* \mathcal{O}_\mathcal{X})$ induces a homomorphism $A_{\mathcal{U}'} \to A_\mathcal{U}$ which is compatible with the homomorphism $A'_{\mathcal{U}'} \otimes_K k \to A_\mathcal{U} \otimes_K k$. In this way we get a system of compatible homomorphisms of K-algebras $A_{\mathcal{U}'} \to A_\mathcal{U}$ for all pairs of connected open affine subschemes $\mathcal{U} \subset \mathcal{X}$

and $\mathcal{U}' \subset \mathcal{X}'$ with $\varphi(\mathcal{U}^{(\alpha)}) \subset \mathcal{U}'^{(\alpha)}$. This system defines a morphism $\mathcal{X} \to \mathcal{X}'$ of schemes over K which gives rise to the morphism φ.

(ii) If $\mathcal{X}$ is a locally connected α-nontrivial scheme over K, the corresponding object of $\mathcal{S}ch_k^{[\alpha]}$ is the quadruple $(\mathcal{X}^{(\alpha)}, \sigma, \pi_\sigma^* \mathcal{O}_\mathcal{X}, \nu)$, described in the proof of (i). The preimages of the connected open affine subschemes of $\mathcal{X}$ form a base b_σ of the topology σ. That b_σ possesses the properties (1) and (2) follows from Lemma 6.4.5.

Suppose now that $(\mathcal{Y}, \sigma, \mathcal{A}, \nu)$ is an object of $\mathcal{S}ch_k^{[\alpha]}$ with a base b_σ of σ that possesses those properties. For $\mathcal{V} \in b_\sigma$, let $\mathcal{X}_\mathcal{V}$ be the affine scheme $\mathrm{Fspec}(\mathcal{A}(\mathcal{V}))$. The property (1) implies that there is a canonical isomorphism $\mathcal{V} \widetilde{\to} \mathcal{X}_\mathcal{V}^{(\alpha)}$. The property (2) implies that, for every pair $\mathcal{V} \subset \mathcal{W}$, the canonical morphism $\mathcal{X}_\mathcal{V} \to \mathcal{X}_\mathcal{W}$ is an open immersion that induces the open immersion $\mathcal{V} \to \mathcal{W}$. For a pair $\mathcal{V}, \mathcal{W} \in b_\sigma$, let $\mathcal{X}_{\mathcal{V},\mathcal{W}}$ denote the open subscheme of $\mathcal{X}_\mathcal{V}$ which is the union of the images of $\mathcal{X}_\mathcal{U}$, where $\mathcal{U}$ runs through all sets from b_σ for which $\mathcal{X}_\mathcal{U}^{(\alpha)} \subset \mathcal{V} \cap \mathcal{W}$. Notice that, since b_σ is a base of the topology σ, such sets $\mathcal{X}_\mathcal{U}^{(\alpha)}$ cover the intersection $\mathcal{V} \cap \mathcal{W}$, and that there is a well defined isomorphism $\nu_{\mathcal{V},\mathcal{W}} : \mathcal{X}_{\mathcal{V},\mathcal{W}} \widetilde{\to} \mathcal{X}_{\mathcal{W},\mathcal{V}}$. The system $\{\nu_{\mathcal{V},\mathcal{W}}\}$ satisfies the conditions of Lemma 5.2.10 and, therefore, we can glue all $\mathcal{X}_\mathcal{V}$'s along $\mathcal{X}_{\mathcal{V},\mathcal{W}}$'s. In this way we get a locally connected α-nontrivial scheme $\mathcal{X}$ over K with $\mathcal{Y} \widetilde{\to} \mathcal{X}^{(\alpha)}$. ∎

6.5. Schemes over k with a prelogarithmic K-structure. In this subsection $\alpha : K \to k^\cdot$ is a homomorphism as in §6.4.

6.5.1. Definition. (i) An α-nontrivial separated scheme $\mathcal{X}$ over K is said to be *α-special* if it admits a net of connected open affine subschemes σ such that every $\mathcal{U} \in \sigma$ possesses the following properties:

(1) $\mathcal{U}$ is reduced and the set of its irreducible components is finite;

(2) the intersection $\mathcal{U}_M$ of the sets $\mathcal{W}_\mathbf{m}$, where $\mathcal{W}$ runs through all irreducible components of $\mathcal{U}$, is nonempty;

(3) for each Zariski prime ideal $\mathfrak{p} \subset A_\mathcal{U}$, the k-algebra $k \otimes_K \kappa(\mathfrak{p})$ is integral.

(ii) The full subcategory of $\mathcal{S}ch_K$ consisting of α-special schemes is denoted by $\mathcal{S}ch_K^{(\alpha)}$.

The properties (1) and (2) do not depend on the homomorphism α. Notice that, since the canonical homomorphism $A_\mathcal{U}/\Pi_\mathfrak{p} \to \kappa(\mathfrak{p})$ is injective, the property (3) implies that the k-algebra $k \otimes_K A_\mathcal{U}/\Pi_\mathfrak{p}$ is integral and, in particular, there is a one-to-one correspondence between the set of irreducible components of $\mathcal{U}$ and that of $\mathcal{U}^{(\alpha)}$. Here is a simple sufficient condition for validity of the property (3).

6.5.2. Lemma. *Suppose that K is an $\mathbf{F}_1$-field and k is an integral domain. If a ϕ-nontrivial K-algebra A is such that, for every Zariski prime ideal $\mathfrak{p} \subset A$, the group $\mathrm{Coker}(K^* \to \kappa(\mathfrak{p})^*)$ has no torsion, then the affine scheme $\mathcal{X} = \mathrm{Fspec}(A)$ possesses the property (3).*

Notice that if the condition of Lemma 6.5.2 is satisfied by those $\mathfrak{p}$ for which $\mathcal{X}^{(\mathfrak{p})}$ is an irreducible component of $\mathcal{X}$, then it is satisfied by all $\mathfrak{p}$'s.

Proof. Since A is α-nontrivial, we may replace A by $A/\mathrm{Ker}(K^* \to A^*)$ and assume that the homomorphism $K \to A^*$ is injective. Furthermore, to verify the required property for a Zariski prime ideal $\mathfrak{p} \subset A$, we may replace A by $A/\Pi_{\mathfrak{p}}$ and assume that A is an integral domain and $\mathfrak{p} = 0$. If F is the fraction $\mathbf{F}_1$-field of A, the homomorphism $k \otimes_K A \to k \otimes_K F$ is injective and, therefore, it suffices to verify the required fact for F instead of A. Finally, if $F = \varinjlim F_i$, then $k \otimes_K F = \varinjlim k \otimes_K F_i$. We may therefore assume that the group F^*/K^* is finitely generated. If $f_1, \dots, f_n$ are elements of F^* whose images form a basis of the free group F^*/K^*, then $k \otimes_K F = k[f_1, \dots, f_n, \frac{1}{f_1}, \dots, \frac{1}{f_n}]$. The latter is an integral domain because k is an integral domain. ∎

6.5.3. Definition. (i) A scheme over k with a *prelogarithmic K-structure* is a triple $(\mathcal{Y}, \mathcal{A}, \nu)$ consisting of a scheme $\mathcal{Y}$ over k, a sheaf of K-algebras $\mathcal{A}$, and a homomorphism of K-algebras $\nu : \mathcal{A} \to \mathcal{O}^{\cdot}_{\mathcal{Y}}$ which is compatible with the homomorphism α.

(ii) A morphism $(\mathcal{Y}, \mathcal{A}, \nu) \to (\mathcal{Y}', \mathcal{A}', \nu')$ is a pair consisting of a morphism of schemes over k, $\varphi : \mathcal{Y} \to \mathcal{Y}'$, and a homomorphism of sheaves of K-algebras $\mathcal{A}' \to \varphi_*\mathcal{A}$ which is compatible with the homomorphism $\mathcal{O}_{\mathcal{Y}'} \to \varphi_*\mathcal{O}_{\mathcal{Y}}$.

(iii) The category of schemes over k with a prelogarithmic K-structure is denoted by $\mathcal{S}ch_k^{(\alpha)}$.

If $(\mathcal{Y}, \mathcal{A}, \nu)$ is an object of $\mathcal{S}ch_k^{(\alpha)}$, every open subscheme $\mathcal{V} \subset \mathcal{Y}$ gives rise to an object of $\mathcal{S}ch_k^{(\alpha)}$, namely, $(\mathcal{V}, \mathcal{A}\big|_{\mathcal{V}}, \nu\big|_{\mathcal{V}})$. (In the formulation of Theorem 6.5.4(ii), $\mathcal{V}$ denotes the latter triple.) A morphism $(\mathcal{Y}', \mathcal{A}', \nu') \to (\mathcal{Y}, \mathcal{A}, \nu)$ in $\mathcal{S}ch_k^{(\alpha)}$ is said to be an *open immersion* if it gives rise to an isomorphism of the first triple with the object of $\mathcal{S}ch_k^{(\alpha)}$ induced by an open subscheme of $\mathcal{Y}$.

6.5.4. Theorem. *(i) The correspondence $\mathcal{X} \mapsto \mathcal{X}^{(\alpha)}$ gives rise to a fully faithful functor*

$$\mathcal{S}ch_K^{(\alpha)} \to \mathcal{S}ch_k^{(\alpha)} \ ;$$

(ii) an object $(\mathcal{Y}, \mathcal{A}, \nu)$ of $\mathcal{S}ch_k^{(\alpha)}$ lies in the essential image of the above functor if and only if the following holds:

(1) the family of open sets $\mathcal{V}$, for which the K-algebra $\mathcal{A}(\mathcal{V})$ is α-special and the canonical morphism $\mathcal{V} \to \mathrm{Spec}(k \otimes_K \mathcal{A}(\mathcal{V}))$ is an isomorphism in $\mathcal{S}ch_k^{(\alpha)}$, forms a net;

(2) for every pair $\mathcal{V} \subset \mathcal{W}$ of sets from (1), the canonical homomorphism $\mathcal{A}(\mathcal{W}) \to \mathcal{A}(\mathcal{V})$ induces an open immersion of affine schemes $\mathrm{Fspec}(\mathcal{A}(\mathcal{V})) \to \mathrm{Fspec}(\mathcal{A}(\mathcal{W}))$ *which is compatible with the open immersion $\mathcal{V} \to \mathcal{W}$.*

6.5.5. Lemma. *Let $\mathcal{X} = \mathrm{Fspec}(A)$ be a connected α-special affine scheme over K. Then for any open subscheme $\mathcal{V} \subset \mathcal{X}^{(\alpha)}$ that contains a point from $\pi^{-1}(\mathcal{X}_M)$, one has $A \stackrel{\sim}{\to} (\pi^*\mathcal{O}_\mathcal{X})(\mathcal{V})$.*

Proof. The sheaf $\pi^*\mathcal{O}_\mathcal{X}$ is associated with the separated presheaf $P = \pi^{-1}\mathcal{O}_\mathcal{X}$ whose value at an open subset $\mathcal{V} \subset \mathcal{X}^{(\alpha)}$ is the inductive limit $\varinjlim \mathcal{O}(\mathcal{U})$ taken over all open subschemes $\mathcal{U}$ of $\mathcal{X}$ that contain the image $p(\mathcal{V})$. Thus, we have to verify that, given a finite covering of $\mathcal{V}$ with $\mathcal{V} \cap \pi^{-1}(\mathcal{X}_M) \neq \emptyset$ by open subsets $\{\mathcal{V}_\mu\}$, one has $A \stackrel{\sim}{\to} L = \mathrm{Ker}(\prod_\mu P(\mathcal{V}_\mu) \stackrel{\to}{\to} \prod_{\mu,\rho} P(\mathcal{V}_\mu \cap \mathcal{V}_\rho))$.

Let $\{\mathcal{X}_i\}_{i\in I}$ be the set of irreducible components of $\mathcal{X}$. For an open subscheme $\mathcal{U} \subset \mathcal{X}$, we set $I(\mathcal{U}) = \{i \in I \big| \mathcal{U}_i \neq \emptyset\}$, where $\mathcal{U}_i = \mathcal{U} \cap \mathcal{X}_i$. Since every open affine subscheme of $\mathcal{X}_i$ is a principal open subset, it follows that if, for open subschemes $\mathcal{U}' \subset \mathcal{U}'' \subset \mathcal{X}$ one has $I(\mathcal{U}') = I(\mathcal{U}'')$, then the canonical homomorphism $\mathcal{O}(\mathcal{U}'') \to \mathcal{O}(\mathcal{U}')$ is injective. This also implies that for every open subscheme $\mathcal{U} \subset \mathcal{X}$ there is a canonical injective homomorphism $\mathcal{O}(\mathcal{U}) \hookrightarrow \prod_{i\in I(\mathcal{U})} F_i$, where F_i is the fraction $\mathbf{F}_1$-field of A_i (with $\mathcal{X}_i = \mathrm{Fspec}(A_i)$).

Furthermore, for an open subset $\mathcal{V} \subset \mathcal{X}^{(\alpha)}$, we set $I(\mathcal{V}) = \bigcap I(\mathcal{U})$, where the intersection is taken over open subschemes of $\mathcal{U}$ that contain the set $\pi(\mathcal{V})$. If such $\mathcal{U}$ is sufficiently small, then $I(\mathcal{V}) = I(\mathcal{U})$. It follows from the previous paragraph that, for any $\mathcal{V}$, there is a canonical injective homomorphism $P(\mathcal{V}) \hookrightarrow \prod_{i\in I(\mathcal{V})} F_i$. Notice that, if $\mathcal{V}$ contains a point from $\pi^{-1}(\mathcal{X}_M)$, then $I(\mathcal{V}) = I$ and $A \stackrel{\sim}{\to} P(\mathcal{V})$.

We now turn back to the covering of $\mathcal{V}$ with $\mathcal{V} \cap \pi^{-1}(\mathcal{X}_M) \neq \emptyset$ by open subsets $\{\mathcal{V}_\mu\}$. Let $(f_\mu)_\mu$ be an element in the above kernel L. For every μ, there is a canonical injective homomorphism $P(\mathcal{V}_\mu) \hookrightarrow \prod_{i\in I(\mathcal{V}_\mu)} F_i$. If $i \in I(\mathcal{V}_\mu)I(\mathcal{V}_n u)$, then the images of the elements f_μ and g_ν under the canonical homomorphisms $P(\mathcal{V}_\mu) \to F_i$ and $P(\mathcal{V}_\mu) \to F_i$ are equal. This means that there is an injective homomorphism $L \hookrightarrow \prod_{i\in I} F_i$. But by the assumption, there exists $\mathcal{V}_\rho$ which contains a point from $\pi^{-1}(\mathcal{X}_M)$ and, therefore, $I(\mathcal{V}_\rho) = I$ and $P(\mathcal{V}_\rho) = A$, i.e., there exists $f \in A$ with $f\big|_{\mathcal{V}_\rho} = f_\rho$. The above remark implies that $f\big|_{\mathcal{V}_\mu} = f_\mu$ for all μ. ∎

Proof of Theorem 6.5.4. (i) The functor considered takes an α-special scheme $\mathcal{X}$ to the triple $(\mathcal{X}^{(\alpha)}, \pi^*\mathcal{O}_\mathcal{X}, \nu)$, where π is the morphism $\mathcal{X}^{(\alpha)} \to \mathcal{X}$ and ν is the canonical homomorphism $\pi^*\mathcal{O}_\mathcal{X} \to \mathcal{O}_{\mathcal{X}^{(\alpha)}}$. Let $\varphi : (\mathcal{X}^{(\alpha)}, \pi^*\mathcal{O}_\mathcal{X}, \nu) \to (\mathcal{X}'^{(\alpha)}, \pi'^*\mathcal{O}_{\mathcal{X}'}, \nu')$ be a morphism in $\mathcal{S}ch_k^{(\alpha)}$. For an α-special open connected affine subscheme $\mathcal{U} \subset \mathcal{X}$, we take a point $x \in \mathcal{U}^{(\alpha)}$ whose image in $\mathcal{U}$ lies in $\mathcal{U}_M$, and an α-special open connected affine subscheme $\mathcal{U}'$ of $\mathcal{X}'$. *We claim that*

$\varphi\big|_{\mathcal{U}^{(\alpha)}} = \psi^{(\alpha)}$ *for a morphism* $\psi : \mathcal{U} \to \mathcal{U}' \subset \mathcal{X}'$. Indeed, let $\mathcal{V} = \mathcal{U}^{(\alpha)} \cap \varphi^{-1}(\mathcal{U}'^{(\alpha)})$. By Lemma 6.5.5, one has $A_{\mathcal{U}} \stackrel{\sim}{\to} (\pi^*\mathcal{O}_{\mathcal{X}})(\mathcal{V})$ and $A'_{\mathcal{U}'} \stackrel{\sim}{\to} (\pi'^*\mathcal{O}_{\mathcal{X}'})(\mathcal{U}'^{(\alpha)})$, and so the homomorphism of sheaves $\pi'^*\mathcal{O}_{\mathcal{X}'} \to \varphi_*(\pi^*\mathcal{O}_{\mathcal{X}})$ gives rise to a homomorphism of K-algebras $A'_{\mathcal{U}'} = (\pi'^*\mathcal{O}_{\mathcal{X}'})(\mathcal{U}'^{(\alpha)}) \to A_{\mathcal{U}} = (\pi^*\mathcal{O}_{\mathcal{X}})(\mathcal{V})$. The latter gives rise to a morphism of schemes $\psi : \mathcal{U} \to \mathcal{U}' \subset \mathcal{X}'$ such that the restriction of $\psi^{(\alpha)} : \mathcal{U}^{(\alpha)} \to \mathcal{U}'^{(\alpha)} \subset \mathcal{X}'^{(\alpha)}$ to the dense open subscheme $\mathcal{V}$ coincides with the morphism $\mathcal{V} \to \mathcal{U}'^{(\alpha)} \subset \mathcal{X}'^{(\alpha)}$ induced by φ. Since the scheme $\mathcal{X}'^{(\alpha)}$ is separated and $\mathcal{U}^{(\alpha)}$ is reduced, we get $\varphi\big|_{\mathcal{U}^{(\alpha)}} = \psi^{(\alpha)}$.

In this way we get a system of compatible morphisms $\psi_{\mathcal{U}} : \mathcal{U} \to \mathcal{X}'$ for all α-special open affine subschemes $\mathcal{U}$ of $\mathcal{X}$. This system defines a morphism $\psi : \mathcal{X} \to \mathcal{X}'$ of schemes over K which induces the morphism φ.

(ii) That the image of an α-special scheme $\mathcal{X}$ in $\mathcal{S}ch^{(\alpha)}$ possesses the required properties follows from Definition 6.5.1 and Lemma 6.5.5.Given an object of the category $\mathcal{S}ch^{(\alpha)}$ with those properties, a construction of the required α-special scheme over K is done in the same way as in the proof of Theorem 6.4.4(ii). ∎

6.6. Classes of morphisms between schemes over $\mathcal{S}ch_{\mathbf{F}_1}$. The existence of fiber products in the category $\mathcal{S}ch$ enables one to extend various classes of morphisms from the category $\mathcal{S}ch_{\mathbf{Z}}$ to the whole category $\mathcal{S}ch$. Namely, let $\mathcal{P}$ be a property of morphisms of schemes over $\mathbf{Z}$ which is local with respect to the target.

6.6.1. Definition. A morphism $\varphi : \mathcal{X}' \to \mathcal{X}$ of schemes over $\mathbf{F}_1$ is said *to have the property* $\mathcal{P}$ if, for any morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ from a scheme over $\mathbf{Z}$, the induced morphism $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}' \to \mathcal{Y}$ of schemes over $\mathbf{Z}$ has the property $\mathcal{P}$.

§6. The category of schemes $\mathcal{S}ch$

In this section we introduce a category $\mathcal{S}ch$ whose family of objects is a disjoint union of those of the categories $\mathcal{S}ch_{\mathbf{Z}}$ of *schemes over* $\mathbf{Z}$ (i.e., classical schemes) and $\mathcal{S}ch_{\mathbf{F}_1}$ of schemes over $\mathbf{F}_1$. The category $\mathcal{S}ch$ in fact contains $\mathcal{S}ch_{\mathbf{Z}}$ and $\mathcal{S}ch_{\mathbf{F}_1}$ as full subcategories. If $\mathcal{X}$ and $\mathcal{Y}$ are schemes over $\mathbf{F}_1$ and $\mathbf{Z}$, respectively, then the set $\mathrm{Hom}(\mathcal{X}, \mathcal{Y})$ of morphisms in $\mathcal{S}ch$ is always empty, but the set $\mathrm{Hom}(\mathcal{Y}, \mathcal{X})$ is not necessarily empty, e.g., $\mathrm{Fspec}(\mathbf{F}_1)$ is the final object of $\mathcal{S}ch$. The main feature of the category $\mathcal{S}ch$ is that it admits fiber products.

6.1. Definition of the category $\mathcal{S}ch$. The family of objects of the category of schemes $\mathcal{S}ch$ is defined as the disjoint union of the families of objects of the category $\mathcal{S}ch_{\mathbf{Z}}$ of schemes over $\mathbf{Z}$

and that of the category $\mathcal{S}ch^p_{\mathbf{F}_1}$ of schemes over $\mathbf{F}_1$. The sets of morphisms between two objects of $\mathcal{S}ch_{\mathbf{Z}}$ or of $\mathcal{S}ch^p_{\mathbf{F}_1}$ are defined as the corresponding sets in their categories. Furthermore, let $\mathcal{X}$ and $\mathcal{Y}$ be schemes over $\mathbf{F}_1$ and $\mathbf{Z}$, respectively. We set $\mathrm{Hom}(\mathcal{X},\mathcal{Y}) = \emptyset$. A morphism from $\mathcal{Y}$ to $\mathcal{X}$ is a pair consisting of a continuous map $\varphi : \mathcal{Y} \to \mathcal{X}$ and a homomorphism $\nu_\varphi : \mathcal{O}_{\mathcal{X}} \to (\varphi_*\mathcal{O}_{\mathcal{Y}})^{\cdot}$ of sheaves of $\mathbf{F}_1$-algebras (in the schematic topology of $\mathcal{X}$) with the following property: for every point $y \in \mathcal{Y}$, there exist an open affine neighborhood $\mathcal{V}$ of y and an open p-affine neighborhood $\mathcal{U}$ of $\varphi(y)$ such that $\varphi(\mathcal{V}) \subset \mathcal{U}$ and the map $\varphi : \mathcal{V} \to \mathcal{U}$ coincides with that induced by the homomorphism of $\mathbf{F}_1$-algebras $A_{\mathcal{U}} \to B^{\cdot}_{\mathcal{V}}$ (which is in its turn induced by ν_φ).

It follows from the definition that the above property holds for every pair consisting of an open affine subscheme $\mathcal{V} \subset \mathcal{Y}$ and an open p-affine subscheme $\mathcal{U} \subset \mathcal{X}$ with $\varphi(\mathcal{V}) \subset \mathcal{U}$. It follows also that for any pair of morphisms $\psi : \mathcal{Y}' \to \mathcal{Y}$ and $\chi : \mathcal{X} \to \mathcal{X}'$ there is a well defined composition morphism $\chi\varphi\psi : \mathcal{Y}' \to \mathcal{X}'$. Thus, $\mathcal{S}ch$ is really a category.

6.1.1. Lemma. *The correspondence* $\mathcal{Y}' \mapsto \mathrm{Hom}(\mathcal{Y}',\mathcal{X})$ *is a sheaf on* $\mathcal{Y}$.

Proof. Let $\{\mathcal{Y}_i\}_{i\in I}$ be a covering of $\mathcal{Y}$ by open subschemes, and suppose we are given a compatible system of morphisms $\varphi_i : \mathcal{Y}_i \to \mathcal{X}$. It is clear that they induce a continuous map $\varphi : \mathcal{Y} \to \mathcal{X}$. Let $\mathcal{V}$ be an open affine subscheme of $\mathcal{Y}$ and $\mathcal{U}$ an open p-affine subscheme of $\mathcal{X}$, and suppose $\varphi(\mathcal{V}) \subset \mathcal{U}$. For every $i \in I$, we take a covering $\{\mathcal{V}_{ij}\}_{j\in J_i}$ of $\mathcal{V} \cap \mathcal{Y}_i$ by open affine subschemes. Then we get a compatible system of homomorphisms of $\mathbf{F}_1$-algebras $A_{\mathcal{U}} \to B^{\cdot}_{\mathcal{V}_{ij}}$. Since $B_{\mathcal{V}} \xrightarrow{\sim} \mathrm{Ker}(\prod B_{\mathcal{V}_{ij}} \rightrightarrows \prod B_{\mathcal{V}_{ij}\cap\mathcal{V}_{kl}})$, that system is induced by a unique homomorphism $A_{\mathcal{U}} \to B^{\cdot}_{\mathcal{V}}$. In this way we get a homomorphism of sheaves of $\mathbf{F}_1$-algebras $\nu_\varphi : \mathcal{O}_{\mathcal{X}} \to (\varphi_*\mathcal{O}_{\mathcal{Y}})^{\cdot}$. That it satisfies the required property is trivial. It follows that the morphisms φ_i's are induced by a unique morphism $\varphi : \mathcal{Y} \to \mathcal{X}$. ∎

6.1.2. Lemma. *If* $\mathcal{X} = \mathrm{Fspec}(A)$ *is affine, then* $\mathrm{Hom}(\mathcal{Y},\mathcal{X}) = \mathrm{Hom}(A, \mathcal{O}(\mathcal{Y})^{\cdot})$.

Proof. Lemma 6.1.1 reduces the situation to the case when $\mathcal{Y} = \mathrm{Spec}(B)$ is also affine. By Proposition 4.4.8, any homomorphism of $\mathbf{F}_1$-algebras $A \to B^{\cdot}$ extends in a unique way to a compatible system of homomorphisms $A_{\mathcal{U}} \to B^{\cdot}_{\mathcal{V}}$ for all pairs of open affine subschemes $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{V} \subset \mathcal{Y}$ with $\varphi(\mathcal{V}) \subset \mathcal{U}$. We have to extend the homomorphisms $A_{\mathcal{U}} \to B^{\cdot}_{\mathcal{V}}$ to similar pairs in which $\mathcal{U}$ is an open p-affine subscheme. For this we take a covering $\{\mathcal{U}_i\}_{i\in I}$ of $\mathcal{U}$ by pairwise disjoint open affine subschemes. By Proposition 4.4.8, each $\mathcal{V}_i = \varphi^{-1}(\mathcal{U}_i)$ is an open affine subscheme of $\mathcal{Y}$, and they form a finite covering of $\mathcal{V}$. One therefore has $B_{\mathcal{V}} \xrightarrow{\sim} \prod_{i\in I} B_{\mathcal{V}_i}$. This gives a homomorphism of $\mathbf{F}_1$-algebras $A_{\mathcal{U}} \to \prod_{i\in I} A_{\mathcal{U}_i} \to B^{\cdot}$ which induces a continuous map $\mathcal{V} \to \mathcal{U}$ that coincides with the map $\varphi\big|_{\mathcal{V}}$. ∎

6.1.3. Proposition. *The category $\mathcal{S}ch$ admits fiber products.*

Proof. First of all, it is trivial that the canonical fully faithful functor $\mathcal{S}ch_{\mathbf{Z}} \to \mathcal{S}ch$ commutes with fiber products. Furthermore, Lemma 6.1.2 implies that the canonical functor $\mathcal{A}sch_{\mathbf{F}_1} \to \mathcal{S}ch$ commutes with fiber products. One deduces from this using the reasoning from the proof of Proposition 5.3.1 that the canonical fully faithful functor $\mathcal{S}ch_{\mathbf{F}_1} \to \mathcal{S}ch$ commutes with fiber products. Finally, suppose we are given a morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ from a scheme $\mathcal{Y}$ over $\mathbf{Z}$ and a morphism $f : \mathcal{X}' \to \mathcal{X}$ of schemes over $\mathbf{F}_1$. Construction of the fiber product $\mathcal{Y}' = \mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ is done in several steps.

Step 1. Suppose that f is a morphism of affine schemes $\mathcal{X}' = \mathrm{Fspec}(A') \to \mathcal{X} = \mathrm{Fspec}(A)$ and φ is a morphism $\mathcal{Y} = \mathrm{Spec}(B) \to \mathcal{X}$. The latter is defined by a homomorphism of $\mathbf{F}_1$-algebras $\varphi^* : A \to B^{\cdot}$ and enables one to view the $\mathbf{F}_1$-algebra $C^{\cdot}$ of every B-algebra C as an A-algebra. It is easy to see that the quotient $B_\varphi[A']$ of the B-algebra of polynomials $B[T_{a'}]_{a' \in A'}$ by the ideal generated by the elements $T_{a'_1 a'_2} - T_{a'_1} T_{a'_2}$ with $a'_1, a'_2 \in A'$ and $T_{f^*(a)} - \varphi^*(a)$ with $a \in A$ represents the covariant functor $C \mapsto \mathrm{Hom}_A(A', C^{\cdot})$. Lemma 6.1.2 implies that $\mathrm{Fspec}(B_\varphi[A'])$ is a fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ in $\mathcal{S}ch$.

Step 2. Suppose that φ is the same as in Step 1, but f is a p-morphism of affine schemes as in Step 1. It is defined by morphisms $f_i : \mathcal{U}'_i \to \mathcal{X}$ for a finite covering $\{\mathcal{U}'_i\}_{i \in I}$ of $\mathcal{X}'$ by pairwise disjoint open affine subschemes. *We claim that the affine scheme $\mathcal{Y}'$ which is a finite disjoint union $\mathcal{Y}'$ of the affine schemes $\mathcal{Y}_i = \mathcal{Y} \times_{\mathcal{X}} \mathcal{U}_i$ is a fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ in $\mathcal{S}ch$.* Indeed, given morphisms $g : \mathcal{Z} \to \mathcal{Y}$ and $\psi : \mathcal{Z} \to \mathcal{X}'$ with $\varphi g = f \psi$, we set $\mathcal{Z}_i = \psi^{-1}(\mathcal{U}'_i)$. By Step 2, there are canonical morphisms $\mathcal{Z}_i \to \mathcal{Y}_i$ which induce a canonical morphism $\mathcal{Z} \to \mathcal{Y}'$ whose composition with the projections to $\mathcal{Y}$ and $\mathcal{X}'$ coincide with g and ψ, respectively. The claim follows.

Notice that in this case, given open subschemes $\mathcal{V} \subset \mathcal{Y}$ and $\mathcal{U}' \subset \mathcal{X}$, the preimage of $\mathcal{V} \times \mathcal{U}'$ in $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ is a fiber product $\mathcal{V} \times_{\mathcal{X}} \mathcal{U}$.

Step 3. *A fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ exists if $\mathcal{X}$ is affine.* Indeed, take coverings $\{\mathcal{V}_i\}$ of $\mathcal{Y}$ and $\{\mathcal{U}'_k\}$ of $\mathcal{X}'$ by open affine and p-affine subschemes, respectively. Lemma 6.1.1 easily implies that the scheme $\mathcal{Y}'$ obtained by gluing all $\mathcal{V}_i \times_{\mathcal{X}} \mathcal{U}'_k$ along $(\mathcal{V}_i \cap \mathcal{V}_j) \times_{\mathcal{X}} (\mathcal{U}'_k \cap \mathcal{U}'_l)$ is a fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$.

Step 4. *A fiber product $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}'$ exists in the general case.* Indeed, if the morphisms φ and f go through a morphisms to an open p-affine subscheme $\mathcal{U}$, then $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}' = \mathcal{Y} \times_{\mathcal{U}} \mathcal{X}'$. In the general case, we take a covering $\{\mathcal{U}_i\}$ of $\mathcal{X}$ by open p-affine subschemes. Then the scheme $\mathcal{Y}'$ obtained by gluing all $\varphi^{-1}(\mathcal{U}_i) \times_{\mathcal{X}} f^{-1}(\mathcal{U}_i)$ along $\varphi^{-1}(\mathcal{U}_i \cap \mathcal{U}_j) \times_{\mathcal{X}} f^{-1}(\mathcal{U}_i \cap \mathcal{U}_j)$ is a fiber product of $\mathcal{Y}$ and $\mathcal{X}'$ over $\mathcal{X}$. ∎

Given morphisms $f : \mathcal{X}' \to \mathcal{X}$ of schemes over $\mathbf{F}_1$ and $\varphi : \mathcal{Y} \to \mathcal{X}$ from a scheme $\mathcal{Y}$ over $\mathbf{Z}$, if $\mathcal{X} = \mathrm{Fspec}(A)$ and $\mathcal{Y} = \mathrm{Spec}(B)$ are affine, the fiber product $\mathcal{X}' \times_{\mathcal{X}} \mathcal{Y}$ will be denoted by $\mathcal{X}' \otimes_A B$. For example, given a scheme $\mathcal{X}$ over $\mathbf{F}_1$, any morphism $\mathcal{Y} \to \mathcal{X}$ from a scheme over $\mathbf{Z}$ goes through a unique morphism $\mathcal{Y} \to \mathcal{X} \otimes_{\mathbf{F}_1} \mathbf{Z}$.

6.2. Lifting of quasi-coherent $\mathcal{O}_{\mathcal{X}}$-modules. Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism from a scheme over $\mathbf{Z}$ to a scheme over $\mathbf{F}_1$. For an $\mathcal{O}_{\mathcal{Y}}$-module $\mathcal{G}$, the direct image $\varphi_*\mathcal{G}$ considered as a sheaf of $\mathcal{O}_{\mathcal{X}}$-modules (in the schematic topology of $\mathcal{X}$) will be denoted by $(\varphi_*\mathcal{G})^{\cdot}$. Given a sheaf of $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$, consider the covariant functor on the category of $\mathcal{O}_{\mathcal{Y}}$-modules that takes $\mathcal{G}$ to the set of homomorphisms of sheaves of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{F} \to (\varphi_*\mathcal{G})^{\cdot}$.

6.2.1. Proposition. *Suppose that $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-module. Then*

(i) the above functor is representable by a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-module denoted by $\varphi^*\mathcal{F}$;

(ii) if $\mathcal{F}$ is of finite type or coherent, then so is $\varphi^*\mathcal{F}$;

(iii) the correspondence $\mathcal{F} \mapsto \varphi^*\mathcal{F}$ is a functor which commutes with direct sums and tensor products;

(iv) if $\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{X}}$-algebra, then $\varphi^*\mathcal{F}$ is a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-algebra, and it represents the covariant functor that takes an $\mathcal{O}_{\mathcal{Y}}$-algebra $\mathcal{G}$ to the set of homomorphism of $\mathcal{O}_{\mathcal{X}}$-algebras $\mathcal{F} \to (\varphi_*\mathcal{G})^{\cdot}$.

6.2.2. Lemma. *If $\mathcal{X} = \mathrm{Fspec}(A)$ is affine and $\mathcal{F} = \mathcal{O}_{\mathcal{X}}(M)$ for an A-module M, then $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, (\varphi_*\mathcal{G})^{\cdot}) \stackrel{\sim}{\to} \mathrm{Hom}_A(M, \mathcal{G}(\mathcal{Y})^{\cdot})$.*

Proof. That the map considered is injective is easy. Suppose we are given a homomorphism of A-modules $M \to \mathcal{G}(\mathcal{Y})^{\cdot}$. It induces a system of compatible homomorphisms of $A_{\mathcal{U}}$-modules $M \otimes_A A_{\mathcal{U}} \to \mathcal{G}(\varphi^{-1}(\mathcal{U}))^{\cdot}$ for open affine subschemes $\mathcal{U} \subset \mathcal{X}$. Given a covering of $\mathcal{U}$ by open affine subschemes $\mathfrak{U} = \{\mathcal{U}_i\}_{i \in I}$, one has $\mathcal{G}(\varphi^{-1}(\mathcal{U})) \stackrel{\sim}{\to} \mathrm{Ker}(\prod_{i \in I} \mathcal{G}(\mathcal{U}_i) \stackrel{\to}{\to} \prod_{i,j \in I} \mathcal{G}(\varphi^{-1}(\mathcal{U}_i \cap \mathcal{U}_j)))$ and, therefore, a homomorphism $M_{\mathfrak{U}} \to \mathcal{G}(\varphi^{-1}(\mathcal{U}))^{\cdot}$. In their turn the latter induce a system of compatible homomorphisms $\mathcal{F}(\mathcal{U}) = \langle M_{\mathcal{U}} \rangle \to \mathcal{G}(\varphi^{-1}(\mathcal{U}))^{\cdot}$ and, therefore, a homomorphism of sheaves of $\mathcal{O}_{\mathcal{X}}$-modules $\mathcal{F} \to (\varphi_*\mathcal{G})^{\cdot}$ which gives rise to the homomorphism we started from. ∎

Notice that if, in the situation of Lemma 6.2.2, M is in fact an A-algebra, the same is true for the sets of homomorphisms of $\mathbf{F}_1$-algebras instead of homomorphisms of modules.

6.2.3. Lemma. *Suppose that both $\mathcal{X} = \mathrm{Fspec}(A)$ and $\mathcal{Y} = \mathrm{Spec}(B)$ are affine. Then*

(i) for any A-module M, the covariant functor $N \to \mathrm{Hom}_A(M, N^{\cdot})$ on the category of B-modules is representable by a B-module denoted by $B \otimes_A M$;

(ii) the correspondence $M \mapsto B \otimes_A M$ is a functor that commutes with direct sums and tensor products;

(iii) if M is an A-algebra, then $B \otimes_A M$ is a B-algebra that represents the covariant functor that takes a B-algebra N to the set of homomorphisms of A-algebras $M \to N^{\cdot}$.

Proof. Let f denote the homomorphism $A \to B^{\cdot}$ that induces φ. The functor considered is representable by the quotient of the free B-module $\oplus_{m\in M} BT_m$ by the B-submodule generated by the elements $T_{am} - f(a)T_m$ with $m \in M$ and $a \in A$, i.e., the statement (i) is true. Notice that, if M is a quotient of a free A-module $A^{(I)}$ by an A-submodule $E \subset M \times M$, then $B \otimes_A M$ is also the quotient of the free B-module $\oplus_{i\in I} BT_i$ by the B-submodule generated by the elements $a'T_i - a''T_j$ with $(a't_i, a''t_j) \in E$, where t_i is the image of the canonical i-th generator of $A^{(I)}$. In particular, if M is finite or finitely presented, then so is $B \otimes_A M$. The statements (ii) and (iii) easily follow from (i). ∎

Proof of Proposition 6.2.1. The situation is easily reduced to the case when both $\mathcal{X} = \mathrm{Fspec}(A)$ and $\mathcal{Y} = \mathrm{Spec}(B)$ are affine.

Step 1. *For any open affine subscheme $\mathcal{U} \subset \mathcal{X}$, one has $(B \otimes_A M)_{\varphi^{-1}(\mathcal{U})} \widetilde{\to} B_{\varphi^{-1}(\mathcal{U})} \otimes_{A_{\mathcal{U}}} M_{\mathcal{U}}$.* Indeed, this is trivial if $\mathcal{U}$ is a principal open subset or defined by vanishing a finite number of idempotents and, therefore, this is true if $\mathcal{U}$ is an elementary open subset. If $\mathcal{U}$ is arbitrary, we take an elementary family $\{\mathcal{U}_i\}_{i\in \check{I}}$ that covers $\mathcal{U}$. Then $B_{\varphi^{-1}(\mathcal{U})} \widetilde{\to} \prod_{i\in\check{I}} B_{\varphi^{-1}(\mathcal{U}_i)}$ and, therefore, $(B \otimes_A M)_{\varphi^{-1}(\mathcal{U})} \widetilde{\to} \prod_{i\in\check{I}} B_{\varphi^{-1}(\mathcal{U}_i)} \otimes_{A_{\mathcal{U}_i}} M_{\mathcal{U}_i}$. For the same reason, the right hand side coincides with $B_{\varphi^{-1}(\mathcal{U})} \otimes_{A_{\mathcal{U}}} M_{\mathcal{U}}$, and the claim follows.

Step 2. *The functor $M \mapsto \mathcal{O}_{\mathcal{Y}}(B \otimes_A M)$ is extended to a functor* $\mathrm{Qcoh}(\mathcal{X}) \to \mathrm{Qcoh}(\mathcal{Y})$ *that takes $\mathcal{F}$ to a quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-module $\varphi^*\mathcal{F}$.* Recall that, for A-modules M and P, one has $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{O}_{\mathcal{X}}(M), \mathcal{O}_{\mathcal{X}}(P)) = \varinjlim \mathrm{Hom}_A(M, P_{\mathfrak{U}})$, where the inductive limit is taken over finite coverings $\mathfrak{U}$ of $\mathcal{X}$ by open affine subschemes. Step 1 and acyclicity of quasi-coherent modules on affine schemes over $\mathbf{Z}$ imply that the canonical homomorphism $P \to (B \otimes_A P)^{\cdot}$ extends in a canonical way to a homomorphism $P_{\mathfrak{U}} \to (B \otimes_A P)^{\cdot}$. This gives the required extension of the functor considered to the essential image of the category of A-modules in $\mathrm{Qcoh}(\mathcal{X})$. Suppose now that $\mathcal{O}_{\mathcal{X}}(M)$ is an arbitrary object of the category $\mathrm{Qcoh}(\mathcal{X})$. We may assume that there is a finite covering of $\mathcal{X}$ by pairwise disjoint open affine subschemes $\{\mathcal{U}_i\}_{i\in I}$ such that $\mathcal{O}_{\mathcal{X}}(M)$ is associated with a system of $A_{\mathcal{U}_i}$-modules $M_{\mathcal{U}_i}$. By the previous case, each $M_{\mathcal{U}_i}$ gives rise to a quasi-coherent $\mathcal{O}_{\varphi^{-1}(\mathcal{U}_i)}$-module on $\varphi^{-1}(\mathcal{U}_i)$, and all of them define the required quasi-coherent $\mathcal{O}_{\mathcal{Y}}$-module.

Step 3. *The coherent $\mathcal{O}_{\mathcal{Y}}$-module $\varphi^*\mathcal{F}$ possesses all of the required properties.* Indeed, it suffices to verify the claim in the case $\mathcal{F} = \mathcal{O}_{\mathcal{X}}(M)$ for an A-module M. In this case $\mathrm{Hom}_{\mathcal{O}_{\mathcal{X}}}(\mathcal{F}, (\varphi_*\mathcal{G})^{\cdot}) \xrightarrow{\sim} \mathrm{Hom}_A(M, \mathcal{G}(\mathcal{Y})^{\cdot})$, by Lemma 6.2.2. By Lemma 6.2.3, the latter coincides with $\mathrm{Hom}_B(B \otimes_A M, \mathcal{G}(\mathcal{Y}))$ and, by quasi-coherence, with $\mathrm{Hom}_{\mathcal{O}_{\mathcal{Y}}}(\mathcal{O}_{\mathcal{Y}}(B \otimes_A M), \mathcal{G})$, i.e., the property (i) holds. The other properties follow from Lemma 6.2.3. ∎

6.3. The image of the map $\mathcal{Y} \to \mathcal{X}$. Recall that an abelian group is said to be locally cyclic if every subgroup of it generated by a finite number of elements is cyclic. For example, the torsion subgroup of the multiplicative group k^* of any field k is locally cyclic. It follows that, given a morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ from a scheme over $\mathbf{Z}$ to a scheme over $\mathbf{F}_1$, the torsion subgroup of $\kappa(x)^*$ of every point x from the image of φ is locally cyclic.

6.3.1. Proposition. *Let $\mathcal{X}$ be a scheme over $\mathbf{F}_1$, and let k be a field of characteristic zero (resp. $p > 0$). Then the image of the map $\mathcal{X} \otimes_{\mathbf{F}_1} k \to \mathcal{X}$ is the set of points $x \in \mathcal{X}$ with the property that the torsion subgroup of $\kappa(x)^*$ is locally cyclic (resp. and has no elements of order p).*

Proof. Let x be a point of $\mathcal{X}$ with that property, and set $K = \kappa(x)$. We notice that it suffices to show that there exists an embedding $K^* \hookrightarrow k'^*$ for an extension k' of k. Indeed, such an embedding gives rise to a morphism $\mathrm{Spec}(k') \to \mathcal{X}$ whose image is the point x and which goes through a morphism $\mathrm{Spec}(k') \to \mathcal{X} \otimes_{\mathbf{F}_1} k$. If x' is the image of the latter morphism, then the induced homomorphism $K^* \to \kappa(x')^*$ is injective. The required fact is a version of a result of Cohn [Cohn], and here is an easy proof of it.

We may assume that K is infinite, and we can increase the field k and assume that it is algebraically closed and its cardinality is greater than that of K. Then there is an emdedding $K^*_{tors} \hookrightarrow k^*$. Let S be the set of pairs (G, α), where G of a subgroup of K^* that contains K^*_{tors} and α is an embedding $G \hookrightarrow k^*$. We provide S with a partial ordering as follows: $(G, \alpha) \le (G', \alpha')$ if $G \subset G'$ and $\alpha'\big|_G = \alpha$. The poset S satisfies the condition of Zorn's Lemma and, therefore, it contains a maximal element (G, α). It suffices to show that $G = K^*$. Suppose this is not true. We then can find an element $\lambda \in K^* \backslash G$. If $\lambda^n \notin G$ for all $n \ge 1$, then we take an arbitrary element $x \in k^*$ transcendental over the subfield of k generated by $\alpha(G)$. (It exists since the cardinality of k is greater than that of K.) If G' is the subgroup of K^* generated by G and λ and α' is the homomorphism $G' \to k^*$ that coincides with α on G and takes λ to x, then the pair (G', α') is an element of S strictly bigger that (G, α), which is a contradiction. Suppose now that $\lambda^n \in G$ for some $n > 1$. We may assume that n is minimal with this property and, therefore, each element of the subgroup G' of K^* generated by G and λ has a unique representation in the form $\lambda^i g$ with $0 \le i \le n-1$ and $g \in G$. Let x be an element of k with $x^n = \alpha(\lambda^n)$. If α' is the homomorphism of $G' \to k^*$ that coincides with α on G and takes λ to x, then the pair (G', α') is an element of S strictly bigger that (G, α), which is a contradiction. ∎

Suppose now we are given an $\mathbf{F}_1$-field K, a commutative ring with unity k, and a homomorphism of $\mathbf{F}_1$-algebras $K \to k^{\cdot}$, i.e., a morphism $\mathrm{Spec}(k) \to \mathrm{Fspec}(K)$. Then for any K-algebra A there are an induced morphism $\mathrm{Spec}(k \otimes_K A) \to \mathrm{Fspec}(A)$ and an induced map $\mathrm{Spec}(k \otimes_K A) \to \mathrm{Zspec}(A)$.

6.3.2. Lemma. *The following properties of a Zariski ideal $\mathfrak{p} \subset A$ are equivalent:*

(a) $\mathfrak{p}$ lies in the image of the map $\mathrm{Spec}(k \otimes_K A) \to \mathrm{Zspec}(A)$*;*

(b) $k \otimes_K \kappa(\mathfrak{p}) \neq 0$;

(c) the stabilizer of every element $f \in A \backslash \mathfrak{p}$ in K^* lies in $\mathrm{Ker}(K^* \to k^*)$.

Proof. We set $B = k \otimes_K A$. A Zariski ideal $\mathfrak{p} \subset A$ lies in the image of $\mathrm{Spec}(B)$ if and only if there exists an ideal $\mathfrak{q} \subset B$ with $\mathfrak{p} = \mathrm{Zker}(A \to (B/\mathfrak{q})^{\cdot})$, The latter is equivalent to the property $B \otimes_A \kappa(\mathfrak{p}) \neq 0$. Since $B \otimes_A \kappa(\mathfrak{p}) = k \otimes_K \kappa(\mathfrak{p})$, the equivalence (a)$\Longleftrightarrow$(b) follows. The equivalence (b)$\Longleftrightarrow$(c) is trivial. ∎

6.3.3. Corollary. *The following properties of a K-algebra A are equivalent:*

(a) the map $\mathrm{Spec}(k \otimes_K A) \to \mathrm{Zspec}(A)$ *is surjective;*

(b) the stabilizer of every non-nilpotent element $f \in A$ in K^* is contained in $\mathrm{Ker}(K^* \to k^*)$. ∎

6.4. Schemes over k with a topologized prelogarithmic K-structure. Suppose now we are given an $\mathbf{F}_1$-algebra K, a commutative ring with unity k, and a homomorphism of $\mathbf{F}_1$-algebras $\alpha : K \to k^{\cdot}$, i.e., a morphism $\mathrm{Spec}(k) \to \mathrm{Fspec}(K)$. For a scheme $\mathcal{X}$ over K, we set $\mathcal{X}^{(\alpha)} = \mathcal{X} \otimes_K k$, and denote by π the morphism $\mathcal{X}^{(\alpha)} \to \mathcal{X}$.

6.4.1. Definition. (i) A scheme $\mathcal{X}$ over K is said to be *α-nontrivial* if every point of $\mathcal{X}$ has an open p-affine neighborhood $\mathcal{U}$ for which the map $\mathcal{U}^{(\alpha)} \to \mathrm{Zspec}(A_{\mathcal{U}})$ is surjective.

(ii) The full subcategory of $\mathcal{S}ch_K$ of schemes over K consisting of locally connected α-nontrivial schemes is denoted by $\mathcal{S}ch_K^{[\alpha]}$.

For example, if K is an $\mathbf{F}_1$-field, Corollary 6.3.3 implies that an affine scheme $\mathcal{X} = \mathrm{Fspec}(A)$ over K is k/K-nontrivial if and only if the stabilizer of every non-nilpotent element $f \in A$ in K^* is contained in $\mathrm{Ker}(K^* \to k^*)$.

6.4.2. Lemma. *Suppose that $\mathcal{X}$ is α-nontrivial. If $\pi^{-1}(\mathcal{U}) \subset \pi^{-1}(\mathcal{V})$ for open subschemes of $\mathcal{X}$, then $\mathcal{U} \subset \mathcal{V}$. In particular, if $\pi^{-1}(\mathcal{U}) = \pi^{-1}(\mathcal{V})$, then $\mathcal{U} = \mathcal{V}$.*

Proof. The situation is easily reduced to the case when $\mathcal{X} = \mathrm{Fspec}(A)$ is affine. Since the map $\mathcal{X} \otimes_K k \to \mathrm{Zspec}(A)$ is surjective, it follows that the images of the sets $\pi^{-1}(\mathcal{U})$ and $\pi^{-1}(\mathcal{V})$ coincide with the sets of Zariski prime ideals $\mathfrak{p} \subset A$ with $\mathcal{U} \cap \mathcal{X}^{(\mathfrak{p})} \neq \emptyset$ and $\mathcal{V} \cap \mathcal{X}^{(\mathfrak{p})} \neq \emptyset$, respectively. It remains to notice that, if $\mathcal{V} \cap \mathcal{X}^{(\mathfrak{p})} \neq \emptyset$, then $\mathcal{X}^{(\mathfrak{p})} \subset \mathcal{V}$. ∎

6.4.3. Definition. (i) A scheme over k with a *topologized prelogarithmic K-structure* is a quadruple $(\mathcal{Y}, \sigma, \mathcal{A}, \nu)$ consisting of a scheme $\mathcal{Y}$ over k, a topology σ on $\mathcal{Y}$ formed by open subschemes, a σ-sheaf of K-algebras $\mathcal{A}$, and a homomorphism of σ-sheaves of K-algebras $\nu : \mathcal{A} \to \mathcal{O}^{\cdot}_{\mathcal{Y}}\big|_\sigma$ which is compatible with the homomorphism α.

(ii) A morphism $(\mathcal{Y}, \sigma, \mathcal{A}, \nu) \to (\mathcal{Y}', \sigma', \mathcal{A}', \nu')$ is a pair consisting of a morphism of schemes over k, $\varphi : \mathcal{Y} \to \mathcal{Y}'$, which is σ-continuous (i.e., $\varphi^{-1}(\mathcal{V}')$ is σ-open for all σ'-open subschemes $\mathcal{V}' \subset \mathcal{Y}'$) and a homomorphism of σ-sheaves of K-algebras $\mathcal{A}' \to \varphi_* \mathcal{A}$ which is compatible with the homomorphism $\mathcal{O}_{\mathcal{Y}'} \to \varphi_* \mathcal{O}_{\mathcal{Y}}$.

(iii) The category of schemes over k with a topologized prelogarithmic K-structure is denoted by $\mathcal{S}ch_k^{[\alpha]}$.

If $(\mathcal{Y}, \sigma, \mathcal{A}, \nu)$ is an object of $\mathcal{S}ch_k^{[\alpha]}$, every σ-open subscheme $\mathcal{V} \subset \mathcal{Y}$ gives rise to an object of $\mathcal{S}ch_k^{[\alpha]}$, namely, $(\mathcal{V}, \sigma\big|_{\mathcal{V}}, \mathcal{A}\big|_{\mathcal{V}}, \nu\big|_{\mathcal{V}})$. (In the formulation of Theorem 6.4.4(ii), $\mathcal{V}$ denotes the latter tuple.) A morphism $(\mathcal{Y}', \sigma', \mathcal{A}', \nu') \to (\mathcal{Y}, \sigma, \mathcal{A}, \nu)$ in $\mathcal{S}ch_k^{[\alpha]}$ is said to be an *open immersion* if it gives rise to an isomorphism of the first tuple with the object of $\mathcal{S}ch_k^{[\alpha]}$ induced by a σ-open subscheme of $\mathcal{Y}$.

6.4.4. Theorem *(i) The correspondence $\mathcal{X} \mapsto \mathcal{X}^{(\alpha)}$ gives rise to a fully faithful functor*

$$\mathcal{S}ch_K^{[\alpha]} \to \mathcal{S}ch_k^{[\alpha]} \ ;$$

(ii) an object $(\mathcal{Y}, \sigma, \mathcal{A}, \nu)$ of $\mathcal{S}ch_k^{[\alpha]}$ lies in the essential image of the above functor if and only if the σ-topology admits a base b_σ with the following properties:

(1) for every $\mathcal{V} \in b_\sigma$, $\mathcal{A}(\mathcal{V})$ is a K-algebra with connected and locally connected spectrum, and the canonical morphism $\mathcal{V} \to \mathrm{Spec}(\mathcal{A}(\mathcal{V})] \otimes_K k)$ is an isomorphism in $\mathcal{S}ch_k^{[\alpha]}$;

(2) for every pair $\mathcal{V} \subset \mathcal{W}$ in b_σ, the canonical homomorphism $\mathcal{A}(\mathcal{W}) \to \mathcal{A}(\mathcal{V})$ induces an open immersion of affine schemes $\mathrm{Fspec}(\mathcal{A}(\mathcal{V})) \to \mathrm{Fspec}(\mathcal{A}(\mathcal{W}))$ which is compatible with the open immersion $\mathcal{V} \to \mathcal{W}$.

Proof. (i) Let $\mathcal{X}$ be locally connected α-nontrivial scheme over K. The scheme $\mathcal{X}^{(\alpha)}$ is provided with the topology σ whose open sets are the preimages of open subschemes of $\mathcal{X}$ with respect to the map $\pi : \mathcal{X}^{(\alpha)} \to \mathcal{X}$. If π_σ denotes the map from $\mathcal{X}^{(\alpha)}$, provided with the σ-topology, to $\mathcal{X}$, provided with the schematic topology, the homomorphism $\mathcal{O}_\mathcal{X} \to \pi_* \mathcal{O}^{\cdot}_{\mathcal{X}^{(\alpha)}}$ induces a homomorphism $\nu : \pi_\sigma^* \mathcal{O}_\mathcal{X} \to \mathcal{O}^{\cdot}_{\mathcal{X}^{(\alpha)}}\big|_\sigma$. The tuple $(\mathcal{X}^{(\alpha)}, \sigma, \pi_\sigma^* \mathcal{O}_\mathcal{X}, \nu)$ is an object of the category $\mathcal{S}ch_k^{[\alpha]}$. That the correspondence $\mathcal{X} \mapsto (\mathcal{X}^{(\alpha)}, \sigma, \pi_\sigma^* \mathcal{O}_\mathcal{X}, \nu)$ is a faithful functor is easy. To show that it is fully faithful (and to prove (ii)), we need the following simple fact.

6.4.5. Lemma. *One has $\mathcal{O}_\mathcal{X} \stackrel{\sim}{\to} \pi_{\sigma *} \pi_\sigma^* \mathcal{O}_\mathcal{X}$.*

Proof. Since $\mathcal{X}$ is locally connected, it suffices to verify that, for a connected open affine subscheme $\mathcal{U}$ of $\mathcal{X}$, one has $(\pi_\sigma^* \mathcal{O}_\mathcal{X})(\mathcal{U}^{(\alpha)}) = A_\mathcal{U}$. The σ-open affine subscheme $\mathcal{U}^{(\alpha)} = \pi^{-1}(\mathcal{U})$ contains a point x with $\pi(x) \in \mathcal{U}_\mathbf{m}$. Since $\mathcal{U}$ is the minimal open subscheme of $\mathcal{X}$ that contains a point from $\mathcal{U}_\mathbf{m}$, Lemma 6.4.2 implies that $\mathcal{U}^{(\alpha)}$ is the minimal σ-open subscheme of $\mathcal{X}^{(\alpha)}$ that contains the point x and, therefore, any σ-open covering $\{\mathcal{V}_i\}_{i \in I}$ of $\mathcal{U}^{(\alpha)}$ one has $\mathcal{U}^{(\alpha)} \subset \mathcal{V}_i$ for some $i \in I$. The required fact follows. ∎

Let $\varphi : (\mathcal{X}^{(\alpha)}, \sigma, \pi^* \mathcal{O}_\mathcal{X}, \nu) \to (\mathcal{X}'^{(\alpha)}, \sigma', \pi'^{\alpha} \mathcal{O}_{\mathcal{X}'}, \nu')$ be a morphism in $\mathcal{S}ch_k^{[\alpha]}$. Given a connected open affine subscheme $\mathcal{U} \subset \mathcal{X}$, take a point $x \in \mathcal{U}^{(\alpha)}$ with $p(x) \in \mathcal{U}_\mathbf{m}$ and a connected open affine subscheme $\mathcal{U}' \subset \mathcal{X}'$ with $\varphi(x) \in \mathcal{U}'^{(\alpha)}$. *We claim that $\varphi(\mathcal{U}^{(\alpha)}) \subset \mathcal{U}'^{(\alpha)}$.* Indeed, since the map φ is σ-continuous, $\varphi^{-1}(\mathcal{U}'^{(\phi)}) = \pi^{-1}(\mathcal{V})$ for some open subscheme $\mathcal{V} \subset \mathcal{X}$ and, since $\mathcal{U}$ is the minimal open subscheme of $\mathcal{X}$ that contains the point $\pi(x)$, it follows that $\mathcal{U} \subset \mathcal{V}$ and, therefore, $\mathcal{U}^{(\alpha)} \subset \varphi^{-1}(\mathcal{U}'^{(\alpha)})$.

Furthermore, suppose we are given connected open affine subschemes $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{U}' \subset \mathcal{X}'$ with $\varphi(\mathcal{U}^{(\alpha)}) \subset \mathcal{U}'^{(\alpha)}$. By Lemma 6.4.5, one has $(\pi_\sigma^* \mathcal{O}_\mathcal{X})(\mathcal{U}^{(\alpha)}) = A_\mathcal{U}$ and $(\pi'^*_{\sigma'} \mathcal{O}_{\mathcal{X}'}) = A'_{\mathcal{U}'}$ and, therefore, the homomorphism $\pi'^*_{\sigma'} \mathcal{O}_{\mathcal{X}'} \to \varphi_*(\pi_\sigma^* \mathcal{O}_\mathcal{X})$ induces a homomorphism $A_{\mathcal{U}'} \to A_\mathcal{U}$ which is compatible with the homomorphism $A'_{\mathcal{U}'} \otimes_K k \to A_\mathcal{U} \otimes_K k$. In this way we get a system of compatible homomorphisms of K-algebras $A_{\mathcal{U}'} \to A_\mathcal{U}$ for all pairs of connected open affine subschemes $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{U}' \subset \mathcal{X}'$ with $\varphi(\mathcal{U}^{(\alpha)}) \subset \mathcal{U}'^{(\alpha)}$. This system defines a morphism $\mathcal{X} \to \mathcal{X}'$ of schemes over K which gives rise to the morphism φ.

(ii) If $\mathcal{X}$ is a locally connected α-nontrivial scheme over K, the corresponding object of $\mathcal{S}ch_k^{[\alpha]}$ is the quadruple $(\mathcal{X}^{(\alpha)}, \sigma, \pi_\sigma^* \mathcal{O}_\mathcal{X}, \nu)$, described in the proof of (i). The preimages of the connected open affine subschemes of $\mathcal{X}$ form a base b_σ of the topology σ. That b_σ possesses the properties (1) and (2) follows from Lemma 6.4.5.

Suppose now that $(\mathcal{Y}, \sigma, \mathcal{A}, \nu)$ is an object of $\mathcal{S}ch_k^{[\alpha]}$ with a base b_σ of σ that possesses those properties. For $\mathcal{V} \in b_\sigma$, let $\mathcal{X}_\mathcal{V}$ be the affine scheme $\mathrm{Fspec}(\mathcal{A}(\mathcal{V}))$. The property (1) implies that there is a canonical isomorphism $\mathcal{V} \stackrel{\sim}{\to} \mathcal{X}_\mathcal{V}^{(\alpha)}$. The property (2) implies that, for every pair $\mathcal{V} \subset \mathcal{W}$, the canonical morphism $\mathcal{X}_{\mathcal{V}} \to \mathcal{X}_{\mathcal{W}}$ is an open immersion that induces the open immersion $\mathcal{V} \to \mathcal{W}$. For a pair $\mathcal{V}, \mathcal{W} \in b_\sigma$, let $\mathcal{X}_{\mathcal{V},\mathcal{W}}$ denote the open subscheme of $\mathcal{X}_{\mathcal{V}}$ which is the union of the images of $\mathcal{X}_{\mathcal{U}}$, where $\mathcal{U}$ runs through all sets from b_σ for which $\mathcal{X}_{\mathcal{U}}^{(\alpha)} \subset \mathcal{V} \cap \mathcal{W}$. Notice that, since b_σ is a base of the topology σ, such sets $\mathcal{X}_{\mathcal{U}}^{(\alpha)}$ cover the intersection $\mathcal{V} \cap \mathcal{W}$, and that there is a well defined isomorphism $\nu_{\mathcal{V},\mathcal{W}} : \mathcal{X}_{\mathcal{V},\mathcal{W}} \stackrel{\sim}{\to} \mathcal{X}_{\mathcal{W},\mathcal{V}}$. The system $\{\nu_{\mathcal{V},\mathcal{W}}\}$ satisfies the conditions of Lemma 5.2.10 and, therefore, we can glue all $\mathcal{X}_{\mathcal{V}}$'s along $\mathcal{X}_{\mathcal{V},\mathcal{W}}$'s. In this way we get a locally connected α-nontrivial scheme $\mathcal{X}$ over K with $\mathcal{Y} \stackrel{\sim}{\to} \mathcal{X}^{(\alpha)}$. ∎

6.5. Schemes over k with a prelogarithmic K-structure. In this subsection $\alpha : K \to k^{\cdot}$ is a homomorphism as in §6.4.

6.5.1. Definition. (i) An α-nontrivial separated scheme $\mathcal{X}$ over K is said to be *α-special* if it admits a net of connected open affine subschemes σ such that every $\mathcal{U} \in \sigma$ possesses the following properties:

(1) $\mathcal{U}$ is reduced and the set of its irreducible components is finite;

(2) the intersection $\mathcal{U}_M$ of the sets $\mathcal{W}_{\mathbf{m}}$, where $\mathcal{W}$ runs through all irreducible components of $\mathcal{U}$, is nonempty;

(3) for each Zariski prime ideal $\mathfrak{p} \subset A_{\mathcal{U}}$, the k-algebra $k \otimes_K \kappa(\mathfrak{p})$ is integral.

(ii) The full subcategory of $\mathcal{S}ch_K$ consisting of α-special schemes is denoted by $\mathcal{S}ch_K^{(\alpha)}$.

The properties (1) and (2) do not depend on the homomorphism α. Notice that, since the canonical homomorphism $A_{\mathcal{U}}/\Pi_{\mathfrak{p}} \to \kappa(\mathfrak{p})$ is injective, the property (3) implies that the k-algebra $k \otimes_K A_{\mathcal{U}}/\Pi_{\mathfrak{p}}$ is integral and, in particular, there is a one-to-one correspondence between the set of irreducible components of $\mathcal{U}$ and that of $\mathcal{U}^{(\alpha)}$. Here is a simple sufficient condition for validity of the property (3).

6.5.2. Lemma. *Suppose that K is an $\mathbf{F}_1$-field and k is an integral domain. If a ϕ-nontrivial K-algebra A is such that, for every Zariski prime ideal $\mathfrak{p} \subset A$, the group* $\mathrm{Coker}(K^* \to \kappa(\mathfrak{p})^*)$ *has no torsion, then the affine scheme* $\mathcal{X} = \mathrm{Fspec}(A)$ *possesses the property (3).*

Notice that if the condition of Lemma 6.5.2 is satisfied by those $\mathfrak{p}$ for which $\mathcal{X}^{(\mathfrak{p})}$ is an irreducible component of $\mathcal{X}$, then it is satisfied by all $\mathfrak{p}$'s.

Proof. Since A is α-nontrivial, we may replace A by $A/\mathrm{Ker}(K^* \to A^*)$ and assume that the homomorphism $K \to A^*$ is injective. Furthermore, to verify the required property for a Zariski prime ideal $\mathfrak{p} \subset A$, we may replace A by $A/\Pi_{\mathfrak{p}}$ and assume that A is an integral domain and $\mathfrak{p} = 0$. If F is the fraction $\mathbf{F}_1$-field of A, the homomorphism $k \otimes_K A \to k \otimes_K F$ is injective and, therefore, it suffices to verify the required fact for F instead of A. Finally, if $F = \varinjlim F_i$, then $k \otimes_K F = \varinjlim k \otimes_K F_i$. We may therefore assume that the group F^*/K^* is finitely generated. If $f_1, \dots, f_n$ are elements of F^* whose images form a basis of the free group F^*/K^*, then $k \otimes_K F = k[f_1, \dots, f_n, \frac{1}{f_1}, \dots, \frac{1}{f_n}]$. The latter is an integral domain because k is an integral domain. ∎

6.5.3. Definition. (i) A scheme over k with a *prelogarithmic K-structure* is a triple $(\mathcal{Y}, \mathcal{A}, \nu)$ consisting of a scheme $\mathcal{Y}$ over k, a sheaf of K-algebras $\mathcal{A}$, and a homomorphism of K-algebras $\nu : \mathcal{A} \to \mathcal{O}^{\cdot}_{\mathcal{Y}}$ which is compatible with the homomorphism α.

(ii) A morphism $(\mathcal{Y}, \mathcal{A}, \nu) \to (\mathcal{Y}', \mathcal{A}', \nu')$ is a pair consisting of a morphism of schemes over k, $\varphi : \mathcal{Y} \to \mathcal{Y}'$, and a homomorphism of sheaves of K-algebras $\mathcal{A}' \to \varphi_*\mathcal{A}$ which is compatible with the homomorphism $\mathcal{O}_{\mathcal{Y}'} \to \varphi_*\mathcal{O}_{\mathcal{Y}}$.

(iii) The category of schemes over k with a prelogarithmic K-structure is denoted by $\mathcal{S}ch_k^{(\alpha)}$.

If $(\mathcal{Y}, \mathcal{A}, \nu)$ is an object of $\mathcal{S}ch_k^{(\alpha)}$, every open subscheme $\mathcal{V} \subset \mathcal{Y}$ gives rise to an object of $\mathcal{S}ch_k^{(\alpha)}$, namely, $(\mathcal{V}, \mathcal{A}\big|_{\mathcal{V}}, \nu\big|_{\mathcal{V}})$. (In the formulation of Theorem 6.5.4(ii), $\mathcal{V}$ denotes the latter triple.) A morphism $(\mathcal{Y}', \mathcal{A}', \nu') \to (\mathcal{Y}, \mathcal{A}, \nu)$ in $\mathcal{S}ch_k^{(\alpha)}$ is said to be an *open immersion* if it gives rise to an isomorphism of the first triple with the object of $\mathcal{S}ch_k^{(\alpha)}$ induced by an open subscheme of $\mathcal{Y}$.

6.5.4. Theorem. *(i) The correspondence $\mathcal{X} \mapsto \mathcal{X}^{(\alpha)}$ gives rise to a fully faithful functor*

$$\mathcal{S}ch_K^{(\alpha)} \to \mathcal{S}ch_k^{(\alpha)} \ ;$$

(ii) an object $(\mathcal{Y}, \mathcal{A}, \nu)$ of $\mathcal{S}ch_k^{(\alpha)}$ lies in the essential image of the above functor if and only if the following holds:

(1) the family of open sets $\mathcal{V}$, for which the K-algebra $\mathcal{A}(\mathcal{V})$ is α-special and the canonical morphism $\mathcal{V} \to \mathrm{Spec}(k \otimes_K \mathcal{A}(\mathcal{V}))$ is an isomorphism in $\mathcal{S}ch_k^{(\alpha)}$, forms a net;

(2) for every pair $\mathcal{V} \subset \mathcal{W}$ of sets from (1), the canonical homomorphism $\mathcal{A}(\mathcal{W}) \to \mathcal{A}(\mathcal{V})$ induces an open immersion of affine schemes $\mathrm{Fspec}(\mathcal{A}(\mathcal{V})) \to \mathrm{Fspec}(\mathcal{A}(\mathcal{W}))$ which is compatible with the open immersion $\mathcal{V} \to \mathcal{W}$.

6.5.5. Lemma. *Let $\mathcal{X} = \mathrm{Fspec}(A)$ be a connected α-special affine scheme over K. Then for any open subscheme $\mathcal{V} \subset \mathcal{X}^{(\alpha)}$ that contains a point from $\pi^{-1}(\mathcal{X}_M)$, one has $A \xrightarrow{\sim} (\pi^*\mathcal{O}_{\mathcal{X}})(\mathcal{V})$.*

Proof. The sheaf $\pi^*\mathcal{O}_{\mathcal{X}}$ is associated with the separated presheaf $P = \pi^{-1}\mathcal{O}_{\mathcal{X}}$ whose value at an open subset $\mathcal{V} \subset \mathcal{X}^{(\alpha)}$ is the inductive limit $\varinjlim \mathcal{O}(\mathcal{U})$ taken over all open subschemes $\mathcal{U}$ of $\mathcal{X}$ that contain the image $p(\mathcal{V})$. Thus, we have to verify that, given a finite covering of $\mathcal{V}$ with $\mathcal{V} \cap \pi^{-1}(\mathcal{X}_M) \neq \emptyset$ by open subsets $\{\mathcal{V}_\mu\}$, one has $A \xrightarrow{\sim} L = \mathrm{Ker}(\prod_\mu P(\mathcal{V}_\mu) \rightrightarrows \prod_{\mu,\rho} P(\mathcal{V}_\mu \cap \mathcal{V}_\rho))$.

Let $\{\mathcal{X}_i\}_{i\in I}$ be the set of irreducible components of $\mathcal{X}$. For an open subscheme $\mathcal{U}\subset\mathcal{X}$, we set $I(\mathcal{U})=\{i\in I\big|\mathcal{U}_i\neq\emptyset\}$, where $\mathcal{U}_i=\mathcal{U}\cap\mathcal{X}_i$. Since every open affine subscheme of $\mathcal{X}_i$ is a principal open subset, it follows that if, for open subschemes $\mathcal{U}'\subset\mathcal{U}''\subset\mathcal{X}$ one has $I(\mathcal{U}')=I(\mathcal{U}'')$, then the canonical homomorphism $\mathcal{O}(\mathcal{U}'')\to\mathcal{O}(\mathcal{U}')$ is injective. This also implies that for every open subscheme $\mathcal{U}\subset\mathcal{X}$ there is a canonical injective homomorphism $\mathcal{O}(\mathcal{U})\hookrightarrow\prod_{i\in I(\mathcal{U})}F_i$, where F_i is the fraction $\mathbf{F}_1$-field of A_i (with $\mathcal{X}_i=\mathrm{Fspec}(A_i)$).

Furthermore, for an open subset $\mathcal{V}\subset\mathcal{X}^{(\alpha)}$, we set $I(\mathcal{V})=\bigcap I(\mathcal{U})$, where the intersection is taken over open subschemes of $\mathcal{U}$ that contain the set $\pi(\mathcal{V})$. If such $\mathcal{U}$ is sufficiently small, then $I(\mathcal{V})=I(\mathcal{U})$. It follows from the previous paragraph that, for any $\mathcal{V}$, there is a canonical injective homomorphism $P(\mathcal{V})\hookrightarrow\prod_{i\in I(\mathcal{V})}F_i$. Notice that, if $\mathcal{V}$ contains a point from $\pi^{-1}(\mathcal{X}_M)$, then $I(\mathcal{V})=I$ and $A\xrightarrow{\sim}P(\mathcal{V})$.

We now turn back to the covering of $\mathcal{V}$ with $\mathcal{V}\cap\pi^{-1}(\mathcal{X}_M)\neq\emptyset$ by open subsets $\{\mathcal{V}_\mu\}$. Let $(f_\mu)_\mu$ be an element in the above kernel L. For every μ, there is a canonical injective homomorphism $P(\mathcal{V}_\mu)\hookrightarrow\prod_{i\in I(\mathcal{V}_\mu)}F_i$. If $i\in I(\mathcal{V}_\mu)I(\mathcal{V}_nu)$, then the images of the elements f_μ and g_ν under the canonical homomorphisms $P(\mathcal{V}_\mu)\to F_i$ and $P(\mathcal{V}_\mu)\to F_i$ are equal. This means that there is an injective homomorphism $L\hookrightarrow\prod_{i\in I}F_i$. But by the assumption, there exists $\mathcal{V}_\rho$ which contains a point from $\pi^{-1}(\mathcal{X}_M)$ and, therefore, $I(\mathcal{V}_\rho)=I$ and $P(\mathcal{V}_\rho)=A$, i.e., there exists $f\in A$ with $f\big|_{\mathcal{V}_\rho}=f_\rho$. The above remark implies that $f\big|_{\mathcal{V}_\mu}=f_\mu$ for all μ. ∎

Proof of Theorem 6.5.4. (i) The functor considered takes an α-special scheme $\mathcal{X}$ to the triple $(\mathcal{X}^{(\alpha)},\pi^*\mathcal{O}_{\mathcal{X}},\nu)$, where π is the morphism $\mathcal{X}^{(\alpha)}\to\mathcal{X}$ and ν is the canonical homomorphism $\pi^*\mathcal{O}_{\mathcal{X}}\to\mathcal{O}_{\mathcal{X}^{(\alpha)}}$. Let $\varphi:(\mathcal{X}^{(\alpha)},\pi^*\mathcal{O}_{\mathcal{X}},\nu)\to(\mathcal{X}'^{(\alpha)},\pi'^*\mathcal{O}_{\mathcal{X}'},\nu')$ be a morphism in $\mathcal{S}ch_k^{(\alpha)}$. For an α-special open connected affine subscheme $\mathcal{U}\subset\mathcal{X}$, we take a point $x\in\mathcal{U}^{(\alpha)}$ whose image in $\mathcal{U}$ lies in $\mathcal{U}_M$, and an α-special open connected affine subscheme $\mathcal{U}'$ of $\mathcal{X}'$. *We claim that* $\varphi\big|_{\mathcal{U}^{(\alpha)}}=\psi^{(\alpha)}$ *for a morphism* $\psi:\mathcal{U}\to\mathcal{U}'\subset\mathcal{X}'$. Indeed, let $\mathcal{V}=\mathcal{U}^{(\alpha)}\cap\varphi^{-1}(\mathcal{U}'^{(\alpha)})$. By Lemma 6.5.5, one has $A_{\mathcal{U}}\xrightarrow{\sim}(\pi^*\mathcal{O}_{\mathcal{X}})(\mathcal{V})$ and $A'_{\mathcal{U}'}\xrightarrow{\sim}(\pi'^*\mathcal{O}_{\mathcal{X}'})(\mathcal{U}'^{(\alpha)})$, and so the homomorphism of sheaves $\pi'^*\mathcal{O}_{\mathcal{X}'}\to\varphi_*(\pi^*\mathcal{O}_{\mathcal{X}})$ gives rise to a homomorphism of K-algebras $A'_{\mathcal{U}'}=(\pi'^*\mathcal{O}_{\mathcal{X}'})(\mathcal{U}'^{(\alpha)})\to A_{\mathcal{U}}=(\pi^*\mathcal{O}_{\mathcal{X}})(\mathcal{V})$. The latter gives rise to a morphism of schemes $\psi:\mathcal{U}\to\mathcal{U}'\subset\mathcal{X}'$ such that the restriction of $\psi^{(\alpha)}:\mathcal{U}^{(\alpha)}\to\mathcal{U}'^{(\alpha)}\subset\mathcal{X}'^{(\alpha)}$ to the dense open subscheme $\mathcal{V}$ coincides with the morphism $\mathcal{V}\to\mathcal{U}'^{(\alpha)}\subset\mathcal{X}'^{(\alpha)}$ induced by φ. Since the scheme $\mathcal{X}'^{(\alpha)}$ is separated and $\mathcal{U}^{(\alpha)}$ is reduced, we get $\varphi\big|_{\mathcal{U}^{(\alpha)}}=\psi^{(\alpha)}$.

In this way we get a system of compatible morphisms $\psi_{\mathcal{U}}:\mathcal{U}\to\mathcal{X}'$ for all α-special open affine subschemes $\mathcal{U}$ of $\mathcal{X}$. This system defines a morphism $\psi:\mathcal{X}\to\mathcal{X}'$ of schemes over K which induces the morphism φ.

(ii) That the image of an α-special scheme $\mathcal{X}$ in $\mathcal{S}ch^{(\alpha)}$ possesses the required properties follows from Definition 6.5.1 and Lemma 6.5.5.Given an object of the category $\mathcal{S}ch^{(\alpha)}$ with those properties, a construction of the required α-special scheme over K is done in the same way as in the proof of Theorem 6.4.4(ii). ∎

6.6. Classes of morphisms between schemes over $\mathcal{S}ch_{\mathbf{F}_1}$. The existence of fiber products in the category $\mathcal{S}ch$ enables one to extend various classes of morphisms from the category $\mathcal{S}ch_{\mathbf{Z}}$ to the whole category $\mathcal{S}ch$. Namely, let $\mathcal{P}$ be a property of morphisms of schemes over $\mathbf{Z}$ which is local with respect to the target.

6.6.1. Definition. A morphism $\varphi : \mathcal{X}' \to \mathcal{X}$ of schemes over $\mathbf{F}_1$ is said *to have the property* $\mathcal{P}$ if, for any morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ from a scheme over $\mathbf{Z}$, the induced morphism $\mathcal{Y} \times_{\mathcal{X}} \mathcal{X}' \to \mathcal{Y}$ of schemes over $\mathbf{Z}$ has the property $\mathcal{P}$.

§7. Schemes of finite type over a valuation $\mathbf{F}_1$-algebra

7.1. Flat and strict schemes of finite type over K°. Let K° be a valuation $\mathbf{F}_1$-algebra with fraction $\mathbf{F}_1$-field K. For a scheme $\mathcal{X}$ over K° and a Zariski prime ideal $\mathfrak{r} \subset K^\circ$, we set $\mathcal{X}_{(\mathfrak{r})} = \mathcal{X} \otimes_{K^\circ} K^\circ_{\mathfrak{r}}$, $\mathcal{X}(\mathfrak{r}) = \mathcal{X} \otimes_{K^\circ} \kappa(\mathfrak{r})$ and $\mathcal{X}^{(\mathfrak{r})} = \mathcal{X} \otimes_{K^\circ} K^\circ/\mathfrak{r}$. Notice that $\mathcal{X}(\mathfrak{r}) = (\mathcal{X}_{(\mathfrak{r})})^{(\mathfrak{r})} = (\mathcal{X}^{(\mathfrak{r})})_{(\mathfrak{r})}$. If $\mathfrak{r} = 0$, one has $K^\circ_{\mathfrak{r}} = K$, and the scheme $\mathcal{X}_\eta = \mathcal{X} \otimes_{K^\circ} K$ over K is said to be the *generic fiber of* $\mathcal{X}$. If $\mathfrak{r} = K^{\circ\circ}$, one has $K^\circ/\mathfrak{r} = \widetilde{K}$, and the scheme $\mathcal{X}_s = \mathcal{X} \otimes_{K^\circ} \widetilde{K}$ over $\widetilde{K}$ is said to be the *closed* (or *special*) *fiber of* $\mathcal{X}$. There is a canonical closed immersion $\mathcal{X}_s \to \mathcal{X}$ and, if there exists a nonzero element $\alpha \in K^\circ$ with $K = K^\circ_\alpha$ (e.g., the valuation on K is of finite rank), the canonical injective map $\mathcal{X}_\eta \to \mathcal{X}$ is an open immersion. Of course, if the valuation on K is trivial, both morphisms $\mathcal{X}_s \to \mathcal{X}$ and $\mathcal{X}_\eta \to \mathcal{X}$ are isomorphisms.

7.1.1. Definition. (i) A K°-module M is said to be *flat* if it possesses the following properties:

(1) if $\alpha m = \beta m$ for $\alpha, \beta \in K^\circ$ and $m \in M$, then either $\alpha = \beta$ or $m = 0$;

(2) if $\alpha m = \alpha n$ for $\alpha \in K^\circ$ and $m, n \in A$, then either $\alpha = 0$ or $m = n$.

(ii) A scheme $\mathcal{X}$ over K° is said to be *flat over* K° if it is covered by a family of open p-affine subschemes $\mathcal{U}$ for which $A_{\mathcal{U}}$ is a flat K°-module.

(iii) A scheme $\mathcal{X}$ over K° is said to be *strict over* K° if it is flat over K° and the $\mathcal{U}$'s from (i) possess the following additional properties:

(3) $\mathbf{m}_{A_\mathcal{U}} \cap K^\circ = K^{\circ\circ}$;

(4) if $\mathcal{V}$ is a principal open subset of $\mathcal{U}$ with $\mathbf{m}_{A_\mathcal{V}} \cap K^\circ = K^{\circ\circ}$, then $I_{A_\mathcal{V}} \cap K^{\circ\circ} A_\mathcal{V} = 0$.

Of course, if the valuation on K is trivial, every scheme over $K^\circ = K$ is strict. If $\mathcal{X}$ is flat over K° then, for every Zariski prime ideal $\mathfrak{r} \subset K^\circ$, $\mathcal{X}_{(\mathfrak{r})}$ and $\mathcal{X}^{(\mathfrak{r})}$ flat over $K^\circ_\mathfrak{r}$ and $K^\circ/\mathfrak{r}$, respectively. If $\mathcal{X} = \mathrm{Fspec}(A)$ is strict over K°, we say that A is a strict K°-algebra.

Let M be a flat K°-module. Then the canonical homomorphism $M \to M \otimes_{K^\circ} K$ is injective. The latter is a free K-vector space which is a direct sum of free K-vector spaces of dimension one. This defines a decomposition of M in a direct sum of Zariski K°-submodules. Let P be such a Zariski K°-submodule. It is a subset of M that contains zero and has the property that, for every pair of nonzero elements $m, n \in P$, there exist nonzero $\alpha, \beta \in K^\circ$ with $\alpha m = \beta n$. Since K° is a valuation $\mathbf{F}_1$-algebra, the latter is equivalent to the property that, for every pair of nonzero elements $m, n \in P$ there exists a nonzero element $\alpha \in K^\circ$ with either $\alpha m = n$ or $m = \alpha n$. By the way, the above decomposition defines a K°-submodule E on M such that $(m, n) \in E$ if either $m, n = 0$, or $m, n \in P \backslash \{0\}$ for P as above. The quotient $\overline{M} = M/E$ is an $\mathbf{F}_1$-module, and its nonzero elements correspond to the above K°-modules P. If M is a K°-algebra, then $\overline{M}$ is an $\mathbf{F}_1$-algebra.

7.1.2. Proposition. *Every flat reduced quasi-irreducible finitely generated K°-algebra A is a free $K^\circ_\mathfrak{r}$-module, where $\mathfrak{r} = \mathbf{m}_A \cap K^\circ$.*

Proof. We can replace K° by $K^\circ_\mathfrak{r}$ and assume that $\mathbf{m}_A \cap K^\circ = K^{\circ\circ}$. It suffices to show that every K°-module P from the decomposition of A as above is free of rank one. For this it suffices to show that $P \neq K^{\circ\circ} P$. Indeed, if this is so, then the above property implies that every element from $P \backslash K^{\circ\circ} P$ is a generator of P.

Consider first the case when the K°-algebra A is integral, and take a surjective homomorphism $K^\circ[T_1, \dots, T_n] \to A : T_i \mapsto f_i$. Notice that if a nonzero element $\alpha f^\mu = \alpha f_1^{\mu_1} \cdot \ldots \cdot f_n^{\mu_n}$ with $\alpha \in K^\circ$ and $\mu \in \mathbf{Z}_+^n$ belongs to P, then $f^\mu \in P$. Thus, the equality $P = K^{\circ\circ} P$ implies that there exist sequences $\mu^{(1)}, \mu^{(2)}, \ldots \in \mathbf{Z}_+^n$ and $\alpha_1, \alpha_2, \ldots \in K^{\circ\circ} \backslash \{0\}$ such that $f^{\mu^{(i)}} = \alpha_i f^{\mu^{(i+1)}}$ for all $i \geq 1$. We prove by induction on n that existence of such sequences is impossible. This is of course impossible if $n = 0$, and so we assume that $n \geq 1$ and that this is impossible for strictly smaller values of n. We may assume that $\mu_1^{(1)}, \ldots, \mu_m^{(1)} \geq 1$ and $\mu_{m+1}^{(1)} = \ldots = \mu_n^{(1)} = 0$. Since $\mathbf{m}_A \cap A = K^{\circ\circ}$, it follows that $m \geq 1$. We may also assume that among sequences with such properties the number $\sum_{k=1}^m \mu_k^{(1)}$ is minimal. *We then claim that there exists $l \geq 2$ such that $\mu_1^{(t)} = 0$ for all $t \geq l$.* Indeed, if there exist an infinite sequence $l_1 = 1 < l_2 < \ldots$ with $\mu_1^{(l_j)} \geq 1$ for all $j \geq 1$, then we can replace our sequences with the subsequences with numbers $l_1, l_2, \dots$ and assume that $\mu_1^{(j)} \geq 1$ for all $j \geq 1$. Since A is an integral domain, we can divide all of the elements $f^{\mu^{(j)}}$ by f_1 and get sequences with the smaller sum $\sum_{k=1}^{m} \mu_k^{(1)}$. The claim follows. Thus, we can replace our sequences with the subsequences with numbers $l, l+1, \dots$, and we get elements $f^{\mu^{(j)}}$ as above which lie in the K°-subalgebra of A generated by $f_2, \dots, f_n$. By induction, existence of such sequences is impossible.

In the general case, let $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ denote the intersections with A of the Zariski prime ideals of the finitely generated K-algebra $A \otimes_{K^\circ} K$. Then $\mathfrak{p}_i \cap K^\circ = 0$ for all $1 \leq i \leq n$ and, since A is reduced, one has $\bigcap_{i=1}^{n} \mathfrak{p}_i = 0$. *We claim that* $P \cap \mathfrak{p}_i = 0$ *for some* $1 \leq i \leq n$. Indeed, assume that, for every $1 \leq i \leq n$, the intersection $P \cap \mathfrak{p}_i$ contains a nonzero element f_i. Let m be the maximal number such that some of the elements f_i's lies in the intersection of m of the Zariski prime ideals $\mathfrak{p}_j$'s. One clearly has $1 \leq m \leq n-1$. We may assume that $f_1 \in \bigcap_{i=1}^{m} \mathfrak{p}_i$. By the property of P, we can find a nonzero element $\alpha \in K^\circ$ with either $f_1 = \alpha f_{m+1}$, or $f_{m+1} = \alpha f_1$. In both cases, either f_1 or f_{m+1} lies in the intersection $\bigcap_{i=1}^{m+1} \mathfrak{p}_i$, which contradicts the minimality of m. The claim implies that the canonical surjective homomorphism $A \to A/\mathfrak{p}_i$ is injective on P. Since the latter quotient satisfies the same assumption as A and is integral, we get $P \neq K^{\circ\circ}P$, by the previous case. ∎

7.1.3. Proposition. *Let $\mathcal{X}$ be a strict scheme of finite type over K°. Then*

(i) the correspondence $\mathcal{U} \mapsto \mathcal{U}_s = \mathcal{U} \cap \mathcal{X}_s$ induces a bijection between the set of strict open subschemes of $\mathcal{X}$ and that of open subschemes of $\mathcal{X}_s$;

(ii) the correspondence $\mathcal{Y} \mapsto \overline{\mathcal{Y}}$ (the closure of $\mathcal{Y}$) induces a bijection between the sets of irreducible components of $\mathcal{X}_\eta$ and of $\mathcal{X}$;

(iii) if $\mathcal{Y}$ is a Zariski closed subset of $\mathcal{X}_\eta$, then $\overline{\mathcal{Y}}$ is Zariski closed in $\mathcal{X}$ and is a strict scheme of finite type over K°;

(iv) $\pi_0(\mathcal{X}_\eta) \xrightarrow{\sim} \pi_0(\mathcal{X})$ and, if $\mathcal{X}$ is affine, then $\pi_0(\mathcal{X}_s) \xrightarrow{\sim} \pi_0(\mathcal{X})$.

An open subscheme $\mathcal{U} \subset \mathcal{X}$ is said to be *strict* if it is strict as a scheme over K°. Notice that the intersection of two strict open subschemes is not necessarily a strict open subscheme. Indeed, let $\mathcal{X} = \mathrm{Fspec}(A)$, where A is the quotient of $K^\circ[T_1, T_2]$ by the ideal generated by the pair (T_1T_2, a) with $a \in K^{\circ\circ} \backslash \{0\}$. If t_i is the image of T_i in A, then the principal open subschemes $D(t_1)$ and $D(t_2)$ are strict open subschemes of $\mathcal{X}$, but their intersection coincides with $D(a)$ and, therefore, it is not strict over K°.

Proof. To prove the statement, it suffices to consider the case when $\mathcal{X} = \mathrm{Fspec}(A)$ is affine with a strict finitely generated K°-algebra A. In this case, one has $\mathcal{X}_s = \mathrm{Fspec}(\widetilde{A})$ with $\widetilde{A} = A \otimes_{K^\circ} \widetilde{K} = A/(K^{\circ\circ})$ and $\mathcal{X}_\eta = \mathrm{Fspec}(\mathcal{A})$ with $\mathcal{A} = A \otimes_{K^\circ} K$.

(i) Let first $\mathcal{U}$ be a nonempty strict principal open subset of $\mathcal{X}$, i.e., $\mathcal{U} = D(f)$ for $f \in A$ with $\mathbf{m}_{A_f} \cap K^\circ = K^{\circ\circ}$. Nonemptyness of $\mathcal{U}$ and the latter equality imply that $f \notin \mathrm{z}\mathbf{r}((K^{\circ\circ}))$. This means that the image $\widetilde{f}$ of f in $\widetilde{A}$ is not nilpotent and, in particular, $\mathcal{U}_s = D(\widetilde{f}) \neq \emptyset$. Suppose that $\mathcal{U}_s \subset \mathcal{V}_s$ for a strict principal open subset $\mathcal{V} = D(g)$ with $g \in A$. Then $\widetilde{f}^{m+n} = \widetilde{g}\widetilde{h}\widetilde{f}^n$ for some $h \in A$ and $m, n \geq 0$. It follows that $f^{m+n} = ghf^n$ and, therefore, $\mathcal{U} \subset \mathcal{V}$. Thus, the correspondence considered induces a bijection between the set of strict principal open subsets of $\mathcal{X}$ and that of principal open subsets of $\mathcal{X}_s$.

Furthermore, let $\mathcal{U}$ and $\mathcal{V}$ be nonempty strict elementary open subsets of $\mathcal{X}$, which are defined by principal open subsets $D_\mathcal{U} = D(f)$ and $D_\mathcal{V} = D(g)$ and finitely generated Zariski ideals $\mathbf{a}_\mathcal{U} \subset I_{A_f}$ and $\mathbf{a}_\mathcal{V} \subset I_{A_g}$ (see §4.2). Suppose that $\mathcal{U}_s \subset \mathcal{V}_s$. Then $D(\widetilde{f}) \subset D(\widetilde{g})$ and, by the previous case, one has $D_\mathcal{U} \subset D_\mathcal{V}$. The assumption on strictness of $\mathcal{U}$ and $\mathcal{V}$ implies that the canonical homomorphisms $A_f \to \widetilde{A}_{\widetilde{f}}$ and $A_g \to \widetilde{A}_{\widetilde{g}}$ induce isomorphisms of the corresponding idempotent $\mathbf{F}_1$-algebras. Since the image of $\mathbf{a}_{\mathcal{V}_s}$ in the idempotent $\mathbf{F}_1$-subalgebra of $\widetilde{A}_{\widetilde{f}}$ lies in $\mathbf{a}_{\mathcal{U}_s}$, it follows that the image of $\mathbf{a}_\mathcal{V}$ in I_{A_f} lies in $\mathbf{a}_\mathcal{U}$ and, therefore, $\mathcal{U} \subset \mathcal{V}$. Thus, the correspondence considered induces a bijection between the set of strict elementary open subsets of $\mathcal{X}$ and that of elementary open subsets of $\mathcal{X}_s$.

Finally, let $\mathcal{U}$ and $\mathcal{V}$ be nonempty open subschemes of $\mathcal{X}$ with $\mathcal{U}_s \subset \mathcal{V}_s$, and let $\{\mathcal{U}_i\}_{i \in I}$ and $\{\mathcal{V}_j\}_{j \in J}$ be coverings of $\mathcal{U}$ and $\mathcal{V}$ by elementary open subsets. Then for every $i \in I$ the elementary open subset $\mathcal{U}_{s,i}$ is covered by the elementary open subsets $\mathcal{U}_{s,i} \cap \mathcal{V}_{s,j} = (\mathcal{U}_i \cap \mathcal{V}_j)_s$ for $j \in J$. This implies that $\mathcal{U}_{s,i} \subset \mathcal{V}_{s,j}$ for some $j \in J$. By the previous case, we get $\mathcal{U}_i \subset \mathcal{V}_j$. It follows that $\mathcal{U} \subset \mathcal{V}$.

(ii) First of all, we notice that the image of $\mathrm{Zspec}(\mathcal{A})$ in $\mathrm{Zspec}(A)$ is the set of Zariski prime ideals $\mathfrak{p} \subset A$ with $\mathfrak{p} \cap K^\circ = 0$. If $\mathfrak{p}$ corresponds to a Zariski prime ideal $\mathfrak{q} \subset \mathcal{A}$, then $\mathfrak{q} = \mathfrak{p}K$, $\mathfrak{p} = \mathfrak{q} \cap A$, $\Pi_\mathfrak{q} = \{(\alpha f, \alpha g) \big| \alpha \in K \text{ and } (f, g) \in \Pi_\mathfrak{p}\}$ and $\Pi_\mathfrak{p} = \Pi_\mathfrak{q} \cap (A \times A)$. It follows that the closure of $\mathcal{X}_\eta^{(\mathfrak{q})}$ in $\mathcal{X}$ coincides with $\mathcal{X}^{(\mathfrak{p})}$. Let $\mathcal{Y}$ be an irreducible component of $\mathcal{X}_\eta$, i.e., $\mathcal{Y} = \mathcal{X}_\eta^{(\mathfrak{q})}$, where $\mathfrak{q}$ is a Zariski prime ideal of $\mathcal{A}$ for which $\Pi_\mathfrak{q}$ is a minimal prime ideal of $\mathcal{A}$. If $\mathfrak{p} = \mathfrak{q} \cap A$, the above remark easily implies that $\Pi_\mathfrak{p}$ is a minimal prime ideal of A and $\mathcal{X}^{(\mathfrak{p})} = \overline{\mathcal{Y}}$, i.e., $\overline{\mathcal{Y}}$ is an irreducible component of $\mathcal{X}$. Furthermore, let $\mathfrak{q}_1, \ldots, \mathfrak{q}_n$ be Zariski prime ideals of $\mathcal{A}$ such that $\Pi_{\mathfrak{q}_1} \cap \ldots \cap \Pi_{\mathfrak{q}_n} = \mathbf{n}(\mathcal{A})$. Then $\Pi_{\mathfrak{p}_1} \cap \ldots \cap \Pi_{\mathfrak{p}_n} = \mathbf{n}(A)$ for $\mathfrak{p}_i = \mathfrak{q}_i \cap A$. Lemma 1.2.7(ii) implies that each prime ideal Π of A contains a prime ideal of the form $\Pi_\mathfrak{p}$ for $\mathfrak{p} = \mathfrak{p}_{i_1} \cup \ldots \cup \mathfrak{p}_{i_k}$. If $\mathfrak{q} = \mathfrak{p}K$, the above remark implies that the point of $\mathcal{X}$ that corresponds to the prime ideal Π lies in the closure of $\mathcal{X}_\eta^{(\mathfrak{q})}$ (since it coincides with $\mathcal{X}^{(\mathfrak{p})}$). Thus, the closure of $\mathcal{X}_\eta$ in $\mathcal{X}$ coincides with $\mathcal{X}$, and the required fact follows.

(iii) We may assume that $\mathcal{X}$ is connected. By (iii), $\mathcal{X}_\eta$ is also connected, and so any Zariski closed subset $\mathcal{Y}$ of $\mathcal{X}_\eta$ is of the form $\mathrm{Fspec}(\mathcal{A}/\mathbf{b})$ for a Zariski ideal $\mathbf{b} \subset \mathcal{A}$. *We claim that* $\overline{\mathcal{Y}} = \mathrm{Fspec}(A/\mathbf{a})$ *for the Zariski ideal* $\mathbf{a} = \mathbf{b} \cap A$. Indeed, since $\mathbf{b} = \mathbf{a}K$, one has $A/\mathbf{a} \otimes_{K^\circ} K \xrightarrow{\sim} \mathcal{A}/\mathbf{a}$, and the claim follows from the statement (ii) applied to the strict scheme $\mathrm{Fspec}(A/\mathbf{a})$ of finite type over K°.

(iv) It suffices to show that the canonical homomorphisms of idempotent $\mathbf{F}_1$-algebras $I_A \to I_{\widetilde{A}}$ and $I_A \to I_{\mathcal{A}}$ are bijections. Bijectivity of the former follows from the property (3) of Definition 7.1.1. Injectivity of the latter is trivial. Let $\frac{f}{\alpha}$ be a nonzero idempotent in $\mathcal{A}$, where $f \in A$ and $\alpha \in K^\circ$. Then $(\alpha f)^2 = \alpha f$, i.e., αf is an idempotent in A. The same property (3) implies that α is invertible in K° and, therefore, $\frac{f}{\alpha} \in I_A$. ∎

7.1.4. Proposition. *Let K'°/K° be an extension of valuation $\mathbf{F}_1$-algebras, and let $\mathcal{X}$ and $\mathcal{Y}$ be nonempty strict schemes over K° and K'°, respectively. Then*

(i) the fiber product $\mathcal{X} \times_{K^\circ} \mathcal{Y}$ is a nonempty strict scheme over K'°;

(ii) every strict open subscheme of $\mathcal{X} \times_{K^\circ} \mathcal{Y}$ is covered by open subschemes of the form $\mathcal{U} \times_{K^\circ} \mathcal{V}$ for strict open subschemes $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{V} \subset \mathcal{Y}$.

Proof. (i) It suffices to consider the case when $\mathcal{X} = \mathrm{Fspec}(A)$ and $\mathcal{Y} = \mathrm{Fspec}(B)$ are affine with strict K° and K'°-algebras A and B, respectively. In this case validity of the properties (1) and (2) of Definition 7.1.1 for $C = A \otimes_{K^\circ} B$ follows from Corollary 2.7.7. To verify the property (3), it suffices to show that in our situation one has $I_C \cap K'^{\circ\circ} C = 0$.

7.1.5. Lemma. *In the above situation, one has $I_A \otimes_{\mathbf{F}_1} I_B \xrightarrow{\sim} I_C$.*

Proof. *The homomorphism is injective.* Indeed, suppose that $f_1 \otimes g_1 = f_1 \otimes g_2$ in C for some nonzero $f_1, f_2 \in I_A$ and $g_1, g_2 \in I_B$. By Corollary 2.7.7(ii), we may assume that there exists an element $\alpha \in K^\circ$ with $f_1 = \alpha f_2$ and $\alpha g_1 = g_2$. Since f_1 and f_2 are nonzero idempotents, we get $\alpha^2 f_2 = \alpha f_2$. The property (1) implies that $\alpha^2 = \alpha$ and, therefore, $\alpha = 1$.

The homomorphism is surjective. Let e be a nonzero idempotent in C. Then $e = f \otimes g$ for some $f \in A$ and $g \in B$. By the equality $e^2 = e$ and Corollary 2.7.7(ii), we may assume that there exists an element $\alpha \in K^\circ$ with $f^2 = \alpha f$ and $\alpha g^2 = g$. Then $(\alpha g)^2 = \alpha g$, i.e., the element αg is a nonzero idempotent in B. If $\alpha \in K^{\circ\circ}$, then $\alpha g \in I_B \cap K^{\circ\circ} B$, which is impossible. Thus, α is invertible in K°. Replacing f by $\alpha^{-1} f$ and g by αg, we may assume that f and g are idempotents, i.e., $e \in I_A \otimes_{\mathbf{F}_1} I_B$. ∎

Suppose that e is a nonzero idempotent in $I_C \cap K'^{\circ\circ} C$. By Lemma 7.1.5, one has $e = f \otimes g$ for $f \in I_A$ and $g \in I_B$. Thus, $f \otimes g = \alpha(f \otimes g)$ for some $\alpha \in K^{\circ\circ}$. Corollary 2.7.7(ii) implies that there exists an element $\beta \in K^\circ$ with either $f = \alpha\beta f$ and $\beta g = g$, or $\beta f = \alpha f$ and $g = \beta g$. The property (3) then implies that β and α are invertible in K°, which is a contradiction.

(ii) It suffices to consider the case when $\mathcal{X} = \mathrm{Fspec}(A)$ and $\mathcal{Y} = \mathrm{Fspec}(B)$ are affine with strict K°-algebras A and B. Let $\mathcal{W}$ be a strict elementary open subset of $\mathcal{X} \otimes_{K^\circ} \mathcal{Y}$, i.e., $\mathcal{W} = \{z \in D(h) \big| e_i(z) = 0 \text{ for } 1 \le i \le n\}$, where $h \in A \otimes_{K^\circ} B$ and $e_1, \dots, e_n$ are idempotents in $(A \otimes_{K^\circ} B)_h$. One has $h = f \otimes g$ for $f \in A$ and $g \in B$. It follows that $D(h) = D(f) \times D(g)$. We may therefore replace $\mathcal{X}$ and $\mathcal{Y}$ by $D(f)$ and $D(g)$, respectively, and assume that $\mathcal{W}$ is defined only by the equalities $e_i = 0$. It suffices to consider the case $n = 1$. If e is a nonzero idempotent in $A \otimes_{K^\circ} B$, Lemma 7.1.5 implies that $e = f \otimes g$ for $f \in I_A$ and $g \in I_B$. We get $\mathcal{W} = \mathcal{U} \times \mathcal{V}$ for $\mathcal{U} = \{x \in \mathcal{X} \big| f(x) = 0\}$ and $\mathcal{V} = \{y \in \mathcal{Y} \big| g(x) = 0\}$. ∎

7.1.6. Proposition. *Let $\mathcal{X}$ and $\mathcal{Y}$ be strict schemes of finite type over K°. If both schemes $\mathcal{X}$ and $\mathcal{Y}$ are Zariski reduced (resp. reduced; resp. connected; resp. irreducible; resp. quasi-integral; resp. integral), then so is the direct product $\mathcal{X} \times_{K^\circ} \mathcal{Y}$.*

Proof. We may assume that $\mathcal{X} = \mathrm{Fspec}(A)$ and $\mathcal{Y} = \mathrm{Fspec}(B)$ are affine with Zariski reduced strict finitely generated K°-algebras A and B. Since the homomorphisms $A \to \mathcal{A} = A \otimes_{K^\circ} K$, $B \to \mathcal{B} = B \otimes_{K^\circ} K$ and $A \otimes_{K^\circ} B \to \mathcal{A} \otimes_K \mathcal{B}$ are injective, the situation is reduced to the case when the valuation on K is trivial, i.e., $K^\circ = K$. In this case, $a \otimes b = c \otimes d$ for some nonzero $a, c \in A$ and $b, d \in B$, then $c = \lambda a$ and $b = \lambda d$ for a unique nonzero element $\lambda \in K$. This immediately implies that $A \otimes_K B$ is Zariski reduced, and reduces the case of quasi-integral $\mathcal{X}$ and $\mathcal{Y}$ to that of integral ones. Suppose that A and B are integral, and assume that $(a \otimes b)(c \otimes d) = (a \otimes b)(c' \otimes d')$ for nonzero elements of $A \otimes_K B$. Then $ac \otimes bd = ac' \otimes bd'$ and, therefore, $ac' = \lambda ac$ and $bd = \lambda bd'$ for some $\lambda \in K^*$. Since A and B are integral, it follows that $c' = \lambda c$ and $d = \lambda d'$ and, therefore, $c \otimes d = c' \otimes d$, i.e., $A \otimes_K B$ is integral. Suppose now that A and B are reduced, and assume that, for some nonzero elements $a \otimes b, c \otimes d \in A \otimes_K B$, there exists $n \ge 1$ such that $a^i \otimes b^i = c^i \otimes d^i$ for all $i \ge n$. Then $c^i = \lambda_i a_i$ and $b^i = \lambda_i d^i$ for some $\lambda_i \in K^*$. *We claim that $c = \lambda a$ and $b = \lambda d$ for $\lambda = \lambda_{n+1} \lambda_n^{-1}$.* Indeed, since A and B are reduced, it suffices to verify that the pairs $(\lambda a, c)$ and $(b, \lambda d)$ lie in the intersection of all of the prime ideals $\Pi_{\mathfrak{p}}$ and $\Pi_{\mathfrak{q}}$, respectively, for Zariski prime ideals $\mathfrak{p} \subset A$ and $\mathfrak{q} \subset B$. If $a \in \mathfrak{p}$, then each of the equalities $c^i = \lambda_i a_i$ implies that $c \in \mathfrak{p}$ and, therefore, $(\lambda a, c) \in \Pi_{\mathfrak{p}}$. If $a \notin \mathfrak{p}$, the same equalities imply that $c \notin \mathfrak{p}$ and, moreover, the equalities for $i = n$ and $n + 1$ imply that the images of the elements λa and c in the quotient $A/\Pi_{\mathfrak{p}}$ are equal, i.e., $(\lambda a, c) \in \Pi_{\mathfrak{p}}$. It follows that $c = \lambda a$. The same reasoning shows that $(b, \lambda d) \in \Pi_{\mathfrak{q}}$ for all Zariski prime ideals $\mathfrak{q} \subset B$ and, therefore, $b = \lambda d$. The claim implies that $A \otimes_K B$ is reduced. Furthermore, assume that $\mathcal{X}$ and $\mathcal{Y}$ are irreducible. Replacing A by $A/\mathbf{n}(A)$ and B by $B/\mathbf{n}(B)$, we may assume that $\mathcal{X}$ and $\mathcal{Y}$ are reduced and, therefore, integral. Then $\mathcal{X} \times_K \mathcal{Y}$ is also integral and, in particular, irreducible. Finally, if $\mathcal{X}$ and $\mathcal{Y}$ are connected, then connectedness of $\mathcal{X} \times_K \mathcal{Y}$ follows by induction from the cases of irreducible and of nonempty $\mathcal{X}$ and $\mathcal{Y}$. ∎

7.1.7. Remarks. (i) The statement of Proposition 7.1.2 is not true in general if A is not reduced. Indeed, let K be an $\mathbf{F}_1$-field with non-discrete valuation of rank one. Take a nonzero element $\alpha \in K^{\circ\circ}$, and consider the K°-algebra $A = K^\circ[T_1, T_2]/E$, where E is the ideal defined by the pairs $(T_2^2, 0)$ and $(T_2, \alpha T_1 T_2)$. If f and g are the images of T_1 and T_2, then $g^2 = 0$ and $g = \alpha f g$. It follows that A is a quasi-integral K°-algebra and, for the K°-module P from the proof of Proposition 7.1.2 that contains the element g, one has $P = K^{\circ\circ}P$, i.e., A is not a free K°-module.

(ii) Suppose that $\mathcal{X}$ is an irreducible scheme flat over K°. If $\mathcal{U}$ is an open affine subscheme of $\mathcal{X}$, then the intersection $A_\mathcal{U} \cap K^\circ$ is a Zariski prime ideal $\mathfrak{r}$ of K°, and $\mathcal{U}$ is a strict scheme over $K^\circ_\mathfrak{r}$. Suppose in addition that there exists an open affine subscheme $\mathcal{U}$ strict over K°. In this case, if the valuation on K is of rank at most one, then $\mathcal{X}$ is strict over K°. But if the valuation is of higher rank, $\mathcal{X}$ is not necessarily strict over K°. Indeed, suppose that there exist nonzero elements $a, b \in K^{\circ\circ}$ such that $|a| < |b^n|$ for all $n \geq 1$. We set $\mathcal{U} = \mathrm{Fspec}(A)$, where $A = K^\circ[T^{\pm 1}]$, and $\mathcal{V} = \mathrm{Fspec}(B)$, where B is the quotient of $K^\circ[T_1, T_2, S]$ by the ideal generated by the pairs $(T_1 T_2, a)$ and $(Sb, 1)$. Notice that there is a canonical isomorphism $A_b \stackrel{\sim}{\to} B_{t_1} : T \mapsto t_1$, where t_i is the image of T_i in B. Then $\mathcal{U}$ is strict over K°, but the irreducible scheme $\mathcal{X}$ obtained by gluing of $\mathcal{U}$ and $\mathcal{V}$ along $D_\mathcal{U}(b) \stackrel{\sim}{\to} D_\mathcal{V}(t_1)$ is not strict over K°.

7.2. Integral flat schemes of finite type over K°. Let $\mathcal{X}$ be an integral flat scheme of finite type over K°. Then the intersection of all nonempty open subschemes of $\mathcal{X}$ is a nonempty connected open affine subscheme of $\mathcal{X}_\eta$ denoted by $\check{\mathcal{X}}_\eta$, and $A_{\check{\mathcal{X}}_\eta}$ is a finitely generated K-field denoted by $K(\mathcal{X})$ and called the *field of rational functions* on $\mathcal{X}$. The complement $\mathcal{X}_\eta \backslash \check{\mathcal{X}}_\eta$ is a maximal proper Zariski closed subset of $\mathcal{X}_\eta$. We say that the generic fiber $\mathcal{X}_\eta$ is *geometrically irreducible* (resp. *reduced*) if, for any homomorphism $K \to k^\cdot$ to the $\mathbf{F}_1$-field of a (usual) field k, the k-scheme $\mathcal{X}_\eta \otimes_K k$ is irreducible (resp. reduced).

7.2.1. Proposition. *In the above situation, $\mathcal{X}_\eta$ is geometrically irreducible (resp. reduced) if and only if K is algebraically closed in $K(\mathcal{X})$ (i.e., the quotient group $K(\mathcal{X})^*/K^*$ has no torsion).*

Proof. We may assume that the valuation on K is trivial, i.e., $K^\circ = K$ and $\mathcal{X}_\eta = \mathcal{X}$. We set $L = K(\mathcal{X})$, and suppose first that the group L^*/K^* has no torsion. To show that $\mathcal{X} \otimes_K k$ is integral, we may assume that $\mathcal{X} = \mathrm{Fspec}(A)$ is affine. Since A is integral, one has $k \otimes_K A \subset k \otimes_K L$, and so it suffices to show that the k-algebra $k \otimes_K L$ is integral. But L is isomorphic to the K-field $K'[T_1, \dots, T_n, T_1^{-1}, \dots, T_n^{-1}]$ and, therefore, $k \otimes_K L = k[T_1, \dots, T_n, T_1^{-1}, \dots, T_n^{-1}]$. Conversely, suppose that the group L^*/K^* has torsion. To show that $\mathcal{X}$ is not geometrically irreducible (resp. reduced), it suffices to find such a homomorphism $K \to k$ that the preimage of $\breve{\mathcal{X}}$ in $\mathcal{X} \otimes_K k$ is nonempty and non-connected (resp. not reduced). We may therefore replace $\mathcal{X}$ by $\breve{\mathcal{X}}$ and assume that $\mathcal{X} = \mathrm{Fspec}(L)$. By the assumption, the quotient group L^*/K^* has a direct factor isomorphic to a cyclic group of order $n > 1$. Let L' be the K-subfield of L for which the group L'^*/K^* coincides with that factor. Furthermore, let k be an arbitrary field of characteristic prime to n that contains all n-th roots of unity (resp. of characteristic that divides n), and let $K \to k^\cdot$ be the homomorphism which is the composition $K \to \mathbf{F}_1 \to k^\cdot$. Then $k \otimes_K L'$ embeds in $k \otimes_K L$ and is isomorphic to the group ring of the cyclic group of order n over k. This group ring is a direct product of n copies of k (resp. contains nilpotent elements). ∎

We say that $\mathcal{X}$ *has good reduction* if it is strict over K° and its closed fiber $\mathcal{X}_s$ is an integral scheme. For example, if $\mathcal{Y}$ is an integral scheme over $\widetilde{K}$, then the scheme $\mathcal{Y} \otimes_{\widetilde{K}} K^\circ$ has good reduction. Integral schemes of this form are said to be *constant*. *We claim that each scheme with good reduction is constant.* Indeed, it suffices to verify that the morphism $\mathcal{X} \to \mathcal{X}_s \otimes_{\widetilde{K}} K^\circ$ induced by the morphism $\mathcal{X} \to \mathcal{X}_s$ from Proposition 5.3.6 is an isomorphism. Notice that, if $\mathcal{X}$ has good reduction, then $\widetilde{K}(\mathcal{X}_s) \otimes_{\widetilde{K}} K \stackrel{\sim}{\to} K(\mathcal{X})$. Notice also that the property to have good reduction is preserved under any extension of valuation $\mathbf{F}_1$-algebras K'°/K°.

More generally, suppose that $\mathcal{X}$ is strict and $\mathcal{X}_s$ is irreducible. Then the reduction $\mathcal{X}_s^{\mathrm{r}}$ of $\mathcal{X}_s$ (which is by the way is coincide with the Zariski reduction $\mathcal{X}_s^{\mathrm{zr}}$) is an integral scheme over $\widetilde{K}$ and so, by Proposition 5.3.6, the canonical closed immersion $\mathcal{X}_s^{\mathrm{r}} \to \mathcal{X}$ has a section $\mathcal{X} \to \mathcal{X}_s^{\mathrm{r}}$. The following statement implies that the induced morphism $\mathcal{X} \to \mathcal{X}_s^{\mathrm{r}} \otimes_{\widetilde{K}} K^\circ$ is finite and surjective.

7.2.2. Proposition. *Let A be a strict integral K°-algebra such that the Zariski radical of $(K^{\circ\circ})$ is a Zariski prime ideal $\mathfrak{p}$. Then*

(i) the homomorphism $A/\mathfrak{p} \otimes_{\widetilde{K}} K^\circ \to A$ is injective;

(ii) if B is the image of the homomorphism from (i), then $B/K^{\circ\circ}B \stackrel{\sim}{\to} A/\mathfrak{p}$;

(iii) there exists $n \geq 1$ such that $f^n \in K^{\circ\circ}B$ for all elements $f \in A \backslash B$ and, in particular, A is a finite B-algebra.

Proof. (i) Suppose that $\alpha f = \beta g$ for some nonzero $\alpha, \beta \in K^\circ$ and $f, g \in A/\mathfrak{p}$ and that $|\alpha| \leq |\beta|$. Then $\beta(\frac{\alpha}{\beta} f) = \beta g$. Since A is integral, it follows that $\frac{\alpha}{\beta} f = g$. This implies that $\frac{\alpha}{\beta} \in (K^\circ)^*$ and, therefore, $f \otimes \alpha = g \otimes \beta$. The statement (ii) is trivial.

(iii) Suppose that A is generated by elements $g_1, \ldots, g_n$ over B. Since $A \backslash B \subset \mathfrak{p}$, we may assume that $g_1, \ldots, g_n \in \mathfrak{p} \backslash \{0\}$. By the assumption, for every $1 \le i \le n$ there exists $k_i \ge 1$ with $g_i^{k_i} \in (K^{\circ\circ})$, i.e., $g_i^{k_i} = \alpha_i f_i \prod_{j=1}^n g_j^{l_{ij}}$ for some $\alpha_i \in K^{\circ\circ}$, $f_i \in A/\mathfrak{p}$, $l_{ij} \ge 0$. We prove by induction on n that the above equalities imply that some powers of each g_i belong to $K^{\circ\circ}B$. Indeed, since A is an integral domain, it follows that $k_i > l_{ii}$. We can therefore divide both sides of the above expression by $g_j^{l_{ij}}$, and so we may assume that $l_{ii} = 0$. Suppose that $l_{in} \ne 0$ for some $1 \le i \le n-1$. We then have

$$g_i^{k_i k_n} = \alpha_i^{k_n} f_i^{k_n} \prod_{j=1}^{n-1} g_j^{l_{ij} k_n} \cdot \left(\alpha_n f_n \prod_{j=1}^{n-1} g_j^{l_{nj}} \right)^{l_{in}} .$$

The powers of g_i on the left and right hand sides are $k_i k_n$ and $l_{ni} l_{in}$, respectively. Since $\alpha_i \in K^{\circ\circ}$, it follows that the latter number is strictly less than the former one. We can therefore divide both sides by the smaller power of g_i and get an expression for g_i's with $1 \le i \le n-1$ as above with $l_{in} = 0$. By the induction hypothesis, we get inclusions $g_i^{k_i} \in K^{\circ\circ}B$ for $1 \le i \le n-1$. Since $g_n^{k_n} = \alpha_n f_n \prod_{j=1}^{n-1} g_j^{l_{nj}}$, we also get an inclusion $g_n^{k_1 \cdot \ldots \cdot k_n} \in K^{\circ\circ}B$. ∎

For every integral flat scheme $\mathcal{X}$ over K° one can construct in the evident way the *integral closure of* $\mathcal{X}$ *in its generic fiber* $\mathcal{X}_\eta$, i.e., a morphism $\mathcal{X}' \to \mathcal{X}$ from an integral scheme such that, for every open affine subscheme $\mathcal{U} \subset \mathcal{X}$, $\varphi^{-1}(\mathcal{U})$ is an affine scheme which is the spectrum of the integral closure of $A_\mathcal{U}$ in $A_\mathcal{U} \otimes_{K^\circ} K$. Notice that the integral closure of an integral flat scheme $\mathcal{X}$ of finite type over K° is not necessarily a scheme of finite type over K° (see Remark 2.7.9).

7.2.3. Corollary. *Given an integral strict scheme* $\mathcal{X}$ *of finite type over* K° *with irreducible closed fiber* $\mathcal{X}_s$, *there exist elements* $\gamma_1, \ldots, \gamma_n \in |K^*|$ *and integers* $l_1, \ldots, l_n \ge 2$ *such that, for any extension of valuation* $\mathbf{F}_1$*-fields* K'/K *with* $\gamma_i \in |K'^*|^{l_i}$ *for all* $1 \le i \le n$, *the integral closure of* $\mathcal{X} \otimes_{K^\circ} K'^\circ$ *in its generic fiber is an integral strict scheme of finite type over* K'° *with good reduction.*

Proof. We may assume that $\mathcal{X} = \mathrm{Fspec}(A)$ is affine, and let $\mathfrak{p} = \mathrm{z}\mathbf{r}((K^{\circ\circ}))$. Proposition 7.2.2 implies that A is generated by elements $g_1, \ldots, g_n \in \mathfrak{p}$ with $g_i^{l_i} = \alpha_i f_i$ for some $l_i \ge 2$, $\alpha_i \in K^{\circ\circ}$ and $f_i \in A/\mathfrak{p}$. *We claim that the required property is achieved for the elements* $\gamma_i = |\alpha_i|$ *and the numbers* l_i. Indeed, let K'° be a valuation K°-algebra with $\gamma_i \in |K'^*|$ for all $1 \le i \le n$. The Zariski radical of the Zariski ideal $(K'^{\circ\circ})$ of $A' = A \otimes_{K^\circ} K'^\circ$ is a Zariski prime ideal $\mathfrak{p}'$ generated by the elements $g_1, \ldots, g_n$. Take elements $\alpha'_i \in K'^{\circ\circ}$ with $|\alpha'_i|^{l_i} = \gamma_i$. Then $\beta_i = \frac{\alpha_i}{\alpha_i'^{l_i}} \in (K'^\circ)^*$ and $\left(\frac{g_i}{\alpha'_i}\right)^{l_i} = \beta_i f_i \in A/\mathfrak{p} \otimes_{\widetilde{K}} \widetilde{K}'$. Thus, the A'-subalgebra A'' of $A \otimes_{K^\circ} K'$ generated by the elements $\frac{g_i}{\alpha'_i}$ lies in the integral closure A''' of A' in $A \otimes_{K^\circ} K'$, and the Zariski ideal $(K'^{\circ\circ})$ of A'' is prime, i.e., $\mathcal{X}'' = \mathrm{Fspec}(A'')$ has good reduction. It remains to notice that A''' is generated by the integral closure of $\widetilde{A}''$ (embedded in A'') in its fraction $\mathbf{F}_1$-field. The latter is a finite $\widetilde{A}''$-algebra, by Proposition 2.6.7. It follows A''' is finitely generated over K'°. ∎

Let again $\mathcal{X}$ be an integral flat scheme of finite type over K°. Since $\mathcal{X}$ is quasicompact, Proposition 4.4.6 implies that the image of the canonical map $\mathcal{X} \to \mathrm{Fspec}(K^{\circ})$ is a principal open subset $D(\alpha)$ of $\mathrm{Fspec}(K^{\circ})$. Given a Zariski prime ideal $\mathfrak{r} \subset K^{\circ}$ with $\alpha \notin \mathfrak{r}$, $\mathcal{X}(\mathfrak{r})$ is a nonempty quasi-irreducible flat scheme of finite type over $\kappa(\mathfrak{r})$. It follows that, for every irreducible component $\mathcal{Y}$ of $\mathcal{X}(\mathfrak{r})$ the intersection of all of the open affine subschemes of $\mathcal{X}(\mathfrak{r})$ that contain the generic point of $\mathcal{Y}$ is the spectrum of a local artinian $\mathbf{F}_1$-algebra which is a finite dimensional vector space of the field of rational functions $\kappa(\mathfrak{r})(\mathcal{Y})$ of $\mathcal{Y}$. Its dimension is said to be the *multiplicity of* $\mathcal{Y}$ *in* $\mathcal{X}(\mathfrak{r})$. Furthermore, let $\breve{\mathcal{X}}_{/\mathcal{Y}}$ denote the intersection of all open subschemes of $\mathcal{X}$ that contain the generic point of $\mathcal{Y}$. Proposition 7.1.3(i) implies that $\breve{\mathcal{X}}_{/\mathcal{Y}}$ is a connected open affine subscheme of $\mathcal{X}_{(\mathfrak{r})}$. Its K°-algebra $A_{\breve{\mathcal{X}}_{/\mathcal{Y}}}$ is denoted by $K^{\circ}(\breve{\mathcal{X}}_{/\mathcal{Y}})$. Notice that $K^{\circ}(\breve{\mathcal{X}}_{/\mathcal{Y}})$ is a finitely generated $K^{\circ}_{\mathfrak{r}}$-algebra, and its fraction field is $K(\mathcal{X})$. If $\mathfrak{r} = K^{\circ\circ}$ and $\mathcal{Y}$ is the only irreducible component of $\mathcal{X}_s$ (i.e., $\mathcal{Y}$ is the reduction of $\mathcal{X}_s$), then $\breve{\mathcal{X}}_{/\mathcal{Y}}$ is denoted by $\breve{\mathcal{X}}$, and $K^{\circ}(\breve{\mathcal{X}}_{/\mathcal{Y}})$ is denoted by $K^{\circ}(\mathcal{X})$.

7.2.5. Corollary. *In the above situation, the following is true:*

(i) $K^{\circ}(\breve{\mathcal{X}}_{/\mathcal{Y}})$ *is a finite free* $L^{\circ}_{\mathcal{Y}}$*-module, where* $L^{\circ}_{\mathcal{Y}}$ *is the unramified valuation* $K^{\circ}_{\mathfrak{r}}$*-subalgebra of* $K^{\circ}(\breve{\mathcal{X}}_{/\mathcal{Y}})$ *generated by* $\kappa(\mathfrak{r})(\mathcal{Y})$*;*

(ii) the multiplicity of $\mathcal{Y}$ *in* $\mathcal{X}(\mathfrak{r})$ *is equal to the degree of the finite extension* $K(\mathcal{X})/L_{\mathcal{Y}}$ *which, in its turn, is equal to the order of the cokernel of the canonical injective homomorphism of groups* $\kappa(\mathfrak{r})(\mathcal{Y})^*/\kappa(\mathfrak{r})^* \to K(\mathcal{X})^*/K^*$*;*

(iii) if $\mathcal{X}$ *is normal, then* $K^{\circ}(\breve{\mathcal{X}}_{/\mathcal{Y}})$ *is a valuation* $K^{\circ}_{\mathfrak{r}}$*-subalgebra of* $K(\mathcal{X})$*,* $L_{\mathcal{Y}}$ *is the maximal unramified subextension of* $K(\mathcal{X})$ *(provided with the induced valuation) over* K *and if, in addition, the group* $|K^*|$ *is divisible then* $K^{\circ}(\breve{\mathcal{X}}_{/\mathcal{Y}}) = L^{\circ}_{\mathcal{Y}}$*;*

(iv) the dimensions of all irreducible components $\mathcal{Y}$ *of* $\mathcal{X}(\mathfrak{r})$ *over* $\widetilde{K}$ *are equal to the dimension of* $\mathcal{X}_\eta$ *over* K *and, if* $\mathcal{X}_\eta$ *is geometrically irreducible, then so are all* $\mathcal{Y}$*'s.*

Proof. We may assume that $\mathcal{X} = \mathrm{Fspec}(A)$ is affine and $\mathfrak{r} = K^{\circ\circ}$. Every Zariski prime ideal of $\widetilde{A}$ can be identified with a Zariski prime ideal $\mathfrak{p}$ of A that contains $K^{\circ\circ}$. Since A is an integral domain, it follows that $A/\mathfrak{p}$ is integral and, therefore, $A/\Pi_{\mathfrak{p}} = A/\mathfrak{p}$. This remark applied to the Zariski prime ideal $\mathfrak{p}$ that corresponds to $\mathcal{Y}$ implies that $\mathcal{Y}$ is Zariski closed in $\mathcal{X}$, the localization $A_{\mathfrak{p}}$ coincides with $K^{\circ}(\breve{\mathcal{X}}_{/\mathcal{Y}})$, and the quotient of the latter by the maximal Zariski ideal coincides with $\kappa(\mathfrak{p})$. Since $A/\Pi_{\mathfrak{p}} = A/\mathfrak{p}$, it follows that $\kappa(\mathfrak{p}) = \widetilde{K}(\mathcal{Y})$. Finally, since $\mathcal{Y}$ is an irreducible component of $\mathcal{X}_s = \mathrm{Fspec}(A/(K^{\circ\circ}))$, the Zariski ideal of $A_{\mathfrak{p}}/(K^{\circ\circ})$ generated by the image of $\mathfrak{p}$ is a unique minimal Zariski prime ideal. This implies that it is Zariski nilpotent, and so the finitely generated K°-algebra $K^\circ(\breve{\mathcal{X}}_{/\mathcal{Y}})$ satisfies the assumptions of Corollary 7.2.4, and all of the statements (i)-(iv) easily follow from that lemma. ∎

Let $\Phi(\mathcal{Y}/\mathcal{X})$ denote the cokernel of the injective homomorphism from (ii). It is a finite group of order equal to the multiplicity of $\mathcal{Y}$ in $\mathcal{X}(\mathfrak{r})$.

7.2.5. Corollary. *In the situation of Corollary 7.2.4, suppose that $\mathfrak{s}$ is a Zariski prime ideal of K° lying in $\mathfrak{r}$ and $\mathcal{Z}$ is an irreducible component of $\mathcal{X}(\mathfrak{s})$ whose closure $\overline{\mathcal{Z}}$ in $\mathcal{X}$ contains $\mathcal{Y}$. Then there is an exact sequence of finite groups*

$$1 \to \Phi(\mathcal{Y}/\overline{\mathcal{Z}}) \to \Phi(\mathcal{Y}/\mathcal{X}) \to \Phi(\mathcal{Z}/\mathcal{X}) \to 1$$

In particular, the multiplicity of $\mathcal{Z}$ in $\mathcal{X}(\mathfrak{s})$ divides the multiplicity of $\mathcal{Y}$ in $\mathcal{X}(\mathfrak{r})$.

Proof. We may assume that $\mathcal{X} = \mathrm{Fspec}(A)$ is affine, and let $\mathfrak{p}$ and $\mathfrak{q}$ be the Zariski prime ideals of A that correspond to $\mathcal{Y}$ and $\mathcal{Z}$, respectively. Then $\mathfrak{q} \subset \mathfrak{p}$, and there is a commutative diagram with exact rows

$$\begin{array}{ccccccccc}
1 & \to & \kappa(\mathfrak{p})(\mathcal{Y})^*/\kappa(\mathfrak{r})^* & \to & K(\mathcal{X})^*/K^* & \to & \Phi(\mathcal{Y}/\mathcal{X}) & \to & 1 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \to & \kappa(\mathfrak{q})(\mathcal{Z})^*/\kappa(\mathfrak{q})^* & \to & K(\mathcal{X})^*/K^* & \to & \Phi(\mathcal{Z}/\mathcal{X}) & \to & 1
\end{array}$$

The cokernel of the first vertical arrow is the group $\Phi(\mathcal{Y}/\overline{\mathcal{Z}})$. Since the second vertical arrow is an isomorphism, the required exact sequence is obtained by the five-lemma. ∎

Recall that the normalization of an integral scheme $\mathcal{X}$ of finite type over K° is not necessarily a scheme of finite type over K° (see Remark 2.7.9).

7.2.6. Corollary. *Given an integral flat scheme $\mathcal{X}$ of finite type over K°, there exist elements $\gamma_1, \dots, \gamma_n \in |K^*|$ and integers $l_1, \dots, l_n \geq 2$ such that, for any extension of valuation $\mathbf{F}_1$-fields K'/K with $\gamma_i \in |K'^*|^{l_i}$ for all $1 \leq i \leq n$, the normalization $\mathcal{X}'$ of $\mathcal{X} \otimes_{K^\circ} K'^\circ$ is an integral flat scheme of finite type over K'° such that the multiplicities of the irreducible components of all of the fibers of the canonical morphism $\mathcal{X}' \to \mathrm{Fspec}(K'^\circ)$ are equal to one.*

Proof. We may assume that $\mathcal{X} = \mathrm{Fspec}(A)$ is affine and $\mathbf{m}_A \cap K^\circ = K^{\circ\circ}$ and, by Corollary 7.2.5, it suffices to show that one can find the above data for every irreducible component $\mathcal{Y}$ of $\mathcal{X}_s$ so that the multiplicity of the preimage $\mathcal{Y}'$ of $\mathcal{Y}$ in $\mathcal{X}'_s$ is one. For this we can replace A by the localization $A_{\mathfrak{p}}$, where $\mathfrak{p}$ is the Zariski prime ideal of A that corresponds to $\mathcal{Y}$, and so we may assume that $A = K^\circ(\breve{\mathcal{X}}_{/\mathcal{Y}})$. By Proposition 7.2.1, A is a finite L_A°-module, where L_A° is the unramified valuation K°-subalgebra of A generated by $\kappa(\mathbf{m}_A)$ and, in particular, the K-field $L = K(\mathcal{X})$ is a finite extension of L_A. It follows that L_A°-algebra A is generated by elements $g_1, \dots, g_n \in \mathbf{m}_A$ with $f_i = g_i^{l_i} \in L_A^\circ$ for some $l_1, \dots, l_n \geq 2$. *We claim that the required property is achieved for the elements* $\gamma_i = |f_i| \in |L_A^*| = |K^*|$ *and the numbers* l_i. Indeed, let K' be a valuation K°-algebra with $\gamma_i \in |K'^*|$ for all $1 \leq i \leq n$. The tensor product $L^\circ \otimes_{K^\circ} K'^\circ$ is a valuation K'°-subalgebra of the K'-field $L' = L \otimes_K K'$, and the tensor product $L_A'^\circ = L_A \otimes_{K^\circ} K'^\circ$ is an unramified K'°-subalgebra of the K'-field $L'_A = L_A \otimes_K K'$. Take elements $\alpha_i \in K'$ with $|\alpha_i|^{l_i} = |f_i|$. Then $\left(\frac{g_i}{\alpha_i}\right)^{l_i} = (L_A'^\circ)^*$. This means that the element $\frac{g_i}{\alpha_i} \in L'$ is integral over $L_A'^\circ$. Thus, the $L_A'^\circ$-subalgebra of L' generated by all of the elements $\frac{g_i}{\alpha_i}$ is an unramified valuation K'°-subalgebra of L'. This implies the claim. ∎

7.3. Algebraic groups over K°. In this subsection we consider schemes over K° and, for brevity, the fiber product over K° of such schemes is denoted as a direct product.

An *algebraic group over* K° is a group object in the category of flat schemes of finite type over K°. Such a scheme $\mathcal{G}$ is defined by the multiplication morphism $m = m_\mathcal{G} : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, the unity morphism $e = e_\mathcal{G} : \mathrm{Fspec}(K^\circ) \to \mathcal{G}$, and the inversion morphism $\imath = \imath_\mathcal{G} : \mathcal{G} \to \mathcal{G}$ that satisfy the well known conditions. It follows that $\mathcal{G}$ is a group object in the category of all schemes over K° and, in particular, for every scheme $\mathcal{X}$ over K° the set of morphisms $\mathrm{Hom}_{K^\circ}(\mathcal{X}, \mathcal{G})$ is provided with the structure of a group. It follows also that $\mathcal{G}_\eta$ and $\mathcal{G}_s$ are algebraic groups over K and $\widetilde{K}$, respectively.

7.3.1. Examples. (i) Suppose we are given a finite group G and a map $r : G \to \mathrm{Zspec}(K^\circ) : \sigma \mapsto \mathfrak{r}_\sigma$ such that $\mathfrak{r}_1 = K^{\circ\circ}$, $\mathfrak{r}_\sigma = \mathfrak{r}_{\sigma^{-1}}$, $\mathfrak{r}_\sigma \cap \mathfrak{r}_\tau \subset \mathfrak{r}_{\sigma\tau}$ and $K^\circ_{\mathfrak{r}_\sigma} = K^\circ_{\alpha_\sigma}$ with $\alpha_\sigma \in K^\circ$ for all $\sigma, \tau \in G$. (Such a map will be said to be *special*.) We associate with these data an algebraic group $\mathcal{G} = G^r_{K^\circ}$ over K° as follows: $\mathcal{G}$ is the disjoint union $\coprod_{\sigma \in G} \mathcal{G}^{(\sigma)}$ with $\mathcal{G}^{(\sigma)} = \mathrm{Fspec}(K^\circ_{\mathfrak{r}_\sigma})$. The multiplication morphism $m : \mathcal{G} \times \mathcal{G} = \coprod_{(\sigma,\tau) \in G \times G} \mathcal{G}^{(\sigma,\tau)} \to \mathcal{G}$ is induced by the canonical morphisms $\mathcal{G}^{(\sigma,\tau)} = \mathrm{Fspec}(K^\circ_{\mathfrak{r}_\sigma \cap \mathfrak{r}_\tau}) \to \mathcal{G}^{(\sigma\tau)} = \mathrm{Fspec}(K^\circ_{\mathfrak{r}_{\sigma\tau}})$. The unity morphism e is the identity morphism $\mathrm{Fspec}(K^\circ) \xrightarrow{\sim} \mathcal{G}^{(1)}$, and the inversion morphism $\imath$ are the identity morphisms $\mathcal{G}^{(\sigma)} \xrightarrow{\sim} \mathcal{G}^{(\sigma^{-1})}$. An algebraic group over K° isomorphic to a group $G^r_{K^\circ}$ of the above form is said to be a *discrete finite algebraic group* over K°. As a scheme such $G^r_{K^\circ}$ is affine (with non uniquely defined K°-algebra).

(ii) More generally, suppose that the above finite group G acts on the right by automorphisms on an algebraic group $\mathcal{H}$ over K°. Then one can construct an algebraic group $\mathcal{G}$ called the *semidirect product* of $G^r_{K^\circ}$ and $\mathcal{H}$. Namely, it is the disjoint union $\coprod_{\sigma \in G} \mathcal{G}^{(\sigma)}$ with $\mathcal{G}^{(\sigma)} = \mathcal{H} \otimes_{K^\circ} K^\circ_{\mathfrak{r}_\sigma}$. The multiplication morphism $m : \mathcal{G} \times \mathcal{G} = \coprod_{(\sigma,\tau) \in G \times G} \mathcal{G}^{(\sigma,\tau)} \to \mathcal{G}$ with $\mathcal{G}^{(\sigma,\tau)} = (\mathcal{H} \times \mathcal{H}) \otimes_{K^\circ} K^\circ_{\mathfrak{r}_\sigma \cap \mathfrak{r}_\tau}$ is induced by the multiplication morphism on $\mathcal{H}$ and the canonical morphisms $\mathrm{Fspec}(K^\circ_{\mathfrak{r}_\sigma \cap \mathfrak{r}_\tau}) \to$

Fspec($K^\circ_{\mathfrak{r}_{\sigma\tau}}$). The unity morphism $e_{\mathcal{G}}$ coincides the morphism $e_{\mathcal{H}} : \mathrm{Fspec}(K^\circ) \to \mathcal{G}^{(1)} = \mathcal{H}$, and the restriction of the inversion morphism $\imath_{\mathcal{G}}$ to $\mathcal{G}^{(\sigma)}$ is the morphism $\mathcal{G}^{(\sigma)} \to \mathcal{G}^{(\sigma^{-1})}$ that corresponds to the composition morphism $\mathcal{H} \stackrel{\sigma^{-1}}{\to} \mathcal{H} \stackrel{\imath_{\mathcal{H}}}{\to} \mathcal{H}$.

(iii) Any finitely generated $\mathbf{F}_1$-field M defines an affine algebraic group $D_{\mathbf{F}_1}(M) = \mathrm{Fspec}(M)$ over $\mathbf{F}_1$ as follows: the multiplication morphism corresponds to the homomorphism $M \to M \otimes_{\mathbf{F}_1} M : f \mapsto f \otimes f$, the identity morphism corresponds to the homomorphism $M \to \mathbf{F}_1 : f \mapsto 1$ for $f \in M^*$, and the inversion morphism corresponds to the homomorphism $M \to M : f \mapsto f^{-1}$ for $f \in M^*$. Notice that $D_{\mathbf{F}_1}(M)$ represents the contravariant functor that takes a scheme $\mathcal{X}$ over $\mathbf{F}_1$ to the group $\mathrm{Hom}(M^*, \mathcal{O}(\mathcal{X})^*)$. The algebraic group $D_{K^\circ}(M) = D_{\mathbf{F}_1}(M) \otimes_{\mathbf{F}_1} K^\circ$ is said to be a *diagonalizable group* (of finite type) over K°. The correspondence $M \mapsto D_{K^\circ}(M)$ gives rise to an anti-equivalence between the category of finitely generated $\mathbf{F}_1$-fields or the equivalent category of finitely generated abelian groups and the category of diagonalizable groups over K° and, therefore, one can view the latter as an abelian category. For example, a homomorphism $D_{K^\circ}(M) \to D_{K^\circ}(N)$ is surjective if the corresponding homomorphism of groups $N^* \to M^*$ is injective, and its kernel is $D_{K^\circ}(L)$ with $L^* = \mathrm{Coker}(N^* \to M^*)$. If M^* has no torsion, $D_{K^\circ}(M)$ is said to be a *torus* over K° and, if M^* is an infinite cyclic group, this torus is denoted by $\mathrm{G}_{\mathrm{m},K^\circ}$. If M^* is finite, $D_{K^\circ}(M)$ is said to be a *connected finite algebraic group* over K°.

A *left action* of an algebraic group $\mathcal{G}$ on a scheme $\mathcal{X}$ over K is a morphism $\mu : \mathcal{G} \times \mathcal{X} \to \mathcal{X}$ which possesses the usual properties. For example, $\mathcal{G}$ acts on itself. An open subscheme $\mathcal{U} \subset \mathcal{X}$ is *$\mathcal{G}$-invariant* if $\mu(\mathcal{G} \times \mathcal{U}) \subset \mathcal{U}$. We say that $\mathcal{X}$ is a *left torsor for* $\mathcal{G}$ if the morphism $(\mu, p_2) : \mathcal{G} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is an isomorphism (where p_i denotes the projection to the i-th multiplier). In the same way one defines right actions and right torsors. By default, the actions and torsors considered are left. Notice that, if $\mathcal{X}$ is a torsor for $\mathcal{G}$, then any $\mathcal{G}$-invariant open subscheme of $\mathcal{X}$ coincides with $\mathcal{X}$.

We say that a torsor $\mathcal{X}$ is *split* if the set $\mathcal{X}(K^\circ) = \mathrm{Hom}_{K^\circ}(\mathrm{Fspec}(K^\circ), \mathcal{X})$ is nonempty. It is easy to see that $\mathcal{X}$ is a split torsor for $\mathcal{G}$ if and only if there is a $\mathcal{G}$-equivariant isomorphism of schemes $\mathcal{G} \stackrel{\sim}{\to} \mathcal{X}$. Notice also that, if $\mathcal{X}$ is a torsor for $\mathcal{G}$, the algebraic group $\mathcal{G}$ is unique up to a unique isomorphism.

7.3.2. Proposition. *The following properties of a strict scheme $\mathcal{X}$ of finite type over K° are equivalent:*

(a) $\mathcal{X}$ is a torsor for a diagonalizable group $D_{K^\circ}(M)$;

(b) $\mathcal{X} = \mathrm{Fspec}(L^\circ)$ for an unramified valuation K°-algebra L°.

Furthermore, in this case there is a canonical isomorphism $M \stackrel{\sim}{\to} \widetilde{L}/\widetilde{K}^* = L/K^*$.

Proof. (b)$\Longrightarrow$(a) and the latter property. Since $\mathcal{X}$ is of finite type over K°, the $\mathbf{F}_1$-field L/K^* is finitely generated over $\mathbf{F}_1$. We set $M = L/K^*$ and denote by A the K°-algebra $K^\circ \otimes_{\mathbf{F}_1} M$ of the diagonalizable group $D_{K^\circ}(M)$. If $\overline{f}$ denote the image of an element $f \in L$ in M, then the homomorphism $L^\circ \to L^\circ \otimes_{K^\circ} A = L^\circ \otimes_{\mathbf{F}_1} M : f \mapsto f \otimes \overline{f}$ defines an action $\mu : D_{K^\circ}(M) \times \mathcal{X} \to \mathcal{X}$ of $D_{K^\circ}(M)$ on $\mathcal{X}$. *We claim that $\mathcal{X}$ is a torsor for $D_{K^\circ}(M)$.* Indeed, the morphism (μ, p_2) : $D_{K^\circ}(M) \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ corresponds to the homomorphism $L^\circ \otimes_{K^\circ} L^\circ \to L^\circ \otimes_{K^\circ} A = L^\circ \otimes_{\mathbf{F}_1} M$: $f \otimes g \mapsto fg \otimes \overline{g}$. That this homomorphism is injective is trivial. Let $f \otimes m$ be a nonzero element of $L^\circ \otimes m$. Since $\widetilde{L}^*/\widetilde{K}^* \xrightarrow{\sim} L^*/K^*$, we can find an element $g \in (L^\circ)^*$ with $\overline{g} = m$. Then $f \otimes m$ is the image of the element $fg^{-1} \otimes g \in L^\circ \otimes_{K^\circ} L^\circ$, and the claim follows.

(a)$\Longrightarrow$(b). Proposition 7.1.6 implies that, if the number of irreducible components of $\mathcal{X}$ is n, then the numbers of irreducible components of $D_{K^\circ}(M) \times \mathcal{X}$ and of $\mathcal{X} \times \mathcal{X}$ are n and n^2, respectively. Since both schemes are isomorphic, it follows that $n = 1$, i.e., $\mathcal{X}$ is irreducible. Applying this to the torsor $\mathcal{X}_s$ for $D_{\widetilde{K}}(M)$, we get that $\mathcal{X}_s$ is also irreducible. Furhermore, Proposition 7.1.4(iii) implies that $D_{K^\circ}(M) \times \check{\mathcal{X}}$ is the minimal open subscheme of $D_{K^\circ}(M) \times \mathcal{X}$ and, therefore, $\mu(D_{K^\circ}(M) \times \check{\mathcal{X}}) \subset \check{\mathcal{X}}$, i.e., $\check{\mathcal{X}}$ is invariant under the action of $D_{K^\circ}(M)$. It follows that $\mathcal{X} = \check{\mathcal{X}}$ and, in particular, $\mathcal{X} = \mathrm{Fspec}(A)$ is affine. The action morphism μ corresponds to a homomorphism $A \to A \otimes_{\mathbf{F}_1} M$: $f \mapsto f \otimes \theta_\mu(f)$ such that $\theta(f) \neq 0$ for any nonzero f. The morphism (μ, p_2) corresponds to a homomorphism $A \otimes_{K^\circ} A \to A \otimes_{\mathbf{F}_1} M : f \otimes g \mapsto fg \otimes \theta_\mu(f)$. Since the former is an isomorphism, then so is the latter. Suppose that $fh = gh$ for $f, g, h \in A$ and $h \neq 0$. Then the above isomorphism takes the elements $f \otimes h$ and $g \otimes h$ to the same element of $A \otimes_{\mathbf{F}_1} M$. This implies that $f = g$, i.e., A and $\mathcal{G}$ are integral. Applying this to the torsor $\mathcal{X}_s$ for $D_{\widetilde{K}}(M)$, we get that $\mathcal{X}_s$ is also integral and, by Corollary 7.2.4, A is an unramified valuation K°-algebra. ∎

7.3.3. Corollary. *In the situation of Proposition 7.3.2, the torsor $\mathcal{X}$ is split if and only if the canonical surjection $\widetilde{L}^* \to M^* = \widetilde{L}^*/\widetilde{K}^*$ is split over the torsion subgroup of M^*. In particular, it is always split if the group M^* has no torsion.*

Proof. By the definition, $\mathcal{X}$ is split if there exists a section $L^\circ \to K^\circ$ of the canonical embedding of valuation $\mathbf{F}_1$-algebras $K^* \hookrightarrow L^\circ$. This is evidently equivalent to existence of a section of the canonical homomorphism of groups $\widetilde{L}^* \to M^* = \widetilde{L}^*/\widetilde{K}^*$. Suppose that the latter surjection is split over the torsion subgroup M^*_{tors} of M^*. Since M^*/M^*_{tors} is a free abelian group of finite rank, we can find a splitting $M^* = M^*_{\mathrm{tors}} \times G$. Since the surjection considered has a section over the free abelian group G, it follows that it has a section over the whole group M. ∎

7.3.4. Corollary. *In the situation of Proposition 7.3.2, the following are equivalent:*

(a) $\mathcal{X}$ is split for $D_{K^\circ}(M)$;

(b) $\mathcal{X}_s$ is split for $D_{\widetilde{K}}(M)$;

(c) $\mathcal{X}_\eta$ is split for $D_K(M)$. ∎

7.3.5. Theorem. *Let $\mathcal{G}$ be an algebraic group K°. Then*

(i) the connected component of the unity of $\mathcal{G}$ is a diagonalizable group $D_{K^\circ}(M)$;

(ii) the set of connected components $\pi_0(\mathcal{G})$ has a canonical structure of a finite group G, and it is provided with a special map $r : G \to$ Zspec(K°);

(iii) the connected component $\mathcal{G}^{(\sigma)}$ of $\mathcal{G}$ that corresponds to an element $\sigma \in \mathcal{G}$ is a left and right torsor for $D_{K^\circ_{\mathfrak{r}_\sigma}}(M)$, and its image in Fspec$(K^\circ)$ *coincides with* Fspec$(K^\circ_{\mathfrak{r}_\sigma})$*;*

(iv) there is a canonical surjective homomorphism of algebraic groups $\mathcal{G} \to G^r_{K^\circ}$.

Proof. Step 1. *If $\mathcal{G}$ is connected and $\mathcal{G}_s$ is irreducible, then $\mathcal{G} = \breve{\mathcal{G}}$.* Indeed, Propositions 7.1.3(i) and 7.1.4(ii) imply that $\breve{\mathcal{G}} \times \breve{\mathcal{G}}$ is the minimal open subscheme $\mathcal{W}$ of $\mathcal{G} \times \mathcal{G}$ with $\mathcal{W}_s \neq \emptyset$ and, therefore, $m(\breve{\mathcal{G}} \times \breve{\mathcal{G}}) \subset \breve{\mathcal{G}}$. For the similar reason, one has $\imath(\breve{\mathcal{G}}) \subset \breve{\mathcal{G}}$. This implies that $\breve{\mathcal{G}}$ is an open affine subgroup of $\mathcal{G}$. Let now g be a point of $\mathcal{G}$, and let g' denote its image under the morphism $(1_\mathcal{G}, \imath) : \mathcal{G} \to \mathcal{G} \times \mathcal{G}$. Then the point $m(g')$ lies in the image of the morphism $e :$ Fspec$(K^\circ) \to \mathcal{G}$ and, in particular, $g' \in m^{-1}(\breve{\mathcal{G}})$. By Proposition 7.1.4(ii), the point g' has an open affine neighborhood in $m^{-1}(\breve{\mathcal{G}})$ of the form $\mathcal{U} \times \mathcal{V}$, where $\mathcal{U}$ and $\mathcal{V}$ are open affine subschemes of $\mathcal{G}$. Since both $\mathcal{U}$ and $\mathcal{V}$ contain $\breve{\mathcal{G}}$ and, in particular, the image of the morphism e, it follows that $m(\mathcal{U} \times \mathcal{V}) \supset \mathcal{U} \cup \mathcal{V}$ and, therefore, $\mathcal{U} = \mathcal{V} = \breve{\mathcal{G}}$. Thus, $g \in \breve{\mathcal{G}}$, i.e., $\mathcal{G} = \breve{\mathcal{G}}$.

Step 2. *If $\mathcal{G}$ is connected and $\mathcal{G}_s$ is irreducible, then $\mathcal{G}$ is integral.* Indeed, since the scheme $\mathcal{G}$ is flat over K°, it suffices to show that $\mathcal{G}_\eta$ is integral. We may therefore assume that the valuation on K is trivial. Step 1 implies that $\mathcal{G}$ is affine, i.e., $\mathcal{G} =$ Fspec(A), and $\mathcal{G} = \breve{\mathcal{G}}$, i.e., $\mathcal{G}$ has no nontrivial open affine subschemes. It follow that all elements of A outside z$\mathbf{n}(A)$ are invertible and, therefore, A is quasi-integral. Consider the homomorphism $\mu : A \to A \otimes_K A$ that corresponds to the multiplication morphism $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, and the homomorphism $\varepsilon : A \to K$ that corresponds to the unity morphism $e :$ Fspec$(K) \to \mathcal{G}$. Let a be an element of z$\mathbf{n}(A)$, and let $\mu(a) = b \otimes c$. Since a is nilpotent, at least one of the elements b or c should be nilpotent. Suppose it is b. Then $\varepsilon(b) = 0$. Since the composition of the homomorphism μ with the homomorphism $A \otimes_K A \to A : x \otimes y \mapsto x\varepsilon(y)$ is the identity on A, we get $a = 0$, i.e., z$\mathbf{n}(A) = 0$. The claim follows.

Step 3. *If $\mathcal{G}$ is connected and $\mathcal{G}_s$ is irreducible, then $\mathcal{G}$ is isomorphic to a diagonalizable group.* By Step 2, $\mathcal{G} =$ Fspec$(K^\circ(\breve{\mathcal{G}}))$ and the K°-algebra $K^\circ(\breve{\mathcal{G}})$ is integral. Applying the same fact to the algebraic group $\mathcal{G}_s$ over $\widetilde{K}$, we get that $\mathcal{G}_s$ is also integral. Corollary 7.2.4 then implies that

$K^\circ(\breve{\mathcal{G}})$ is an unramified valuation K°-algebra. Let M be the finitely generated $\mathbf{F}_1$-field $K^\circ(\breve{\mathcal{G}})/K^*$. By Proposition 7.3.2, $\mathcal{G}$ is a torsor for the diagonalizable group $D_{K^\circ}(M)$. Since $\mathcal{G}$ is also a torsor for itself, the claim follows.

Step 4. *If $\mathcal{G}$ is connected, $\mathcal{G}_s$ is irreducible.* Indeed, by Proposition 7.1.3(iv), $\mathcal{G}_s$ is connected. Since $\mathcal{G}_s$ is an algebraic group over $\widetilde{K}$, this reduces the situation to the case when the valuation on K is trivial. Let $\mathcal{X}$ be an irreducible component of $\mathcal{G}$ that contains the image of the morphism $e : \mathrm{Fspec}(K) \to \mathcal{G}$. Since the scheme $\mathcal{X}\times\mathcal{X}$ is also irreducible. This implies that the image $m(\mathcal{X}\times\mathcal{X})$ lies in an irreducible component of $\mathcal{G}$. But that image contains $\mathcal{X}$ and, therefore, $m(\mathcal{X}\times\mathcal{X}) = \mathcal{X}$. Since $\mathcal{X}\times\mathcal{X}$ is reduced, it follows that the morphism m gives rise to a morphism $m : \mathcal{X}\times\mathcal{X} \to \mathcal{X}$. For the same reason, the isomorphism $\imath$ gives rise to an isomorphism $\imath : \mathcal{X} \xrightarrow{\sim} \mathcal{X}$. Thus, $\mathcal{X}$ is an irreducible algebraic group. By Step 3, $\mathcal{X}$ is a diagonalizable group and, in particular, $\mathcal{X}$ is the spectrum of a K-field. Suppose that $\mathcal{G} \neq \mathcal{X}$. Since $\mathcal{G}$ is connected, we can find an irreducible component $\mathcal{Y}$ of $\mathcal{G}$ which has nonempty intersection with $\mathcal{X}$ and does not coincide with $\mathcal{X}$. As above, the scheme $\mathcal{X}\times\mathcal{Y}$ is irreducible and its image under the morphism m contains $\mathcal{Y}$. It follows that $m(\mathcal{X}\times\mathcal{Y}) = \mathcal{Y}$ and, therefore, $m(\mathcal{X}\times(\mathcal{X}\cap\mathcal{Y})) \subset \mathcal{X}\cap\mathcal{Y}$. The latter is possible only if $\mathcal{X}\cap\mathcal{Y} = \mathcal{X}$ which contradicts the assumption $\mathcal{G} \neq \mathcal{X}$.

Thus, if $\mathcal{G}$ is connected, it is isomorphic to a diagonalizable group, and (i) is true.

Step 5. Consider now the general case. Let G denote the set $\pi_0(\mathcal{G})$ of connected components of $\mathcal{G}$. For an element $\sigma \in G$, let $\mathcal{G}^{(\sigma)}$ denote the corresponding connected component. For every pair of elements $\sigma, \tau \in G$, the direct product $\mathcal{G}^{(\sigma)} \times \mathcal{G}^{(\tau)}$ is connected and, therefore, its image under the morphism m lies in some $\mathcal{G}^{(\rho)}$. We define a binary operation on G by $\sigma\tau = \rho$. It is easy to see that this operation provides G with the structure of a group, the unity element of which is the connected component $\mathcal{G}^{(1)}$ that contains the image of the morphism e. The morphism m induces morphisms $\mathcal{G}^{(\sigma)} \times \mathcal{G}^{(\tau)} \to \mathcal{G}^{(\sigma\tau)}$ and isomorphisms $\imath : \mathcal{G}^{(\sigma)} \to \mathcal{G}^{(\sigma^{-1})}$ for all $\sigma, \tau \in G$. In particular, $\mathcal{G}^{(1)}$ is a connected irreducible algebraic group, and it acts on each $\mathcal{G}^{(\sigma)}$ from the left and the right. By the previous steps, $\mathcal{G}^{(1)}$ is a diagonalizable group $D_{K^\circ}(M)$. The isomorphisms $(m, 1_{\mathcal{G}})$ and $(1_{\mathcal{G}}, m) : \mathcal{G}\times\mathcal{G} \xrightarrow{\sim} \mathcal{G}\times\mathcal{G}$ give rise to isomorphisms of connected schemes $\mathcal{G}^{(\sigma)}\times\mathcal{G}^{(\tau)} \xrightarrow{\sim} \mathcal{G}^{(\sigma\tau)}\times\mathcal{G}^{(\tau)}$ and $\mathcal{G}^{(\sigma)}\times\mathcal{G}^{(\tau)} \xrightarrow{\sim} \mathcal{G}^{(\sigma)}\times\mathcal{G}^{(\sigma\tau)}$ for all $\sigma, \tau \in G$ and, in particular, to isomorphisms $D_{K^\circ}(M)\times\mathcal{G}^{(\sigma)} \xrightarrow{\sim} \mathcal{G}^{(\sigma)}\times\mathcal{G}^{(\sigma)}$ and $\mathcal{G}^{(\sigma)}\times D_{K^\circ}(M) \xrightarrow{\sim} \mathcal{G}^{(\sigma)}\times\mathcal{G}^{(\sigma)}$. This means that each connected component $\mathcal{G}^{(\sigma)}$ is a left and right torsor for $D_{K^\circ}(M)$. Furthermore, Proposition 4.4.6 implies that the image of the canonical map $\mathcal{G}^{(\sigma)} \to \mathrm{Fspec}(K^\circ)$ is a principal open subset $D(\alpha_\sigma)$ for some $\alpha_\sigma \in K^\circ$, and this subset coincides with $\mathrm{Fspec}(K^\circ_{\mathfrak{r}_\sigma})$, where $\mathfrak{r}_\sigma$ is the maximal Zariski prime ideal of K° that does not contain the element α_σ. One evidently has $\mathfrak{r}_1 = K^\circ$, and the isomorphisms $\mathcal{G}^{(\sigma)}\times\mathcal{G}^{(\tau)} \xrightarrow{\sim} \mathcal{G}^{(\sigma)}\times\mathcal{G}^{(\sigma\tau)}$

easily imply that $\mathfrak{r}_\sigma = \mathfrak{r}_{\sigma^{-1}}$ and $\mathfrak{r}_\sigma \cap \mathfrak{r}_\tau \subset \mathfrak{r}_{\sigma\tau}$, i.e., the map $r : G \to \mathrm{Zspec}(K^\circ)$ that takes σ to $\mathfrak{r}_\sigma$ is special. Validity of the statements (ii)-(iv) easily follows. ∎

7.4. Separated integral flat schemes of finite type over K°. For a finitely generated K-field L that contains K, the quotient $\overline{L} = L/K^*$ is a finitely generated $\mathbf{F}_1$-field, and so it defines a diagonalizable group $D_{K^\circ}(\overline{L})$. The correspondence $L \mapsto D_{K^\circ}(\overline{L})$ is a contravariant functor. If $\mathcal{X}$ is an integral flat scheme of finite type over K°, the diagonalizable group $D_{K^\circ}(\overline{K(\mathcal{X})})$ is denoted by $\mathcal{D}(\mathcal{X})$. By Proposition 7.2.1, $\mathcal{D}(\mathcal{X})$ is a torus if and only if $\mathcal{X}_\eta$ is geometrically irreducible. In this case $\mathcal{D}(\mathcal{X})$ will be denoted by $\mathcal{T}(\mathcal{X})$. Notice that $\mathcal{D}(\mathcal{X})_\eta = \mathcal{D}(\mathcal{X}_\eta) = D_K(\overline{K(\mathcal{X})})$ and $\mathcal{D}(\mathcal{X})_s = D_{\widetilde{K}}(\overline{K(\mathcal{X})})$.

7.4.1. Definition. A flat scheme $\mathcal{X}$ of finite type over K° is said to be a *generic torsor for* an algebraic group $\mathcal{G}$ over K° if there is a $\mathcal{G}_\eta$-invariant dense open subscheme $\mathcal{U} \subset \mathcal{X}_\eta$ which is a torsor for $\mathcal{G}_\eta$.

7.4.2. Theorem. *Let $\mathcal{X}$ be a separated integral flat scheme of finite type over K. Then*

(i) there is a canonical action $\mu : \mathcal{D}(\mathcal{X}) \times \mathcal{X} \to \mathcal{X}$ which makes $\mathcal{X}$ a generic torsor for $\mathcal{D}(\mathcal{X})$;

(ii) any action of a diagonalizable group $D_{K^\circ}(M)$ on $\mathcal{X}$ is induced by a unique homomorphism $D_{K^\circ}(M) \to \mathcal{D}(\mathcal{X})$ and the canonical action of $\mathcal{D}(\mathcal{X})$ on $\mathcal{X}$;

(iii) each irreducible component $\mathcal{Y}$ of $\mathcal{X}_s$ is $D(\mathcal{X})_s$-invariant, the induced homomorphism $D(\mathcal{X})_s \to D(\mathcal{Y})$ is surjective, and its kernel is finite of order equal to the multiplicity of $\mathcal{Y}$.

Suppose a diagonalizable group $D_{K^\circ}(M)$ acts on a connected flat affine scheme $\mathcal{X} = \mathrm{Fspec}(A)$ over K°. Then the scheme $D_{K^\circ}(M) \times \mathcal{X}$ is also connected, and so the action μ of $D_{K^\circ}(M)$ on $\mathcal{X}$ defines a homomorphism of K°-algebras $\mu^* : A \to A \otimes_{\mathbf{F}_1} M$. It is easy to see that the properties for the group action are equivalent to the fact that $\mu^*(a) = a \otimes \theta_\mu(a)$ for all $a \in A$, where $\widetilde{\theta}_\mu : A \to M$ is a quasi-homomorphism of $\mathbf{F}_1$-algebras (see §3.4.1) with the property that $\widetilde{\theta}_\mu(\alpha) = 1$ for all $\alpha \in K^\circ \backslash \{0\}$. It follows that $\widetilde{\theta}_\mu$ induces a quasi-homomorphism $\theta_\mu : \overline{A} \to M$.

7.4.3. Lemma. *The correspondence $\mu \mapsto \theta_\mu$ gives rise to a bijection between the set of actions of $D_{K^\circ}(M)$ on $\mathcal{X}$ and the set of quasi-homomorphisms $\overline{A} \to M$.*

Proof. Given a quasi-homomorphism $\theta : A \to M$ with the above property, the map $A \to A \otimes_{\mathbf{F}_1} M : a \mapsto a \otimes \theta(a)$ is a homomorphism of K°-algebras that corresponds to an action of $D_{K^\circ}(M)$ on $\mathcal{X}$. ∎

7.4.4. Lemma. *In the above situation, suppose that $\mathcal{X}$ is integral. Then* *(i) all open subschemes of $\mathcal{X}$ are $D_{K^\circ}(M)$-invariant;*

(ii) $\mathcal{X}$ is a generic torsor for $D_{K^\circ}(M)$ if and only if θ_μ is injective;

(iii) $\mathcal{X}$ is a torsor for $D_{K^\circ}(M)$ if and only if θ_μ is bijective.

Proof. (i) It suffices to verify the statement for open affine subschemes. Recall that any such nonempty subscheme is a principal open subset $D(f)$ for some $f \in A\backslash\{0\}$. Let y be a point of $D_{K^\circ}(M) \times D(f)$ whose projections are points $g \in D_{K^\circ}(M)$ and $x \in D(f)$. Then $f(\mu(y)) = \theta_\mu(f)(y) = f(x)\theta_\mu(f)(g)$. Since $x \in D(f)$, one has $f(x) \neq 0$ and, since M is an $\mathbf{F}_1$-field and $\theta_\mu(f)$ is its nonzero element, one has $\theta_\mu(f)(g) \neq 0$. Thus, $f(\mu(y)) \neq 0$, i.e., $\mu(y) \in D(f)$.

(ii) and (iii). The morphism $(\mu, p_2) : D_{K^\circ}(M)\times\mathcal{X} \to \mathcal{X}\times\mathcal{X}$ corresponds to the homomorphism $\psi : A \otimes_{K^\circ} A \to A \otimes_{\mathbf{F}_1} M : a \otimes b \mapsto ab \otimes \theta_\mu(a)$. By Theorem 3.2.2(ii), the former has dense image (resp. is an isomorphism) if and only if the latter is injective (resp. bijective). Suppose first that ψ is injective and that $\theta_\mu(a) = \theta_\mu(b)$ for $a, b \in A$. If $b = 0$, then $\psi(a\otimes 1) = 0$ and, therefore, $a = 0$. If $a, b \neq 0$, then $\psi(a\otimes b) = \psi(b\otimes a)$. The injectivity assumption implies that $a\otimes b = b\otimes a$. It follows that there exists $\alpha \in K^\circ\backslash\{0\}$ with either $a = \alpha b$, or $b = \alpha a$. If ψ is also surjective, then for every element $m \in M$ there exists an element $a\otimes b \in A\otimes_{K^\circ} A$ with $\psi(a\otimes b) = 1\otimes m$. This implies that $\theta_\mu(a) = m$. Conversely, suppose that θ_μ possesses the above property. If $\psi(a\otimes b) = ab\otimes\theta_\mu(a) = 0$, then either $\theta_\mu(a) = 0$ and, therefore, $a = 0$, or $ab = 0$ and, therefore, $a = 0$ or $b = 0$. If $\psi(a \otimes b) = \psi(a' \otimes b')$ for $a, b, a', b' \neq 0$, then $\theta_\mu(a) = \theta_\mu(a')$ and $ab = a'b'$. The property implies that there exists $\alpha \in K^\circ\backslash\{0\}$ with either $a = \alpha a'$, or $a' = \alpha a$. In the former (resp. latter) case, we have $\alpha a'b = a'b'$ (resp. $\alpha ab' = ab$) and, therefore, $b' = \alpha b$ (resp. $b = \alpha b'$). In both cases, we get $a \otimes b = a' \otimes b'$. Finally, suppose that θ_μ is bijective. Given a nonzero element $a \otimes m \in A \otimes_{\mathbf{F}_1} M$, take an element $b \in A^*$ with $\theta_\mu(b) = m$. Then $\psi(b \otimes ab^{-1}) = a \otimes m$, i.e., the homomorphism ψ is surjective. ∎

Proof of Theorem 7.4.2. (i) Given a strict open affine subscheme $\mathcal{U} \subset \mathcal{X}$, the injective homomorphism $A_\mathcal{U} \to A_\mathcal{U} \otimes_{\mathbf{F}_1} \overline{K(\mathcal{X})} : a \mapsto a \otimes \overline{a}$ gives rise to an action $\mu : \mathcal{D}(\mathcal{X}) \times \mathcal{U} \to \mathcal{U}$ of the diagonalizable group $\mathcal{D}(\mathcal{X}) = D_{K^\circ}(\overline{K(\mathcal{X})})$ on $\mathcal{U}$. These actions are compatible on intersections and, therefore, they give rise to a canonical action $\mu : \mathcal{D}(\mathcal{X}) \times \mathcal{X} \to \mathcal{X}$ of $\mathcal{D}(\mathcal{X})$ on $\mathcal{X}$. The minimal open subscheme $\breve{\mathcal{X}}_\eta$ is clearly a torsor for $\mathcal{D}(\mathcal{X}_\eta) = \mathcal{D}(\mathcal{X})_\eta$ and, therefore, $\mathcal{X}$ is a generic torsor for $\mathcal{D}(\mathcal{X})$.

(ii) Suppose we are given an action $\nu : D_{K^\circ}(M) \times \mathcal{X} \to \mathcal{X}$ of a diagonalizable group $D_{K^\circ}(M)$ on $\mathcal{X}$. *We claim that the minimal open subscheme $\breve{\mathcal{X}}_\eta$ of $\mathcal{X}_\eta$ is $D_K(M)$-invariant.* Indeed, the K-algebra $K(\mathcal{X}) \otimes_{\mathbf{F}_1} M$ of the affine scheme $D_K(M) \times \breve{\mathcal{X}}_\eta$ is a K-field and, therefore, any nonempty open subscheme of $D_K(M) \times \check{\mathcal{X}}_\eta$ coincides with it. Since the intersection $\nu^{-1}(\check{\mathcal{X}}_\eta) \cap (D_K(M) \times \check{\mathcal{X}}_\eta)$ is nonempty, it follows that $\nu(D_K(M) \times \check{\mathcal{X}}_\eta) \subset \check{\mathcal{X}}_\eta$, i.e., $\check{\mathcal{X}}_\eta$ is $D_K(M)$-invariant. The action of $D_K(M)$ on $\check{\mathcal{X}}$ defines (and is defined by) a homomorphism $\overline{K(\mathcal{X})} = K(\mathcal{X})/K^* \to M$, and so it is induced by that homomorphism and the canonical action of $\mathcal{D}(\mathcal{X})$ on $\check{\mathcal{X}}_\eta$. Thus, we have two morphisms μ and $\nu : D_{K^\circ}(M) \times \mathcal{X} \to \mathcal{X}$ that coincide on the dense subset $D_K(M) \times \check{\mathcal{X}}_\eta$. Since $\mathcal{X}$ is separated, it follows that $\mu = \nu$.

(iii) That all irreducible components of $\mathcal{X}_s$ are $D(\mathcal{X})$ is trivial. The homomorphism of diagonalizable groups $D(\mathcal{X})_s \to D(\mathcal{Y})$ corresponds to the homomorphism $\widetilde{K}(\mathcal{Y})^*/\widetilde{K}^* \to K(\mathcal{X})^*/K^*$ and, by Corollary 7.2.4(ii), the cokernel of the latter homomorphism has order equal to the multiplicity of $\mathcal{Y}$. ∎

A strict scheme $\mathcal{X}$ of finite type over K° is said to be a *homogeneous* (resp. *generically homogeneous*) *space for* $\mathcal{G}$ if the morphism $(\mu, p_2) : \mathcal{G} \times \mathcal{X} \to \mathcal{X} \times \mathcal{X}$ is surjective (resp. has dense image).

7.4.5. Corollary. *If $\mathcal{X}$ is a generically homogeneous space for a diagonalizable group $D_{K^\circ}(M)$, then $\check{\mathcal{X}}_\eta$ is a homogeneous space for $D_K(M)$.*

Proof. It suffices to consider the case when the valuation on K is trivial. The assumption implies that the morphism $(\mu, p_2) : D_K(M) \times \check{\mathcal{X}} \to \check{\mathcal{X}} \times \check{\mathcal{X}}$ has dense image. Since $A_{\check{\mathcal{X}}}$ is an K-field, it follows that $A_{\check{\mathcal{X}}} \otimes_K A_{\check{\mathcal{X}}}$ is also a K-field, and Corollary 1.2.6 implies that the above morphism is surjective, i.e., $\check{\mathcal{X}}$ is a homogeneous space for $D_K(M)$. ∎

Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism between separated flat integral schemes of finite type over K°, and denote by $\mathcal{Z}$ the Zariski closure of the set $\varphi(\mathcal{Y})$ in $\mathcal{X}$. Since $\mathcal{Z}_\eta$ is the Zariski closure of $\varphi(\mathcal{Y}_\eta)$ in $\mathcal{X}_\eta$, Proposition 7.1.3(iv) implies that $\mathcal{Z}$ is also a strict separated integral scheme of finite type over K°. Then φ is a composition of the Zariski dominant morphism $\psi : \mathcal{Y} \to \mathcal{Z}$ and the Zariski closed immersion $\chi : \mathcal{Z} \to \mathcal{X}$. Since $\psi(\check{\mathcal{Y}}_\eta) \subset \check{\mathcal{Z}}_\eta$, there are induced homomorphisms of K-fields $K(\mathcal{Z}) \to K(\mathcal{Y})$ and of diagonalizable groups $\jmath_\psi : \mathcal{D}(\mathcal{Y}) \to \mathcal{D}(\mathcal{Z})$. Furthermore, χ identifies $\mathcal{Z}$ with an irreducible Zariski closed subset of $\mathcal{X}$. If $\mathcal{U}$ is a nonempty open affine subscheme of $\mathcal{X}$, then $\mathcal{Z} \cap \mathcal{U} = \mathcal{U}^{(\mathfrak{p})}$ for a Zariski prime ideal $\mathfrak{p} \subset A_\mathcal{U}$, and $K(\mathcal{Z})$ is identified with the K-field $\kappa(\mathfrak{p})$. The canonical homomorphism $A_\mathcal{U}/\mathfrak{p} \hookrightarrow A_\mathcal{U}$ induces an embedding of K-fields $K(\mathcal{Z}) = \kappa(\mathfrak{p}) \hookrightarrow K(\mathcal{X})$, and the latter gives rise to a surjective homomorphism of diagonizable groups $\imath_\chi : \mathcal{D}(\mathcal{X}) \to \mathcal{D}(\mathcal{Z})$.

7.4.6. Corollary. *In the above situation, the following diagram is commutative*

$$\begin{array}{ccccc}
\mathcal{D}(\mathcal{X}) & \times & \mathcal{X} & \stackrel{\mu}{\to} & \mathcal{X} \\
\downarrow \imath_\chi & & \uparrow \chi & & \uparrow \chi \\
\mathcal{D}(\mathcal{Z}) & \times & \mathcal{Z} & \stackrel{\mu}{\to} & \mathcal{Z} \\
\uparrow \jmath_\psi & & \uparrow \psi & & \uparrow \psi \\
\mathcal{D}(\mathcal{Y}) & \times & \mathcal{Y} & \stackrel{\mu}{\to} & \mathcal{Y}
\end{array}$$

Proof. It suffices to consider two cases: (1) φ is Zariski dominant, and (2) φ is a Zariski closed immersion.

(1) If $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{V} \subset \mathcal{Y}$ are strict open affine subschemes with $\varphi(\mathcal{V}) \subset \mathcal{U}$, then the homomorphism $A_\mathcal{U} \to B_\mathcal{V}$ is compatible with the homomorphism $K(\mathcal{X}) \to K(\mathcal{Y})$ and, therefore, the homomorphisms $\theta_\mu : A_\mathcal{U} \to \overline{K(\mathcal{X})}$ and $\theta_\nu : B_\mathcal{V} \to \overline{K(\mathcal{Y})}$ are compatible. This implies the required fact.

(2) Let $\mathcal{U}$ be a strict open affine subscheme of $\mathcal{X}$. Then $\mathcal{V} = \varphi^{-1}(\mathcal{U})$ coincides with $\mathcal{U}^{(\mathfrak{p})} = \mathrm{Fspec}(A_\mathcal{U}/\mathfrak{p})$ for a Zariski prime ideal $\mathfrak{p} \subset A_\mathcal{U}$. The canonical injective homomorphisms $A_\mathcal{U}/\mathfrak{p} \to A_\mathcal{U}$ and $\kappa(\mathfrak{p}) \to K(\mathcal{X})$ are compatible, and this implies the required fact. ∎

7.5. Connection with toric schemes. Let k be a (usual) valuation field. The ring $k^\circ = \{a \in k \big| |a| \le 1\}$, called the *ring of integers of* k, has a unique maximal ideal $k^{\circ\circ} = \{a \in k \big| |a| < 1\}$. The quotient $\widetilde{k} = k^\circ/k^{\circ\circ}$ is called the *residue field of* k. Notice that $k^{\cdot\circ} = k^{\circ\cdot}$ and $k^{\cdot\circ\circ} = k^{\circ\circ\cdot}$. On the other hand, $\widetilde{k^\cdot}$ is the $\mathbf{F}_1$-field whose group of invertible elements is $(k^\circ)^*$, and so the canonical homomorphism of $\mathbf{F}_1$-fields $\widetilde{k^\cdot} \to \widetilde{k}^\cdot$ gives rise to an isomorphism $\widetilde{k^\cdot}/k^1 \stackrel{\sim}{\to} \widetilde{k}^\cdot$, where k^1 is the group $\{a \in k^\circ \big| |a-1| < 1\}$. Notice also that the canonical map $\mathrm{Spec}(k^\circ) \to \mathrm{Fspec}(k^{\cdot\circ}) : \mathfrak{r} \mapsto \mathfrak{r}^\cdot$ is a bijection.

7.5.1. Definition. (i) A *toric scheme* over k° is a separated integral scheme $\mathcal{Y}$ flat and of finite type over k° provided with an action of a split k°-torus $\mathcal{T} = \mathcal{T}(\mathcal{Y})$ such that

(1) $\mathcal{Y}_\eta$ has an open dense orbit which is a torsor for $\mathcal{T}_\eta$;

(2) $\mathcal{Y}$ is covered by $\mathcal{T}$-invariant open affine subschemes.

(ii) A *closed toric subscheme* is a nonempty irreducible $\mathcal{T}$-invariant closed subset flat over k°.

If the valuation on k is trivial, the condition (2) is automatically satisfied if $\mathcal{Y}$ is normal, by a theorem of Sumihiro [Sum]. In this case, toric schemes are called toric varieties, and a closed toric subscheme is just the closure of an orbit of the torus. In the general case, every closed toric subscheme $\mathcal{Z}$ of $\mathcal{Y}$ is a toric scheme over k° for the torus $\mathcal{T}(\mathcal{Z})$ which is the quotient of $\mathcal{T}$ by the stabilizer of all points of $\mathcal{Z}$.

Let $\mathcal{Y} = \mathrm{Spec}(B)$ be an affine toric scheme over k° for a torus $\mathcal{T}$ with the character group $\check{M}$, and let M be the corresponding $\mathbf{F}_1$-field $\{0\} \cup \check{M}$. (In such a situation we will write $\mathcal{T} = D_{k^\circ}(M)$.) The action $\mu : \mathcal{T} \times \mathcal{Y} \to \mathcal{Y}$ of $\mathcal{T}$ on $\mathcal{Y}$ defines a decomposition $B = \oplus_{\chi \in \check{M}} B_\chi$, where B_χ are k°-submodules of B with $B_{\chi'} \cdot B_{\chi''} \subset B_{\chi'\chi''}$. Namely, one has $B_\chi = \{f \in B \big| \mu^*(f) = f \otimes \chi\}$. We set $\check{S} = \{\chi \in \check{M} \big| B_\chi \neq 0\}$, $S = \{0\} \cup \check{S}$, and denote by $\mathfrak{r}$ the set of all elements of k° which are non-invertible in B. Then $\mathfrak{r}$ is a prime ideal of k° and all B_χ's are $k^\circ_\mathfrak{r}$-modules.

7.5.2. Proposition. *In the above situation, the following is true:*

(i) S is a finitely generated $\mathbf{F}_1$-algebra with fraction $\mathbf{F}_1$-field M;

(ii) for every $\chi \in \check{S}$, B_χ is a free $k^\circ_\mathfrak{r}$-module of rank one;

(iii) $A = \cup_{\chi \in \check{S}} B_\chi$ is a strict finitely generated $(k^\circ_\mathfrak{r})^\cdot$-subalgebra of $B^\cdot$, and $A \otimes_{k^{\cdot\circ}} k^\circ \xrightarrow{\sim} B$;

(iv) if A is integrally closed and $\widetilde{A} = A/(k^{\cdot\circ\circ}_\mathfrak{r})$ is reduced, then B is integrally closed;

(v) there is a canonical bijection between the set of closed toric subschemes and the Zariski spectrum $\mathrm{Zspec}(S)$*;*

(vi) all closed toric subschemes of $\mathcal{Y}$ are faithfully flat over $k^\circ_\mathfrak{r}$.

Proof. Take a surjective homomorphism $C = k^\circ[T_1, \dots, T_n] \to B : T_i \mapsto g_i$. If $g_i = \sum_{l=1}^m g_{i,\chi_l}$, we replace the above homomorphism by a similar homomorphism in which instead of the variable T_i there are variables T_{il} that go to the elements g_{i,χ_l}. In this way we get a surjective homomorphism $C \to B$ as above with $g_i \in B_{\chi_i}$ for all $1 \leq i \leq n$ which induces a surjective homomorphism of $k^{\cdot\circ}$-algebras $C^\cdot \to A$. In particular, A is a finitely generated $k^{\cdot\circ}$-subalgebra of $B^\cdot$ and $A \otimes_{k^{\cdot\circ}} k^\circ \xrightarrow{\sim} B$. It follows also that S is a finitely generated $\mathbf{F}_1$-subalgebra of M. Since $\mathcal{Y}_\eta$ has a dense orbit which is a torsor for $\mathcal{T}_\eta$, the fraction $\mathbf{F}_1$-field of S is M, i.e., (i) is true. Furthermore, since $\mathbf{m}_A \cap k^{\cdot\circ} = \mathfrak{r}^\cdot$, Proposition 7.1.2 implies that, for each $\chi \in \check{S}$, A_χ is a free $(k^\circ_\mathfrak{r})^\cdot$-module of rank one and, therefore, B_χ is a free $k^\circ_\mathfrak{r}$-module of rank one, i.e., (ii) and (iii) are true. Finally, let $\mathcal{Z}$ be a closed toric subscheme of $\mathcal{Y}$. It is the closure of a unique dense orbit of $\mathcal{T}_\eta$ in the toric variety $\mathcal{Y}_\eta$. It follows that the set $\mathfrak{p} = \{0\} \cup \{\chi \in \check{S} \big| f(z) = 0$ for all $z \in \mathcal{Z}$ and $f \in B_\chi\}$ is a Zariski prime ideal of S. This implies (v). Since $\mathcal{Z} = \mathrm{Spec}(B/\mathfrak{q})$, where $\mathfrak{q} = \oplus_{\chi \in \mathfrak{p} \setminus \{0\}} B_\chi$, and there is an isomorphism of $k^\circ_\mathfrak{r}$-modules $B/\mathfrak{q} \xrightarrow{\sim} \oplus_{\chi \in S \setminus \mathfrak{p}} B_\chi$, (vi) is true.

It remains to verify the statement (iv). The converse implication is trivial. Suppose that A is integrally closed. If the valuation on K is trivial, this implies that the semigroup $\check{S}$ is saturated in the group $\check{M}$, and the statement is well known. In the general case, we apply the previous one to $A' = A \otimes_{k^{\cdot\circ}} k^\cdot$ and $B' = B \otimes_{k^\circ} k$. Since $B' = A' \otimes_{k^\cdot} k$, it follows that every element of the fraction field of B integral over B is of the form $\lambda^{-1} f$ for some $\lambda \in k^\circ_\mathfrak{r}$ and an element f which is a generator of the free $k^\circ_{\mathfrak{r}}$-module $B_\chi = A_\chi$ for some $\chi \in \check{S}$. If $T^n + g_{n-1}X^{n-1} + \ldots + g_0 = 0$ is an equation on integral dependence of $\lambda^{-1}f$ over B, we may consider only homogeneous summands g_{i,χ_i} of g_i with $\chi^n = \chi_i\chi^{n-1}$, and we get an equality $f^n + \lambda g_{n-1,\chi_{n-1}} f^{n-1} + \ldots + \lambda^n g_{0,\chi_0} = 0$ in B_{χ^n}. The latter is a free $k^\circ_{\mathfrak{r}}$-module of rank. Since $\widetilde{A}$ is reduced, it follows that the element f^n is a generator of B_{χ^n}. The equality now implies that λ is invertible in $k^\circ_{\mathfrak{r}}$, i.e., the element considered lies in B. ∎

Let $\mathcal{Y}$ be a toric scheme over k° for a torus $\mathcal{T} = D_{k^\circ}(M)$.

7.5.3. Corollary. *The correspondence $\mathcal{Z} \mapsto \overline{\mathcal{Z}}$ (the closure of $\mathcal{Z}$ in $\mathcal{Y}$) gives rise to a bijection between the set of $\mathcal{T}_\eta$-orbits in $\mathcal{Y}_\eta$ and the set of closed toric subschemes of $\mathcal{Y}$.* ∎

7.5.4. Corollary. *Let $\mathcal{Z}$ be a $\mathcal{T}$-invariant irreducible closed subset of $\mathcal{Y}$, and let $\mathfrak{r}$ be the prime ideal of k° which is the image of the generic point of $\mathcal{Z}$. Then*

(i) $\mathcal{Z}$ is a toric scheme over the quotient (valuation) ring $k^\circ/\mathfrak{r}$;

(ii) if $\mathcal{Y}$ is faithfully flat over k°, then so is $\mathcal{Z}$ over $k^\circ/\mathfrak{r}$.

Proof. We may assume that $\mathcal{Y} = \mathrm{Spec}(B)$ is affine. By Proposition 7.5.2, one has $B = A \otimes_{k^{\cdot\circ}} k^\circ$, where A is an integral finitely generated k°-algebra and, therefore, $\mathcal{Y} = \mathcal{X} \otimes_{k^{\cdot\circ}} k^\circ$ for $\mathcal{X} = \mathrm{Fspec}(A)$. We set $\mathcal{X}' = \mathcal{X} \otimes_{k^{\cdot\circ}} (k^\circ_{\mathfrak{r}})^\cdot$ and $\mathcal{Y}' = \mathcal{Y} \otimes_{k^\circ} k^\circ_{\mathfrak{r}}$. Then the generic point of $\mathcal{Z}$ lies in the closed fiber $\mathcal{Y}'_s$ of $\mathcal{Y}'$. Since $\mathcal{Y}' = \mathcal{X}' \otimes_{(k^\circ_{\mathfrak{r}})^\cdot} k^\circ_{\mathfrak{r}}$, it follows that each irreducible component $\mathcal{W}$ of $\mathcal{Y}'_s$ is a toric variety over $\kappa(\mathfrak{r})$, the fraction field of $k^\circ/\mathfrak{r}$. More precisely, $\mathcal{W}$ is $D_{\kappa(\mathfrak{r})}(M)$-invariant and the canonical homomorphism $D_{\kappa(\mathfrak{r})}(M) \to \mathcal{T}(\mathcal{W}) = D_{\kappa(\mathfrak{r})}(M')$ is an isogeny. It follows that the closure $\overline{\mathcal{W}}$ of $\mathcal{W}$ in $\mathcal{Y} \otimes_{k^\circ} k^\circ/\mathfrak{r}$ is a toric scheme over $k^\circ/\mathfrak{r}$ for the torus $\mathcal{T}(\overline{\mathcal{W}}) = D_{k^\circ/\mathfrak{r}}(M')$. Since $\mathcal{Z}_\eta = \mathcal{Z} \otimes_{k^\circ} \kappa(\mathfrak{r})$ is irreducible, it lies in such an irreducible component $\mathcal{W}$ and, therefore, it is the closure of a $\mathcal{T}(\mathcal{W})$-orbit. Corollary 7.5.3 implies that $\mathcal{Z}$, which coincides with the closure of the latter in $\overline{\mathcal{W}}$, is a closed toric subscheme in $\overline{\mathcal{W}}$. Furthermore, suppose $\mathcal{Y}$ is faithfully flat over k°. By Proposition 7.5.2(vi), to show that $\mathcal{Z}$ is faithfully flat over $k^\circ/\mathfrak{r}$, it suffices to verify this property for $\overline{\mathcal{W}}$, i.e., we may assume that $\mathcal{Z}_\eta = \mathcal{W}$. For this we notice that $\mathcal{W}$ is the preimage of an irreducible component $\mathcal{V}$ of $\mathcal{X}'_s$. By **Proposition 7.2.1(i)**, $\mathcal{V}$ is Zariski closed in $\mathcal{X}'$, i.e., $\mathcal{V} = \mathrm{Fspec}(A'/\mathfrak{p}')$ for a Zariski prime ideal $\mathfrak{p}' \subset A'$. If $\mathfrak{p}$ is the preimage of $\mathfrak{p}'$ in A, then the closure $\overline{\mathcal{V}}$ of $\mathcal{V}$ in $\mathcal{X}$ coincides with $\mathrm{Fspec}(A/\mathfrak{p})$. Since $\mathfrak{p} \cap k^{\cdot\circ} = \mathfrak{r}$, Corollary 2.8.2 implies that $A/\mathfrak{p}$ is a free $k^{\cdot\circ}/\mathfrak{r}$-module and, therefore, $C = A/\mathfrak{p} \otimes_{k^{\cdot\circ}} k^\circ$ is a free $k^\circ/\mathfrak{r}$-module. Since $\mathcal{Z} = \mathrm{Spec}(C)$, the required fact follows. ∎

Let $\mathcal{Y}$ and $\mathcal{Y}'$ be toric schemes over k° for tori $\mathcal{T}$ and $\mathcal{T}'$, respectively. Let $\varphi : \mathcal{Y}' \to \mathcal{Y}$ be a morphism of schemes over k°. We say that φ is a *toric morphism* if the image of $\mathcal{Y}'_\eta$ lies in a

$\mathcal{T}_\eta$-orbit in $\mathcal{Y}_\eta$ and, if $\mathcal{Z}$ is its closure in $\mathcal{Y}$, there is a homomorphism of tori $\mathcal{T}' \to \mathcal{T}(\mathcal{Z})$ which is compatible with the induced morphism $\mathcal{Y}' \to \mathcal{Z}$. It is easy to see that one can compose toric morphism, and so we get a *category of toric schemes over* k°.

7.5.5. Theorem. *The correspondence $\mathcal{X} \mapsto \mathcal{X} \otimes_{k^{\cdot\circ}} k^\circ$ gives rise to an equivalence between the category of separated integral schemes of finite type over $k^{\cdot\circ}$ with geometrically irreducible generic fiber and the category of toric schemes over k°.*

Proof. *The functor considered is fully faithful.* Let $\mathcal{X}$ and $\mathcal{X}'$ be separated integral schemes of finite type over $k^{\cdot\circ}$ with geometrically irreducible generic fibers, and let φ be a toric morphism $\mathcal{Y}' = \mathcal{X}' \otimes_{k^{\cdot\circ}} k^\circ \to \mathcal{Y} = \mathcal{X} \otimes_{k^{\cdot\circ}} k^\circ$. Suppose $\mathcal{T}(\mathcal{X}) = D_{k^{\cdot\circ}}(M)$ and $\mathcal{T}(\mathcal{X}') = D_{k^{\cdot\circ}}(M')$. Then $\mathcal{T}(\mathcal{Y}) = D_{k^\circ}(M)$ and $\mathcal{T}(\mathcal{Y}') = D_{k^\circ}(M')$.

Consider first the case when φ is Zariski dominant, i.e., $\varphi_\eta(\mathcal{Y}'_\eta) \subset \mathcal{Y}_\eta$. Since both groups $\mathrm{Hom}(\mathcal{T}(\mathcal{X}'), \mathcal{T}(\mathcal{X}))$ and $\mathrm{Hom}(\mathcal{T}(\mathcal{Y}'), \mathcal{T}(\mathcal{Y}))$ are canonically isomorphic to the group $\mathrm{Hom}(M, M')$, they are canonically isomorphic. In particular, the homomorphism of tori $\mathcal{T}(\mathcal{Y}') \to \mathcal{T}(\mathcal{Y})$ that corresponds to the morphism φ comes from a homomorphism of tori $\mathcal{T}(\mathcal{X}') \to \mathcal{T}(\mathcal{X})$ which is induced by a homomorphism $f : M \to M'$.

If $\mathcal{X} = \mathrm{Fspec}(A)$ and $\mathcal{X}' = \mathrm{Fspec}(A)$ are affine, then $\mathcal{Y} = \mathrm{Spec}(B)$ and $\mathcal{Y}' = \mathrm{Spec}(B')$ for $B = A \otimes_{k^{\cdot\circ}} k^\circ$ and $B' = A' \otimes_{k^{\cdot\circ}} k^\circ$. Compatibility of φ with the homomorphism of tori implies that $f(S) \subset S' = \{\chi' \in \check{M}' \big| B'_{\chi'} \neq 0\}$ and $\varphi^*(B_\chi) \subset B'_{f(\chi)}$ for all $\chi \in \check{S}$. Thus, the morphism φ is induced by a homomorphism of $k^{\cdot\circ}$-algebras $A \to A'$, i.e., by a Zariski dominant morphism $\mathcal{X}' \to \mathcal{X}$ of schemes over $k^{\cdot\circ}$.

If $\mathcal{X}$ and $\mathcal{X}'$ are arbitrary, the morphism φ is defined by a compatible system of morphisms of affine toric varieties $\varphi_{\mathcal{V}'/\mathcal{V}} : \mathcal{V}' \to \mathcal{V}$ for all pairs of invariant open affine subschemes $\mathcal{V} \subset \mathcal{Y}$ and $\mathcal{V}' \subset \mathcal{Y}'$ with $\varphi(\mathcal{V}') \subset \mathcal{V}$. By Proposition 7.5.2, one has $\mathcal{V} = \mathcal{U} \otimes_{k^\cdot} k$ and $\mathcal{V}' = \mathcal{U}' \otimes_{k^\cdot} k$ for open affine subschemes $\mathcal{U} \subset \mathcal{X}$ and $\mathcal{U}' \subset \mathcal{X}'$ and, by the previous case, the morphism of affine toric varieties is induced by a unique morphism of affine schemes $\psi_{\mathcal{U}'/\mathcal{U}} : \mathcal{U}' \to \mathcal{U}$. It is easy to see that the morphisms $\psi_{\mathcal{U}'/\mathcal{U}}$ are compatible, and so they define a morphism $\psi : \mathcal{X}' \to \mathcal{X}$ which induces the morphism φ.

To prove the required statement in the general case, we need the following fact.

7.5.6. Lemma. *For any separated integral scheme $\mathcal{X}$ of finite type over $k^{\cdot\circ}$, the correspondence $\mathcal{Z} \mapsto \overline{\mathcal{Z}} \otimes_{k^{\cdot\circ}} k^\circ$ gives rise to a bijection between the set of irreducible Zariski closed subsets $\mathcal{Z} \subset \mathcal{X}_\eta$ and the set of closed toric subschemes of $\mathcal{Y} = \mathcal{X} \otimes_{k^{\cdot\circ}} k^\circ$.*

Proof. First of all, Proposition 7.1.3(iii) implies that the closure $\overline{\mathcal{Z}}$ of $\mathcal{Z}$ is Zariski closed in $\mathcal{Y}$, and so it is an integral closed subscheme of $\mathcal{Y}$. It follows that $\overline{\mathcal{Z}}\otimes_{k^{\cdot\circ}}k^\circ$ is a closed toric subscheme of $\mathcal{Y}$. On the other hand, let $\mathcal{W}$ be a closed toric subscheme of $\mathcal{Y}$. To verify the required fact, we may assume that $\mathcal{X}=\mathrm{Fspec}(A)$ is affine. By Proposition 7.5.2(v) (and in its notation), $\mathcal{W}$ corresponds to a Zariski prime ideal of S. Since $\mathcal{A}/k^*\stackrel{\sim}{\to}S$, where $\mathcal{A}=A\otimes_{k^{\cdot\circ}}k^{\cdot}$, we get $\mathrm{Zspec}(S)\stackrel{\sim}{\to}\mathrm{Zspec}(\mathcal{A})$, and the required fact follows. ∎

Consider now the general case. Let $\mathcal{W}$ be the closure in $\mathcal{Y}$ of the $\mathcal{T}_\eta$-orbit in $\mathcal{Y}_\eta$ that contains $\varphi_\eta(\mathcal{Y}'_\eta)$. By Lemma 7.5.6, we have $\mathcal{W}=\overline{\mathcal{Z}}\otimes_{k^{\cdot\circ}}k^\circ$, where $\mathcal{Z}$ is a Zariski closed subset of $\mathcal{X}$. By the previous case, the morphism $\mathcal{Y}'\to\mathcal{W}$ is induced by a morphism $\mathcal{X}'\to\overline{\mathcal{Z}}$. It follows that φ is induced by a morphism $\mathcal{X}'\to\mathcal{X}$.

The functor considered is essentially surjective. To prove this, we need the following fact.

7.5.7. Lemma. *For any separated integral scheme $\mathcal{X}$ of finite type over $k^{\cdot\circ}$ with geometrically irreducible fiber, the correspondence $\mathcal{U}\mapsto\mathcal{U}\otimes_{k^{\cdot\circ}}k^\circ$ gives rise to a bijection between the set of open (resp. open affine) subschemes of $\mathcal{X}$ and the set of $\mathcal{T}(\mathcal{Y})$-invariant open (resp. open affine) subschemes of $\mathcal{Y}=\mathcal{X}\otimes_{k^{\cdot\circ}}k^\circ$.*

Proof. Step 1. We may assume that $\mathcal{X}=\mathrm{Fspec}(A)$ is affine and strict over $k^{\cdot\circ}$, and set $\mathcal{T}=\mathcal{T}(\mathcal{Y})$. *It suffices to show that any $\mathcal{T}$-invariant open subscheme $\mathcal{V}\subset\mathcal{Y}=\mathrm{Spec}(B)$, where $B=A\otimes_{k^{\cdot\circ}}k^\circ$, is of the form $\mathcal{U}\otimes_{k^{\cdot\circ}}k^\circ$ for an open subscheme $\mathcal{U}\subset\mathcal{X}$.* Indeed, suppose this is true, and let $\mathcal{V}$ be a $\mathcal{T}$-invariant open affine subscheme of $\mathcal{Y}$. By Proposition 7.5.2, one has $\mathcal{V}=\mathcal{U}'\otimes_{k^{\cdot\circ}}k^\circ$ for an affine scheme $\mathcal{U}'$ of finite type over k°. The fully faithfulness already established implies that $\mathcal{U}$ is isomorphic to $\mathcal{U}'$, i.e., $\mathcal{U}$ is in fact an open affine subscheme of $\mathcal{X}$.

Step 2. Let $\mathfrak{r}$ be the prime ideal of k° that consists of the elements which are non-invertible in $B_\mathcal{V}$. *We claim that $k^\circ_\mathfrak{r}=k^\circ_\alpha$ for some nonzero element $\alpha\in k^\circ$.* Indeed, since $B_\mathcal{V}$ is an integral finitely generated k°-algebra, it is a free $k^\circ_\mathfrak{r}$-module, by Proposition 7.5.2. This implies that the image of the canonical map $\mathcal{V}\to\mathrm{Spec}(k^\circ)$ coincides with $\mathrm{Spec}(k^\circ_\mathfrak{r})$. On the other hand, since B is a flat finitely generated k°-algebra, it is finitely presented over k°, by a result of Raynaud-Gruson [RG, Corollary 3.4.7]. This implies that the morphism $\mathcal{Y}\to\mathrm{Spec}(k^\circ)$ is an open map, by [EGAIV, Theorem 2.4.6]. Thus, $\mathrm{Spec}(k^\circ_\mathfrak{r})$ is an open subset of $\mathrm{Spec}(k^\circ)$. Since it is quasi-compact, it is a finite union of principal open subsets $D(\alpha_1)\cup\ldots\cup D(\alpha_n)$. If $\max\{|\alpha_1|,\ldots,|\alpha_n|\}=|\alpha_i|$, it follows that $\mathrm{Spec}(k^\circ_\mathfrak{r})=D(\alpha_i)$ and, therefore, $k^\circ_\mathfrak{r}=k^\circ_{\alpha_i}$. The claim follows. We can therefore replace $\mathcal{Y}$ by the principal open subset $D_\mathcal{Y}(\alpha)$ and assume that $\mathcal{V}$ is also faithfully flat over k°.

Step 3. *The map $\mathcal{V}\mapsto\mathcal{V}_s$ from the set of $\mathcal{T}$-invariant open subschemes of $\mathcal{Y}$, which are faithfully flat over k°, to that of $\mathcal{T}_s$-invariant open subschemes of $\mathcal{Y}_s$ is injective.* Indeed, it suffices to verify that, if for $\mathcal{T}$-invariant open affine subscheme $\mathcal{V}', \mathcal{V}'' \subset \mathcal{Y}$ one has $\mathcal{V}' \subset \mathcal{V}''$ and $\mathcal{V}'_s = \mathcal{V}''_s$, then $\mathcal{V}' = \mathcal{V}''$. Suppose that $\mathcal{V}' \neq \mathcal{V}''$, and set $\mathcal{Z}' = \mathcal{V}''\backslash\mathcal{V}'$. Let $\mathfrak{r}$ be a prime ideal of k° from the image of $\mathcal{Z}'$. The affine scheme $\mathcal{Z}'_\mathfrak{r} = \mathcal{Z}' \otimes_{k^\circ} \kappa(\mathfrak{r})$ is of finite type over the field $\kappa(\mathfrak{r})$, the torus $\mathcal{T}'_\mathfrak{r}$ acts on it, and all its irreducible components are invariant under $\mathcal{T}'_\mathfrak{r}$. Let $\mathcal{Z}$ be the closure in $\mathcal{Y}$ of an irreducible component of $\mathcal{Z}'_\mathfrak{r}$. Then $\mathcal{Z}$ is a nonempty $\mathcal{T}$-invariant irreducible closed subset of $\mathcal{Y}$, and Corollary 7.5.4(ii) implies that $\mathcal{Z}$ is faithfully flat over $k^\circ/\mathfrak{r}$. It follows that $\mathcal{Z}_s \neq \emptyset$ which contradicts the equality $\mathcal{V}' = \mathcal{V}''$.

Step 4. *The statement of the lemma is true.* Indeed, let $\mathcal{V}$ be a $\mathcal{T}$-invariant open affine subscheme of $\mathcal{Y}$. By Step 2, we may assume that it is faithfully flat over k°. Each $\mathcal{T}_s$-orbit in $\mathcal{Y}_s$ either lies in $\mathcal{V}_s$, or does not intersect $\mathcal{V}_s$. If P and Q are two $\mathcal{T}_s$-orbits in $\mathcal{Y}_s$, we write $P \leq Q$ if $P \subset \overline{Q}$. Let P be a minimal $\mathcal{T}_s$-orbit in $\mathcal{Y}_s$ which lies in $\mathcal{V}_s$. Since $\overline{P}$ is a $\mathcal{T}_s$-invariant closed subset of $\mathcal{Y}_s$ and a toric variety, there exist $\chi \in \check{S}$ and $f \in B_\chi$ with $\widetilde{f} \neq 0$ such that P is the principal open subset $D_{\overline{P}}(\widetilde{f})$. (Here we use notations from the proof of Proposition 7.5.2.) Since $\mathcal{V}_s$ is open in $\mathcal{Y}_s$, then for every $\mathcal{T}_s$-orbit Q with $P \leq Q$ one has $Q \subset \mathcal{V}_s$. It follows that $D_{\mathcal{Y}_s}(\widetilde{f}) \subset \mathcal{V}_s$. This implies that $\mathcal{V}_s = \bigcup_{i=1}^n D_{\mathcal{Y}_s}(\widetilde{f_i})$ for some $f_i \in B_{\chi_i}$, $\chi_i \in \check{S}$. Each f_i can be considered as an element of A_{χ_i}, and so we can consider the open subscheme $\mathcal{U} = \bigcup_{i=1}^n D_\mathcal{X}(f_i)$. Since the closed fibers of the open subschemes $\mathcal{V}_s$ and $\mathcal{U} \otimes_{k^{\cdot\circ}} k^\circ$ coincide, Step 3 implies that they coincide. ∎

7.5.8. Corollary. *In the situation of Lemma 7.5.7, the correspondence $\mathcal{Z} \mapsto \mathcal{Z} \otimes_{k^{\cdot\circ}} k^\circ$ gives rise to a bijection between the family of Zariski closed (resp. closed affine) subsets of $\mathcal{X}$ and the family of $\mathcal{T}(\mathcal{Y})$-invariant closed (resp. closed affine) subsets of $\mathcal{Y} = \mathcal{X} \otimes_{k^{\cdot\circ}} k^\circ$.*

Proof. Let $\mathcal{W}$ be a $\mathcal{T}(\mathcal{Y})$-invariant closed subset of $\mathcal{Y}$. Then $\mathcal{V} = \mathcal{Y}\backslash\mathcal{W}$ is a $\mathcal{T}(\mathcal{Y})$-invariant open subscheme of $\mathcal{Y}$. By Lemma 7.5.7, one has $\mathcal{V} = \mathcal{U} \otimes_{k^{\cdot\circ}} k^\circ$ for an open subscheme $\mathcal{U}$ of $\mathcal{X}$. Since any open subscheme of the integral scheme $\mathcal{X}$ is Zariski open, it follows that the set $\mathcal{Z} = \mathcal{X}\backslash\mathcal{U}$ is Zariski closed, and we get $\mathcal{W} = \mathcal{Z} \otimes_{k^{\cdot\circ}} k^\circ$. If $\mathcal{W}$ is affine then, by Proposirion 7.5.2, $\mathcal{W} = \mathcal{Z}' \otimes_{k^{\cdot\circ}} k^\circ$ for an affine scheme of finite type over k°. The fully faithfulness of the functor from Theorem 7.5.5 implies that $\mathcal{Z}$ is isomorphic to $\mathcal{Z}'$ and, therefore, $\mathcal{Z}$ is affine. ∎

Let $\mathcal{Y}$ be a toric scheme over k°. By Proposition 7.1.3, for every $\mathcal{T}$-invariant open affine subscheme $\mathcal{V}$ of $\mathcal{Y}$ one has $\mathcal{V} = \mathcal{U} \otimes_{k^{\cdot\circ}} k^\circ$ for an affine scheme $\mathcal{U}$ of finite type over $k^{\cdot\circ}$. If for such subschemes one has $\mathcal{V}' \subset \mathcal{V}''$, Lemma 7.5.7 implies that, for the corresponding affine schemes over $k^{\cdot\circ}$, $\mathcal{U}'$ is an open affine subscheme of $\mathcal{U}''$. Thus, we can glue all such affine scheme $\mathcal{U}$ along the intersections, and we get an integral scheme $\mathcal{X}$ of finite type over $k^{\cdot\circ}$ with $\mathcal{Y} \xrightarrow{\sim} \mathcal{X} \otimes_{k^{\cdot\circ}} k^\circ$. Finally, since $\mathcal{Y}$ is separated, for any pair of $\mathcal{T}$-invariant open affine subschemes with nonempty intersection

$\mathcal{V}', \mathcal{V}'' \subset \mathcal{Y}$, the canonical homomorphism of k°-algebras $B_{\mathcal{V}'} \otimes_{k^\circ} B_{\mathcal{V}''} \to B_{\mathcal{V}' \cap \mathcal{V}''}$ is surjective. This easily implies that the corresponding homomorphism of $k^{\cdot\circ}$-algebras $A_{\mathcal{U}'} \otimes_{k^{\cdot\circ}} A_{\mathcal{U}''} \to A_{\mathcal{U}' \cap \mathcal{U}''}$ is surjective and, therefore, $\mathcal{X}$ is separated. ∎

7.5.9. Corollary. *Let $\mathcal{Y}$ be a faithfully flat toric scheme over k° for a torus $\mathcal{T}$. Then every irreducible component $\mathcal{Z}$ of the closed fiber $\mathcal{Y}_s$ is a toric variety over $\widetilde{k}$ and the canonical homomorphism of tori $\mathcal{T}_s \to \mathcal{T}(\mathcal{Z})$ is an isogeny with kernel of order equal to the multiplicity of $\mathcal{Z}$ in $\mathcal{Y}_s$.* ∎

7.5.10. Corollary. *Given a toric scheme $\mathcal{Y}$ over k°, there exists a finite separable extension k' of k such that the valuation on k has a unique extension to k' and the normalization $\mathcal{Y}'$ of $\mathcal{Y} \otimes_{k^\circ} k'^\circ$ is a toric scheme over k'° such that the multiplicities of the irreducible components of all of the fibers of the canonical morphism $\mathcal{Y}' \to \mathrm{Spec}(k'^\circ)$ are equal to one.*

Proof. By Theorem 7.5.5, one has $\mathcal{Y} = \mathcal{X} \otimes_{k^{\cdot\circ}} k^\circ$ for a separated strict integral scheme $\mathcal{X}$ of finite type over k° with geometrically irreducible generic fiber. Let $\gamma_1, \dots, \gamma_n \in |k^*|$ and $l_1, \dots, l_n \geq 2$ be as in Corollary 7.2.6 for $\mathcal{X}$. *We claim that there exists a finite separable extension k' of k such that the integral closure of k° in k' is a valuation ring and $\gamma_i \in |k'^*|^{l_i}$ for all $1 \leq i \leq n$.* Indeed, it suffices to consider the case when $n = 1$ and $l_1 = p$ is a prime number. We may assume that $\gamma = \gamma_1 < 1$ and $\gamma \notin |k^*|^p$. Take an element $\alpha \in k^{\circ\circ}$ with $|\alpha| = \gamma$, and consider an extension k' of k of degree p that contains a root β of the separable polynomial $T^p + \alpha^2 T + \alpha$. The valuation on k admits an extension to k', and let k'° be its ring of integers. Then $\beta \in k'^\circ$ and $|\beta| = |\alpha|^{\frac{1}{p}}$. It follows that the extension of the valuation to k' is unique and, therefore, k'° is the integral closure of k° in k', i.e., the claim is true. By Corollary 7.2.6, the normalization $\mathcal{X}'$ of $\mathcal{X} \otimes_{k^{\cdot\circ}} (k'^\circ)^{\cdot}$ is a scheme of finite type over $(k'^\circ)^{\cdot}$ with reduced fibers of the canonical morphism $\mathcal{X}' \to \mathrm{Fspec}((k'^\circ)^{\cdot})$. Notice that all of the irreducible components of the fibers are toric varieties. It follows that the toric scheme $\mathcal{Y}' = \mathcal{X}' \otimes_{(k'^\circ)^{\cdot}} k'^\circ$ over k'° has the required property, and Proposition 7.5.2(iv) implies that $\mathcal{Y}'$ is normal. Since $\mathcal{Y}'$ lies in the normalization of $\mathcal{Y} \otimes_{k^\circ} k'^\circ$, it coincides with it. ∎

7.6. The case of trivial valuation on K and normal $\mathcal{X}$. The theory of toric varieties describes normal toric varieties over a field k in terms of fans, and so Theorem 7.5.5 implies that separated geometrically irreducible normal schemes of finite type over the $\mathbf{F}_1$-field $k^{\cdot}$ can be described in the same terms. In this subsection we give a direct description of the category of separated irreducible (not necessarily geometrically irreducible) normal schemes over an arbitrary $\mathbf{F}_1$-field K in terms of fans.

For an $\mathbf{F}_1$-field L, let $\mathbf{V}_L$ denote the set of all real valuations on L. We denote points of $\mathbf{V}_L$ by the letters x, y and so on and, for $x \in \mathbf{V}_L$, the image of $f \in L$ under the corresponding homomorphism $L \to \mathbf{R}_+$ is denoted by $f(x)$. The set $\mathbf{V}_L$ is provided with the weakest topology with respect to which all functions $\mathbf{V}_L \to \mathbf{R}_+$ of the form $x \mapsto f(x)$ with $f \in L^*$ are continuous. Notice that $\mathbf{V}_L$ is a vector space over $\mathbf{R}$ with respect to the group structure defined by multiplication and the action $\mathbf{R} \times \mathbf{V}_L \to \mathbf{V}_L : (r, x) \mapsto x^r$ defined as follows: $f(x^r) = f(x)^r$ for all $f \in L$. Furthermore, for a subset $F \subset L$, let $\mathbf{V}_L\{F\}$ denote the the subset $\{x \in \mathbf{V}_L \big| f(x) \leq 1 \text{ for all } f \in F\}$ provided with the induced topology.

Notice that, if M is an $\mathbf{F}_1$-field that contains L, then the canonical map $\mathbf{V}_M \to \mathbf{V}_L$ is surjective. Indeed, the spaces $\mathbf{V}_L$ and $\mathbf{V}_M$ coincide with the sets of homomorphisms of abelian groups $\mathrm{Hom}(L^*, \mathbf{R}_+^*)$ and $\mathrm{Hom}(M^*, \mathbf{R}_+^*)$, respectively, and the required surjectivity follows from the fact the abelian group $\mathbf{R}_+^*$ is injective. Notice also that, given homomorphisms of $\mathbf{F}_1$-fields $L \to M$ and $L \to N$, there is a canonical homeomorphism $\mathbf{V}_{M \otimes_L N} \overset{\sim}{\to} \mathbf{V}_M \times_{\mathbf{V}_L} \mathbf{V}_N$.

7.6.1. Proposition. *Let L/K be an extension of $\mathbf{F}_1$-field. Then for any K-subalgebra $A \subset L$, the integral closure of of A in L coincides with the set of $f \in L$ such that $f(x) \leq 1$ for all $x \in \mathbf{V}_L\{A\}$.*

7.6.2. Lemma. *Given a Zariski prime ideal $\mathfrak{p} \subset A$, there exists a point $x \in \mathbf{V}_L\{A\}$ with $\mathfrak{p} = \{f \in A \big| f(x) < 1\}$.*

Proof. By Proposition 2.7.2, we may assume that A is a valuation $\mathbf{F}_1$-algebra in L and $\mathfrak{p} = \mathbf{m}_A$. Furthermore, let K' is the $\mathbf{F}_1$-field $K^*A^* \cup \{0\}$. Then the homomorphism $K \to \mathbf{R}_+ : f \mapsto |f|$ extends in a unique way to a homomorphism $K' \to \mathbf{R}_+ : f \mapsto |f|$ that takes all elements of A^* to 1. Thus, we can replace K by K', and we may assume that $A^* = K^*$. If now $\{f_i\}_{i \in I}$ is a system of nonzero elements in $\mathbf{m}_A$ whose images in L^*/K^* form a basis of the $\mathbf{Q}$-vector space $L^*/K^* \otimes_{\mathbf{Z}} \mathbf{Q}$, then any system of numbers $\{r_i\}_{i \in I}$ with $0 < r_i < 1$ defines a unique homomorphism $L \to \mathbf{R}_+ : f \mapsto f(x)$ that extends the real valuation $K \to \mathbf{R}_+ : f \mapsto |f|$ and takes each f_i to r_i. The point $x \in \mathbf{V}_L$ possesses the required properties. ∎

Proof of Proposition 7.6.1. That the integral closure is contained in that set is trivial. Let f be an element of L which is not integral over A. As in the proof of Corollary 2.7.4, one shows that the Zariski ideal $\mathbf{b}$ of $C = A[f^{-1}]$ generated by $\mathbf{m}_A$ and f^{-1} is nontrivial. It follows that $\mathbf{b} \subset \mathbf{m}_C$, and Lemma 7.6.2 implies that there exists $x \in \mathbf{V}_L\{C\}$ such that $\mathbf{m}_C = \{g \in C \big| g(x) < 1\}$. Since $f^{-1} \in \mathbf{m}_C$, it follows that $f(x) > 1$. ∎

For a finitely generated K-field L, we denote by $V_{L/K}$ the $\mathbf{R}$-vector subspace $\mathbf{V}_L\{K'\}$ of $\mathbf{V}_L$, where K' is the image of K in L. Notice that any system of n elements of L^* whose images form a basis of the $\mathbf{Q}$-vector space $L^*/K'^* \otimes_{\mathbf{Z}} \mathbf{Q}$ defines a homeomorphism between $\mathbf{V}_{L/K}$ and the $\mathbf{R}$-vector space $(\mathbf{R}_+^*)^n$. A convex polyhedral cone in $\mathbf{V}_{L/K}$ is said to be *rational* if it is defined by a finite number of inequalities of the form $f(x) \le 1$ with $f \in L^*$. Thus, the set $\mathbf{V}_L\{A\}$ of any finitely generated K-subalgebra $A \subset L$ with a rational convex polyhedral cone in $\mathbf{V}_{L/K}$ and, conversely, any rational convex polyhedral cone in $\mathbf{V}_{L/K}$ has such a form. Furthermore, if the fraction $\mathbf{F}_1$-field of A coincides with L, then the $\mathbf{V}_L\{A\}$ is strongly convex, i.e., $\mathbf{V}_L\{A\} \cap \mathbf{V}_L\{A\}^{-1} = \{1\}$. (Notice that $\mathbf{V}_L\{A\}^{-1} = \mathbf{V}_L\{A'\}$, where $A' = \{0\} \cup (\check{A})^{-1}$.) The following facts easily follow from the properties of rational convex polyhedral cones (see [Ful, section 1.2]).

7.6.3. Proposition. *(i) The correspondences*

$$A \mapsto \sigma^{(A)} = \mathbf{V}_L\{A\} \ \ \textit{and} \ \ \sigma \mapsto A^{(\sigma)} = \{f \in L \big| f(x) \le 1 \ \ \textit{for all} \ \ x \in \sigma\}$$

are inverse bijections between the set of integrally closed finitely generated K-subalgebras of L with the fraction $\mathbf{F}_1$-field L and the set of rational strongly convex polyhedral cones in $\mathbf{V}_{L/K}$;

(ii) $\sigma^{(B)}$ is a face of $\sigma^{(A)}$ if and only if $B = A_f$ for some $f \in A$;

(iii) if $\sigma \cap \tau$ is a face of both σ and τ, then $A^{(\sigma\cap\tau)} = A^{(\sigma)} \cdot A^{(\tau)}$. ∎

7.6.4. Proposition. *Let A be an integrally closed finitely generated K-algebra with $\mathbf{F}_1$-fraction field L. Then*

(i) the correspondences

$$\begin{gathered}\tau \mapsto \mathfrak{p}_\tau = \{f \in A \big| f(x) < 1 \ \ \textit{for some point} \ \ x \in \tau\} \ \ \textit{and} \\ \mathfrak{p} \mapsto \tau_{\mathfrak{p}} = \{x \in \sigma^{(A)} \big| f(x) = 1 \ \ \textit{for all} \ f \in A \backslash \mathfrak{p}\}\end{gathered}$$

are inverse bijections between the set of faces of $\sigma^{(A)}$ and the set of Zariski prime ideals of A;

(ii) given a Zariski prime ideal $\mathfrak{p} \subset A$, the kernel of the canonical surjection $\mathbf{V}_{L/K} \to \mathbf{V}_{\kappa(\mathfrak{p})/K}$ is the vector subspace of $\mathbf{V}_{L/K}$ generated by the face $\tau_{\mathfrak{p}}$, and the image of the cone $\sigma^{(A)}$ coincides with the cone $\sigma^{(A/\mathfrak{p})}$ in $\mathbf{V}_{\kappa(\mathfrak{p})/K}$. ∎

Let K be an $\mathbf{F}_1$-field.

7.6.5. Definition. (i) A *K-fan* is a pair (L, Δ), consisting of a finitely generated K-field L and a finite family Δ of rational strongly convex polyhedral cones in $\mathbf{V}_{L/K}$ with the following properties:

(1) each face of a cone in Δ is also a cone in Δ;

(2) the intersection of two cones in Δ is a face of both of them.

(ii) For a K-fan (L, Δ), the union of all of the cones from Δ in $\mathbf{V}_{L/K}$ is denoted by $|\Delta|$.

We are going to associate to every separated irreducible normal scheme $\mathcal{X}$ of finite type over K a K-fan $\Delta(\mathcal{X})$ as follows. Let L be the field $K(\mathcal{X})$ of rational functions on $\mathcal{X}$. For an open affine subscheme $\mathcal{U}$ of $\mathcal{X}$, let $\sigma_{\mathcal{U}}$ denote the rational strongly convex polyhedral cone $\sigma^{(A_{\mathcal{U}})}$ (i.e., $\sigma_{\mathcal{U}} = \{x \in \mathbf{V}_{L/K} \big| f(x) \leq 1 \text{ for all } f \in A_{\mathcal{U}}\}$). Furthermore, if $\mathcal{V}$ is an open affine subscheme of $\mathcal{X}$ lying in $\mathcal{U}$, then $\mathcal{V}$ is a principal open subset of $\mathcal{U}$, i.e., $A_{\mathcal{V}} = (A_{\mathcal{U}})_f$ for some $f \in A_{\mathcal{U}}$. This implies that $\sigma_{\mathcal{V}} = \{x \in \sigma_{\mathcal{U}} \big| f(x) = 1\}$ and, therefore, $\sigma_{\mathcal{V}}$ is a face of $\sigma_{\mathcal{U}}$. Proposition 7.6.3 implies that the correspondence $\mathcal{V} \mapsto \sigma_{\mathcal{V}}$ gives rise to an isomorphism between the poset of open affine subschemes of $\mathcal{U}$ and the poset of faces of $\sigma_{\mathcal{U}}$. Finally, given $\mathcal{U}$ and $\mathcal{V}$ are open affine subschemes of $\mathcal{X}$, one has $\sigma_{\mathcal{U} \cap \mathcal{V}} = \sigma_{\mathcal{U}} \cap \sigma_{\mathcal{V}}$. Indeed, let x be a point from the right hand side. Since the homomorphism $A_{\mathcal{U}} \otimes_K A_{\mathcal{V}} \to A_{\mathcal{U} \cap \mathcal{V}}$ is surjective, it follows that every element $h \in A_{\mathcal{U} \cap \mathcal{V}}$ is of the form fg with $f \in A_{\mathcal{U}}$ and $g \in A_{\mathcal{V}}$. We get $h(x) = f(x)g(x) \leq 1$, i.e., $x \in \sigma_{\mathcal{U} \cap \mathcal{V}}$. Thus, the set of the rational strongly convex polyhedral cones $\Delta(\mathcal{X}) = \{\sigma_{\mathcal{U}}\}$ is a fan in $\mathbf{V}_{L/K}$.

Let $\mathcal{Y}$ be an irreducible Zariski closed subset of $\mathcal{X}$. If $\mathcal{U}$ is an open affine subscheme of $\mathcal{X}$ that contains $\check{\mathcal{Y}}$, then $\mathcal{Y} \cap \mathcal{U} = \mathrm{Fspec}(A_{\mathcal{U}}/\mathfrak{p})$ for a Zariski prime ideal $\mathfrak{p} \subset A_{\mathcal{U}}$. By Proposition 7.6.4, one associates to $\mathfrak{p}$ a face $\tau_{\mathfrak{p}}$ of $\sigma_{\mathcal{U}}$, namely, $\tau_{\mathfrak{p}} = \{x \in \sigma_{\mathcal{U}} \big| f(x) = 1 \text{ for all } f \in A_{\mathcal{U}} \backslash \mathfrak{p}\}$. It is easy to see that the cone $\tau_{\mathfrak{p}}$ does not depend on the choice of $\mathcal{U}$; it is therefore denoted by $\tau_{\mathcal{Y}}$.

7.6.6. Lemma. *In the above situation, the following is true:*

(i) the correspondence $\mathcal{Y} \mapsto \tau_{\mathcal{Y}}$ is a bijection between the set of irreducible Zariski closed subsets of $\mathcal{X}$ and the set of cones in Δ;

(ii) $\Delta(\mathcal{Y})$ consists of the images of the cones $\sigma \in \Delta(\mathcal{X})$ with $\tau_{\mathcal{Y}} \subset \sigma$ under the surjection $\mathbf{V}_{L/K} \to \mathbf{V}_{K(\mathcal{Y})/K}$ (induced by the canonical embedding $K(\mathcal{Y}) \hookrightarrow L$).

Proof. The statement (i) straightforwardly follows from Proposition 7.6.4(i).

(ii) Suppose σ is a cone in $\Delta(\mathcal{X})$ that contains $\tau_{\mathcal{Y}}$, and let $\mathcal{U}$ be an open affine subscheme of $\mathcal{X}$ with $\sigma = \sigma_{\mathcal{U}}$. Then $\mathcal{Y} \cap \mathcal{U} = \mathrm{Fspec}(A_{\mathcal{U}}/\mathfrak{p})$ for a Zariski prime ideal $\mathfrak{p} \subset A_{\mathcal{U}}$. By Proposition 7.6.4, the cone $\sigma_{\mathcal{Y} \cap \mathcal{U}}$ in $\Delta(\mathcal{Y})$ is the image of σ under the considered surjection. Furthermore, let $\sigma_{\mathcal{V}}$ be a cone in $\Delta(\mathcal{Y})$ that corresponds to an open affine subscheme $\mathcal{V} \subset \mathcal{Y}$. By Proposition 5.2.4, we can find an open affine subset $\mathcal{U} \subset \mathcal{X}$ with $\mathcal{V} \subset \mathcal{Y} \cap \mathcal{U}$. It follows that $\sigma_{\mathcal{V}}$ is a face of $\sigma_{\mathcal{Y} \cap \mathcal{U}}$. It remains to use the simple fact that, for any fan Δ in a vector space V and any cone τ in Δ, the images of all cones from Δ that contain τ under the canonical map $V \to V/V(\tau)$, where $V(\tau)$ is the vector subspace generated by τ, form a fan in $V/V(\tau)$. ∎

For a K-fan (L, Δ) and a cone $\tau \in \Delta$, we denote by $K(\tau) = K_L(\tau)$ the K-subfield of L

that consists of zero and the elements $f \in L$ with $f(x) = 1$ for all $x \in \tau$. (For example, if τ is the minimal cone in Δ, then $K(\tau) = L$.) Notice that the kernel of the canonical surjective map $\mathbf{V}_{L/K} \to \mathbf{V}_{K(\tau)/K}$ is the vector subspaces generated by τ. We denote by $\widetilde{\mathrm{Star}}(\tau) = \widetilde{\mathrm{Star}}_\Delta(\tau)$ the set of the cones $\sigma \in \Delta$ that contain τ, and by $\mathrm{Star}(\tau) = \mathrm{Star}_\Delta(\tau)$ the set of the images of the cones from $\widetilde{\mathrm{Star}}(\tau)$ in $\mathbf{V}_{K(\tau)/K}$ under the latter surjective map. Notice that the canonical map $\widetilde{\mathrm{Star}}(\tau) \to \mathrm{Star}(\tau)$ is a bijection. We already mentioned (and used in the proof of Lemma 7.6.6) the fact that the pair $(K(\tau), \mathrm{Star}(\tau))$ is a K-fan. For example, in the situation of Lemma 7.6.6, one has $(K(\tau_{\mathcal{Y}}), \mathrm{Star}(\tau_{\mathcal{Y}})) = (K(\mathcal{Y}), \Delta(\mathcal{Y}))$.

7.6.7. Definition. (i) A *dominant morphism of K-fans* $\varphi_\alpha : (L', \Delta') \to (L, \Delta)$ is a homomorphism of K-fields $\alpha : L \to L'$ such that, for every $\sigma' \in \Delta'$ there exists $\sigma \in \Delta$ with $v_\alpha(\sigma') \subset \sigma$, where v_α is the induced surjective map $\mathbf{V}_{L'/K} \to \mathbf{V}_{L/K}$.

(ii) A *morphism of K-fans* $\varphi : (L', \Delta') \to (L, \Delta)$ is a pair (τ, φ_α) consisting of a cone $\tau \in \Delta$ and a dominant morphism $\varphi_\alpha : (L', \Delta') \to (K(\tau), \mathrm{Star}(\tau))$.

Dominant morphisms are precisely morphisms in which τ is the minimal cone in Δ (i.e., the origin of $\mathbf{V}_{L/K}$). It is clear that one can compose dominant morphisms, and so K-fans with dominant morphisms as morphisms form a category which is denoted by $K\text{-}\mathcal{F}ans^{\mathrm{dom}}$. We are now going to explain how to compose arbitrary morphisms.

Let first $\varphi_\alpha : (L', \Delta') \to (L, \Delta)$ be a dominant morphism. For $\tau' \in \Delta'$, let τ be the minimal cone in Δ with $v_\alpha(\tau') \subset \tau$. Then the restriction of α to the K-field $K(\tau)$ induces a homomorphism $\alpha(\tau') : K(\tau) \to K(\tau')$. *We claim that the latter gives rise to a dominant morphism* $\varphi_{\alpha(\tau')} : (K(\tau'), \mathrm{Star}(\tau')) \to (K(\tau), \mathrm{Star}(\tau))$. Indeed, let a cone $\sigma' \in \mathrm{Star}(\tau')$ be the image of a cone $\gamma' \in \Delta'$ with $\tau' \subset \gamma'$. We can find a cone $\gamma \in \Delta$ with $v_\alpha(\gamma') \subset \gamma$. Since τ is the minimal cone that contains $v_\alpha(\tau')$ and $v_\alpha(\tau') \subset \gamma$, it follows that $\tau \subset \gamma$. If σ is the image of γ in $\mathrm{Star}(\tau)$, we get $v_{\alpha(\tau')}(\sigma') \subset \sigma$, i.e., $\varphi_{\alpha(\tau')}$ is really a dominant morphism of K-fans.

Suppose now we are given two morphisms $\varphi' = (\tau', \varphi_{\alpha'}) : (L'', \Delta'') \to (L', \Delta')$ and $\varphi = (\tau, \varphi_\alpha) : (L', \Delta') \to (L, \Delta)$, i.e., two dominant morphisms $\varphi_{\alpha'} : (L'', \Delta'') \to (K(\tau'), \mathrm{Star}(\tau'))$ and $\varphi_\alpha : (L', \Delta') \to (K(\tau), \mathrm{Star}(\tau))$. By the above claim, if σ is the minimal cone in $\mathrm{Star}(\tau)$ with $v_\alpha(\tau') \subset \sigma$, then the restriction of α to $K(\sigma)$ gives rise to a dominant morphism $\varphi_{\alpha(\tau')} : (K(\tau'), \mathrm{Star}(\tau')) \to (K_{K(\tau)}(\sigma), \mathrm{Star}_{\mathrm{Star}(\tau)}(\sigma))$. If now γ is the cone in $\widetilde{\mathrm{Star}}(\tau)$ whose image in $\mathrm{Star}(\tau)$ is σ, then $K_{K(\tau)}(\sigma) = K(\gamma)$ and $\mathrm{Star}_{\mathrm{Star}(\tau)}(\sigma) = \mathrm{Star}(\gamma)$. Thus, the latter morphism is in fact a dominant morphism $\varphi_{\alpha(\tau')} : (K(\tau'), \mathrm{Star}(\tau')) \to (K(\gamma), \mathrm{Star}(\gamma))$, and so the composition of dominant morphisms $\varphi_{\alpha(\tau')} \circ \varphi_{\alpha'} = \varphi_{\alpha' \circ \alpha(\tau')}$ is well defined. We define the composition $\varphi \circ \varphi'$ as the pair $(\gamma, \varphi_{\alpha' \circ \alpha(\tau')})$. The category of K-fans is denoted by K-$\mathcal{F}ans$.

7.6.8. Theorem. *The correspondence $\mathcal{X} \mapsto (K(\mathcal{X}), \Delta(\mathcal{X}))$ gives rise to an equivalence between the category of separated irreducible normal schemes of finite type over K (resp. with Zariski dominant morphisms as morphisms) and the category K-$\mathcal{F}ans$ (resp. K-$\mathcal{F}ans^{\mathrm{dom}}$).*

Proof. Step 1. *The correspondence $\mathcal{X} \mapsto \Delta(\mathcal{X})$ is a functor.* Indeed, let first $\varphi : \mathcal{X}' \to \mathcal{X}$ be a Zariski dominant morphism of between schemes considered. The assumption on φ implies that $\varphi(\check{\mathcal{X}}') \subset \check{\mathcal{X}}$ and, in particular, φ induces a homomorphism of K-fields $\alpha : K(\mathcal{X}) \to K(\mathcal{X}')'$. Furthermore, by Proposition 5.2.4, for every open affine subscheme $\mathcal{U}' \subset \mathcal{X}'$ one can find an open affine subscheme $\mathcal{U} \subset \mathcal{X}$ with $\varphi(\mathcal{U}') \subset \mathcal{U}$ and, therefore, $v_\alpha(\sigma_{\mathcal{U}'}) \subset \sigma_{\mathcal{U}}$. This means that φ induces a dominant morphism of K-fans $(K(\mathcal{X}'), \Delta(\mathcal{X}')) \to (K(\mathcal{X}), \Delta(\mathcal{X}))$. Suppose now that $\varphi : \mathcal{X}' \to \mathcal{X}$ is an arbitrary morphism, and let $\mathcal{Y}$ be the Zariski closure of $\varphi(\mathcal{X}')$ in $\mathcal{X}$. By the previous case, the induced Zariski dominant morphism $\varphi : \mathcal{X}' \to \mathcal{Y}$ gives rise to a dominant morphism of K-fans $\varphi_\alpha : (K(\mathcal{X}), \Delta(\mathcal{X})) \to (K(\mathcal{Y}), \Delta(\mathcal{Y}))$. Since $(K(\mathcal{Y}), \Delta(\mathcal{Y})) = (K(\tau_{\mathcal{Y}}), \mathrm{Star}(\tau_{\mathcal{Y}}))$, the morphism the morphism φ gives rise to a morphism $(K(\mathcal{X}'), \Delta(\mathcal{X}')) \to (K(\mathcal{X}, \Delta(\mathcal{X}))$ represented by the pair $(\tau_{\mathcal{Y}}, \varphi_\alpha)$.

Step 2. *The functor $\mathcal{X} \mapsto (K(\mathcal{X}), \Delta(\mathcal{X}))$ is fully faithful.* Indeed, that the functor is faithful follows from separatedness of the schemes considered. Let first $\varphi_\alpha : (K(\mathcal{X}'), \Delta(\mathcal{X}')) \to (K(\mathcal{X}), \Delta(\mathcal{X}))$ be a dominant morphism of K-fans, i.e., a homomorphism of K-fields $\alpha : K(\mathcal{X}) \to K(\mathcal{X}')$ with the property of Definition 7.5.3(iii). It follows that, for every open affine subscheme $\mathcal{U}' \subset \mathcal{X}'$, there exists an open affine subscheme $\mathcal{U} \subset \mathcal{X}$ with $v_\alpha(\sigma_{\mathcal{U}'}) \subset \sigma_{\mathcal{U}}$. The latter inclusion and Corollary 2.7.7 imply that $\alpha(A_{\mathcal{U}}) \subset A_{\mathcal{U}'}$, i.e., α induces a homomorphism of K-algebras $A_{\mathcal{U}} \to A_{\mathcal{U}'}$ and, therefore, a Zariski dominant morphism of affine schemes $\mathcal{U}' \to \mathcal{U}$. All theses morphisms are compatible on intersections and define a Zariski dominant morphism of schemes $\varphi : \mathcal{X}' \to \mathcal{X}$ which give rise to the homomorphism α. Suppose now that $\varphi = (\tau, \varphi_\alpha) : (K(\mathcal{X}'), \Delta(\mathcal{X}')) \to (K(\mathcal{X}), \Delta(\mathcal{X}))$ is an arbitrary morphism of K-fans. By Lemma 7.5.2(i), one has $\tau = \tau_{\mathcal{Y}}$ for an irreducible Zariski closed subset $\mathcal{Y} \subset \mathcal{X}$, and so φ_α is a dominant morphism $(K(\mathcal{X}'), \Delta(\mathcal{X}')) \to (K(\mathcal{Y}), \Delta(\mathcal{Y}))$. By the previous case, the latter is induced by a morphism of schemes $\mathcal{X}' \to \mathcal{Y}$ which, in its turn, induces a morphism $\mathcal{X}' \to \mathcal{X}$ that gives rise to the morphism of K-fans φ.

Step 3. *The functor $\mathcal{X} \mapsto \Delta(\mathcal{X})$ is essentially surjective.* Indeed, let (L, Δ) be a K-fan. By Proposition 7.6.3, for every $\sigma \in \Delta$, $A^{(\sigma)} = \{f \in L \big| f(x) \le 1 \text{ for all } x \in \sigma\}$ is a normal finitely generated K-algebra with the fraction $\mathbf{F}_1$-field L. We set $\mathcal{X}^{(\sigma)} = \mathrm{Fspec}(A^{(\sigma)})$. If τ is a face of σ, then by the same Proposition 7.6.3(ii) one has $A^{(\tau)} = (A^{(\sigma)})_g$ for some $g \in A^{(\sigma)}$ and, therefore,

$\mathcal{X}^{(\tau)}$ is an open affine subscheme of $\mathcal{X}^{(\sigma)}$. Let $\mathcal{X}$ be the scheme which is obtained by gluing $\mathcal{X}^{(\sigma)}$ along $\mathcal{X}^{(\sigma\cap\tau)}$ for $\sigma, \tau \in \Delta$. It is clear that $\mathcal{X}$ is an irreducible normal scheme of finite type over K. Proposition 7.6.3(iii) implies that, for every air $\sigma, \tau \in \Delta$, the canonical homomorphism $A^{(\sigma)} \otimes_K A^{(\tau)} \to A^{(\sigma\cap\tau)}$ is surjective and, therefore, $\mathcal{X}$ is separated. *We claim that* $\Delta(\mathcal{X}) = \Delta$. Indeed, it suffices to verify that every open affine subscheme $\mathcal{U} \subset \mathcal{X}$ is of the form $\mathcal{X}^{(\sigma)}$ for some $\sigma \in \Delta$. By the separatedness of $\mathcal{X}$, all of the intersections $\mathcal{U} \cap \mathcal{X}^{(\sigma)}$ are open affine subschemes of $\mathcal{X}$ and, since $\mathcal{U}$ is covered by them, one has $\mathcal{U} \subset \mathcal{X}^{(\sigma)}$ for some $\sigma \in \Delta$, i.e., $\mathcal{U}$ is an open affine subscheme of $\mathcal{X}^{(\sigma)}$. It follows that $\mathcal{U}$ is a principal open subset of $\mathcal{X}^{(\sigma)}$ and, therefore, $\mathcal{U} = \mathcal{X}^{(\tau)}$ for some face τ of σ. ∎