Étale equivariant sheaves on p-adic analytic spaces

Vladimir G. Berkovich

§1. G-spaces and G-sheaves

1.1. **G-spaces.** Recall that a non-Archimedean field is a field complete with respect to a *fixed* non-Archimedean valuation (which is not assumed to be non-trivial). Furthermore, a (non-Archimedean) analytic space is a pair (k, X), where k is a non-Archimedean field and X is a k-analytic space, and a morphism $(K, Y) \to (k, X)$ is a pair consisting of an isometric embedding $k \hookrightarrow K$ and a morphism of K-analytic spaces $Y \to X\widehat{\otimes}_k K$ (see [Ber2], §1.4). The category of analytic spaces $\mathcal{A}n$ is a fibred category over the category of non-Archimedean fields. (The fibre category over k is the category of k-analytic spaces k-$\mathcal{A}n$.) For brevity the pair (k, X) is denoted by X and is called an analytic space. When we talk about an étale, quasi-étale, smooth, proper (and so on) morphism between two analytic spaces, we assume that the both spaces are from k-$\mathcal{A}n$ for some field k.

Given analytic spaces X and Y, let $\mathrm{Mor}(Y, X)$ denote the set of morphisms $Y \to X$, and let $\mathcal{G}(X)$ denote the group of automorphisms of X. (If X is k-analytic, then such an automorphism induces an isometric automorphism of the field k.) If X and Y are over an analytic space T, then $\mathrm{Mor}_T(Y, X)$ (resp. $\mathcal{G}_T(X)$) denotes the subset of T-morphisms (resp. T-automorphisms). In what follows all analytic spaces considered are assumed to be Hausdorff. Given an analytic function f on an analytic space X, one sets $\rho(f) = \max_{x\in X} |f(x)|$.

Let X be an analytic space. One introduces a set $\mathfrak{E}(X)$ as follows (see [Ber3], §6). An element ε of $\mathfrak{E}(X)$ consists of a finite family $s(\varepsilon) = \{U_i\}_{i\in I}$ of compact analytic domains in X and, for each $i \in I$, of finite sets of analytic functions $\{f_{ij}\}_{j\in J_i}$ on U_i and of positive numbers $\{t_{ij}\}_{j\in J_i}$. Such an element ε defines, for each analytic space Y, a relation on the set $\mathrm{Mor}(Y, X)$ as follows. Given two morphisms $\varphi, \psi : Y \to X$, we write $d(\varphi, \psi) < \varepsilon$ if $\varphi^{-1}(U_i) = \psi^{-1}(U_i)$ and $\rho(\varphi_i^* f_{ij} - \psi_i^* f_{ij}) \le t_{ij}$ for all $i \in I$ and $j \in J_i$, where φ_i and ψ_i are the induced morphisms $\varphi^{-1}(U_i) \to U_i$ (if $\varphi^{-1}(U_i)$ is empty, the above inequality is assumed to hold). The relations $d(\varphi, \psi) < \varepsilon$ define a uniform space structure and, in particular, a topology on $\mathrm{Mor}(Y, X)$. The group $\mathcal{G}(X)$ is endowed with the topology induced from $\mathrm{Mor}(X,X)$. It is a topological group whose topology is defined by the system of the subgroups $\mathcal{G}_\varepsilon(X) = \{\sigma \in \mathcal{G}(X) \big| \sigma(U_i) = U_i, \rho(\sigma_i^* f_{ij} - f_{ij}) \le t_{ij}\}$ for $\varepsilon \in \mathfrak{E}(X)$ as above. We say that the action of a topological group G on an analytic space is *continuous* if the induced homomorphism $G \to \mathcal{G}(X)$ is continuous.

1.1.1. Examples. (i) If X is k-analytic, then the evident action of the Galois group $\mathrm{Gal}_k = \mathrm{Gal}(k^{\mathrm{s}}/k)$ on $\overline{X} = X\widehat{\otimes}\widehat{k^{\mathrm{a}}}$ is continuous. Moreover, if a k-analytic group G acts on X, then the actions of $G(k)$ on X and $\overline{X}$ are continuous ([Ber3], 6.4).

(ii) Let $\mathfrak{X}$ be a formal scheme locally finitely presented over k° (resp. a special formal scheme over k° if the valuation on k is discrete, see [Ber4]) Then the group of automorphisms of $\mathfrak{X}$ over k°, $\mathcal{G}(\mathfrak{X}/k^\circ)$, is endowed with a topology as follows. If $\mathfrak{X}$ is quasicompact, then the topology of $\mathcal{G}(\mathfrak{X}/k^\circ)$ is defined by the subgroups $\mathcal{G}_{\mathcal{J}}(\mathfrak{X}/k^\circ)$ consisting of the automorphisms trivial modulo an ideal of definition $\mathcal{J} \subset \mathcal{O}_{\mathfrak{X}}$. In the general case, $\mathcal{G}(\mathfrak{X}/k^\circ)$ is endowed with the weakest topology with respect to which for any open quasicompact subscheme $\mathfrak{Y} \subset \mathfrak{X}$ the stabilizer of $\mathfrak{Y}$ in $\mathcal{G}(\mathfrak{X}/k^\circ)$ is open and the homomorphism from it to $\mathcal{G}(\mathfrak{Y}/k^\circ)$ is continuous. It is easy to verify that the homomorphism $\mathcal{G}(\mathfrak{X}/k^\circ) \to \mathcal{G}_k(\mathfrak{X}_\eta)$ is continuous. In particular, if a topological group G acts continuously on $\mathfrak{X}$, then it acts continuously on $\mathfrak{X}_\eta$.

An analytic space X endowed with a continuous action of a topological group G will be called a *G-space*. A G-equivariant morphism between two G-spaces will be called a *G-morphism*.

1.1.2. Construction. Let H be an open subgroup of a topological group G, and let X be an H-space. Then there is a unique (up to a canonical isomorphism) an H-morphism $f : X \to X'$ to a G-space X' such that any H-morphism $\varphi : X \to Y$ to a G-space Y extends in a unique way to a G-morphism $\varphi' : X' \to Y$. Indeed, let $G = \coprod_{i\in G/H} g_i H$ with $g_0 = 1$ (for the coset H), X' the disjoint union $\coprod_{i\in G/H} X_i$, where X_i is a copy of X identified by an isomorphism $f_i : X \widetilde{\to} X_i$. An element $g \in G$ acts on X' as follows: if $gg_i = g_j h$, $h \in H$, then $g\big|_{X_i} = f_j h f_i^{-1} : X_i \to X_j \subset X$. The morphism $f = f_0 : X \to X'$ satisfies the universal property. Indeed, if $\varphi : X \to Y$ is a H-morphism to a G-space Y, then the required morphism $\varphi' : X' \to Y$ is defined by $\varphi'\big|_{X_i} = g_i \varphi f_i^{-1}$. We remark that the morphism $f' : X' \to X$ that coincides with f_i^{-1} on X_i is an H-morphism with $f' \circ f = 1\big|_X$. (If X is in fact a G-space, then f' is a unique G-morphism with the property $f' \circ f = 1\big|_X$.) The analytic space X' will be denoted by $X_{G/H}$ We remark that for any subgroup $H \subset H' \subset G$ there is a canonical isomorphism $(X_{H'/H})_{G/H'} \widetilde{\to} X_{G/H}$.

The *category of analytic spaces with operators $\mathcal{A}nop$* is the category of pairs $X(G)$ where G is a topological group and X is a G-space. A morphism between such spaces $\varphi : X'(G') \to X(G)$ is a pair consisting of a continuous homomorphism of topological groups $\nu_\varphi : G' \to G$ and a morphism of analytic spaces $\varphi : Y \to X$ compatible with the homomorphism ν_φ. For example, a G-morphism $\varphi : Y \to X$ between G-spaces gives rise to a morphism $\varphi : Y(G) \to X(G)$ for which ν_φ is the identity map on G. If X is a G-space, then the action of G on X extends to a natural action of G on $X(G)$ for which $\nu_g(g') = gg'g^{-1}$. The category $\mathcal{A}nop$ is a fibred category over the category of topological groups. The fibre over G will be denoted by G-$\mathcal{A}n$.

1.1.3. Example. Let X be a G-space. In what follows we'll use the following two morphisms $a : X(G^d) \to X(G)$ and $b : X \to X(G)$, where G^d denotes the group G endowed with the discrete topology and $X = X(\{1\})$.

1.2. The quasi-étale and étale topologies on a G-space. For a G-space X, let $\text{Qét}(X(G))$ (resp. $\text{Ét}(X(G))$) denote the category of quasi-étale (resp. étale) morphisms $U(G) \to X(G)$. The *quasi-étale* (resp. *étale*) *topology* on $X(G)$ is the Grothendieck topology on the category $\text{Qét}(X(G))$ (resp. $\text{Ét}(X(G))$) generated by the pretopology for which the set of coverings of $(U(G) \to X(G)) \in \text{Qét}(X(G))$ (resp. $\text{Ét}(X(G))$) consists of the families $\{U_i(G) \to U(G)\}_{i\in I}$ such that $\{U_i \to U\}_{i\in I}$ is a covering in the quasi-étale (resp. étale) topology of X. We denote by $X(G)_{\text{qét}}$ (resp. $X(G)_{\text{ét}}$) the site obtained in this way, by $X(G)_{\widetilde{\text{qét}}}$ (resp. $X(G)_{\widetilde{\text{ét}}}$) the corresponding topos, and by $\mathbf{S}(X(G)_{\text{qét}})$ (resp. $\mathbf{S}(X(G)_{\text{ét}})$) the corresponding category of abelian sheaves. There is a morphism of sites $\mu_G : X(G)_{\text{qét}} \to X(G)_{\text{ét}}$, and any morphism $\varphi : X'(G') \to X(G)$ gives rise in the evident way to morphisms of sites $X'(G')_{\text{qét}} \to X(G)_{\text{qét}}$ and $X'(G')_{\text{ét}} \to X(G)_{\text{ét}}$ and to morphisms of the corresponding topoi. For a quasi-étale (resp. étale) sheaf F' on $X'(G')$, the value of $\varphi_* F$ on $U(G)$ over $X(G)$ is $F'((X' \times_X U)(G'))$ and, for a quasi-étale (resp. étale) sheaf F on $X(G)$ the sheaf $\varphi^* F$ is described as follows. For a morphism $Y(H) \to X(G)$, let $C(Y(H)/X(G)$ denote the category of morphisms $Y(H) \to V(G)$ over $X(G)$, where V is quasi-étale (resp. étale) over X. Then $\varphi^* F$ is the sheaf associated with the presheaf

$$(U'(G') \to X'(G')) \mapsto \varphi^p F(U'(G')) := \varinjlim F(V(G)) \ ,$$

where the limit is taken over the dual category $C(U'(G')/X(G))^\circ$.

1.2.1. Examples. (i) Let N be an open subgroup of a topological group G, and let X be an N-space. Then the morphism $X(N) \to X_{G/N}(G)$ induces an isomorphism of sites $X(N)_{\text{qét}} \widetilde{\to} X_{G/N}(G)_{\text{qét}}$ (resp. $X(N)_{\text{ét}} \widetilde{\to} X_{G/N}(G)_{\text{ét}}$) and, therefore, an isomorphism of the corresponding topoi.

(ii) If X is a k-analytic space, $\overline{X} = X\widehat{\otimes}\widehat{k}^{\rm a}$, and $G = {\rm Gal}(k^{\rm s}/k)$, then the inverse image functor for the morphism $\overline{X}(G) \to X$ induces an isomorphism of topoi $X_{\widetilde{\rm q\acute{e}t}} \widetilde{\to} \overline{X}(G)_{\widetilde{\rm q\acute{e}t}}$ (resp. $X_{\widetilde{\rm \acute{e}t}} \widetilde{\to} \overline{X}(G)_{\widetilde{\rm \acute{e}t}}$).

(iii) Let a discrete group Γ act discretely on a k-analytic space X, and assume that the conditions of Lemma 4 from [Ber5] for the existence of the quotient space $\Gamma\backslash X$ are satisfied and the morphism $X \to \Gamma\backslash X$ is étale. (For example, this is true if the action of Γ on X is free.) Then the inverse image functor for the morphism $X(\Gamma) \to \Gamma\backslash X$ induces an isomorphism of topoi $(\Gamma\backslash X)_{\widetilde{\rm q\acute{e}t}} \widetilde{\to} X(\Gamma)_{\widetilde{\rm q\acute{e}t}}$ (resp. $(\Gamma\backslash X)_{\widetilde{\rm \acute{e}t}} \widetilde{\to} X(\Gamma)_{\widetilde{\rm \acute{e}t}}$).

(iv) Let X be a G-space and let φ be the morphism $X(G') \to X(G)$ that corresponds to a surjective continuous homomorphism $G' \to G$. Then the functor $\varphi^* : {\bf S}(X(G)_{\widetilde{\rm q\acute{e}t}}) \to {\bf S}(X(G')_{\widetilde{\rm q\acute{e}t}})$ (resp. ${\bf S}(X(G)_{\widetilde{\rm \acute{e}t}}) \to {\bf S}(X(G')_{\widetilde{\rm \acute{e}t}})$) has a left adjoint functor θ which is describes as follows. Let $F \in {\bf S}(X(G')_{\widetilde{\rm q\acute{e}t}})$ (resp. ${\bf S}(X(G')_{\widetilde{\rm \acute{e}t}})$). Then for each quasi-étale (resp. étale) morphism $U(G) \to X(G)$ the group $H = {\rm Ker}(G' \to G)$ acts on $F(U(G))$. The sheaf $\theta(F)$ is the sheaffification of the presheaf that associate with $U(G)$ the maximal quotient of $F(U(G))$ where the group H acts trivially.

We denote by $\Gamma_{X(G)}$ the global sections functor on $X(G)_{\widetilde{\rm q\acute{e}t}}$ (resp. $X(G)_{\widetilde{\rm \acute{e}t}}$), i.e., $\Gamma_{X(G)}(F) = F(X(G))$. The high direct images of $\Gamma_{X(G)}$ on the category of abelian sheaves will be denoted by $H^q(X(G), F)$. We also denote by $\Gamma_{X\{G\}}$ the functor from $X(G)_{\widetilde{\rm q\acute{e}t}}$ (resp. $X(G)_{\widetilde{\rm \acute{e}t}}$) to the category of (discrete) G-sets by

$$\Gamma_{X\{G\}}(F) = \varinjlim_{N} F(X_{G/N}(G)) \ ,$$

where N runs through open subgroups of G. The simplest way to see that $\Gamma_{X\{G\}}(F)$ is really a G-set is as follows.

Let $P(G)$ denote the category of G-sets endowed with the Grothendieck topology generated by the pretopology for which the sets of coverings consist of surjective families of G-maps. It is well known that any sheaf $F \in P(G)^{\widetilde{}}$ is representable by the G-set $\cup_N F(G/N)$, where N runs through open subgroups of G. In particular, there is an equivalence of categories $P(G) \widetilde{\to} P(G)^{\widetilde{}}$. (We also remark that there is an equivalence of categories $P(G) \widetilde{\to} P(\widehat{G})$, where $\widehat{G}$ is the completion of G with respect to open subgroups.) For a G-set Σ, let X_Σ denote the disjoint union $\coprod_{\sigma\in\Sigma} X_\sigma$ (X_σ are copies of X) provided with the following action of G: an element $g \in G$ takes X_σ to $X_{g\sigma}$ by the action of g on X. (For example, the space $X_{G/N}$ associated with the G-set G/N, where N is an open subgroup of G, coincides with the space constructed in 1.1.2.) The correspondence $\Sigma \mapsto X_\Sigma(G)$ defines a morphism of sites $\gamma : X(G)_{\rm q\acute{e}t} \to P(G)$ (resp. $X(G)_{\rm \acute{e}t} \to P(G)$), and we see that $\Gamma_{X\{G\}}(F)$ is exactly the G-set that represents the sheaf $\gamma_* F$. It follows also that for any open subgroup $N \subset G$ one has $\Gamma_{X\{G\}}(F)^N = F(X_{G/N}(G))$. The high direct images of the functor $\Gamma_{X\{G\}}$ on the category of abelian sheaves will be denoted by $H^q(X\{G\}, F)$. For any abelian sheaf F there is a spectral sequence

$$E_2^{p,q} = H^p(G, H^q(X\{G\}, F)) \Longrightarrow H^{p+q}(X(G), F) \ .$$

1.3. The stalk of an étale sheaf at a point. Consider first the case when $X = \mathbf{p}_k$, the spectrum of a non-Archimedean field k. (In this case $\text{Qét}(\mathbf{p}_k(G)) = \text{Ét}(\mathbf{p}_k(G))$.) For a field K over k with a valuation that extends the valuation on k, let $\mathcal{G}al(K/k)$ denote the group of isometric automorphisms of K that take k onto k. It is a topological group whose topology is defined by subgroups of the form $\{g \in \mathcal{G}al(K/k) || {}^g\alpha_i - \alpha_i| \le r_i, 1 \le i \le n\}$, where $\alpha_1, \dots, \alpha_n \in K$ and $r_1, \dots, r_n > 0$. We set $\mathcal{G}(k) = \mathcal{G}al(k/k)$, $\mathcal{G}al_k = \mathcal{G}al(k^{\text{s}}/k)$ and $\text{Gal}_k = \text{Gal}(k^{\text{s}}/k)$, where k^{s} is a separable closure of k. Then there is an exact sequence of topological groups

$$1 \longrightarrow \text{Gal}_k \longrightarrow \mathcal{G}al_k \longrightarrow \mathcal{G}(k) \longrightarrow 1$$

The action of G on $\mathbf{p}_k$ is a continuous homomorphism $G \to \mathcal{G}(k)$. The latter gives rise to an exact sequence of topological groups

$$1 \longrightarrow \text{Gal}_k \longrightarrow \mathcal{G} \xrightarrow{\nu} G \longrightarrow 1$$

Furthermore, let $\overline{\mathbf{p}}_k = \mathbf{p}_{\widehat{k^{\text{a}}}}$. For an étale morphism $U(G) \to \mathbf{p}_k(G)$, let $\Sigma_{U(G)}$ denote the set of all morphisms $\overline{\mathbf{p}}_k \to U$ over $\mathbf{p}_k$. If $\sigma \in \Sigma_{U(G)}$ and $g \in \mathcal{G}$, then the formula $g\sigma = \nu(g) \circ \sigma \circ g^{-1}$ defines an action of $\mathcal{G}$ on $\Sigma_{U(G)}$.

1.3.1. Proposition *(equivariant Galois theory). The correspondence $U(G) \mapsto \Sigma_{U(G)}$ gives rise to an equivalence of categories $\text{Ét}(\mathbf{p}_k(G)) \widetilde{\to} P(\mathcal{G})$. In particular, there is an equivalence of categories $\mathbf{p}_k(G)^{\sim}_{\text{ét}} \widetilde{\to} P(\mathcal{G})$.*

Proof. First of all, we have to verify that the action of $\mathcal{G}$ on $\Sigma_{U(G)}$ is discrete. Given $\sigma : \overline{\mathbf{p}}_k \to U$, let V be the connected component of U that contains the image of $\overline{\mathbf{p}}_k$. Then $V = \mathcal{M}(K)$, where K is a finite separable extension of k. Let α be an element K that generates it over k, β the image of α in k^{s} under σ, and r the minimum of the distances from β to its congugates in k^{s}. Then the stabilizer of σ contains the open subgroup of $\mathcal{G}$ that consists of the elements g with $|{}^g\beta - \beta| < r$. Furthermore, it follows from the construction that the connected components of $U(G)$ correspond bijectively to the G-orbits in $\Sigma_{U(G)}$. This easily implies that the functor considered is fully faithful. Finally, let Σ be a transitive G-set. Then the stabilizer $\mathcal{N}$ of a fixed element $\sigma \in \Sigma$ is open in $\mathcal{G}$ and the field $K = (k^{\mathrm{s}})^{\mathcal{N} \cap \mathrm{Gal}_k}$ is finite over k. It follows that the action of N, the image of $\mathcal{N}$ in G, on k extends to an action of N on K. If $V = \mathcal{M}(K)$ and $U = V_{G/N}$, then $\Sigma_{U(G)} \widetilde{\to} \Sigma$, and therefore the functor is essentially surjective. ∎

If F is a sheaf on $\mathbf{p}_k(G)$ and Σ is the corresponding $\mathcal{G}$-set, then $\Gamma_{\mathbf{p}_k(G)}(F) = \Sigma^{\mathcal{G}}$ and $\Gamma_{\mathbf{p}_k\{G\}}(F) = \Sigma^{\mathrm{Gal}_k}$. In particular, if F is abelian, then

$$H^q(\mathbf{p}_k(G), F) = H^q(\mathcal{G}, F) \text{ and } H^q(\mathbf{p}_k\{G\}, F) = H^q(\mathrm{Gal}_k, \Sigma) \ .$$

Let X now be a G-space and $x \in X$. Then there is a canonical morphism $\mathbf{p}_x(G_x) \to X(G)$, where $\mathbf{p}_x = \mathbf{p}_{\mathcal{H}(x)}$ and $G_x = \{g \in G \big| gx = x\}$. For an étale sheaf F on $X(G)$, let F_x denote the pullback of F on $\mathbf{p}_x(G_x)$. By Proposition 1.3.1, F_x can be considered as a $\mathcal{G}_x$-set, where $\mathcal{G}_x$ is the extension of G_x by $\mathrm{Gal}_{\mathcal{H}(x)}$ constructed above, and it is called the *stalk of F at the point x*. Furthermore, a *geometric point of* $X(G)$ is a morphism of the form $\overline{x} : \mathbf{p}_{\overline{x}} \to X(G)$, where $\mathbf{p}_{\overline{x}}$ is the spectrum of an algebraically closed non-Archimedean field $\mathcal{H}(\overline{x})$. If the image of $\overline{x}$ is a point $x \in X$, we say that $\overline{x}$ *is over* x. For an étale sheaf F on $X(G)$, let $F_{\overline{x}}$ denote the pullback of F on $\mathbf{p}_{\overline{x}}$. It can be considered as a set and is called the *stalk of F at the geometric point $\overline{x}$*. If $\overline{x}$ is over x, then any embedding of fields $\mathcal{H}(x)^{\mathrm{s}} \hookrightarrow \mathcal{H}(\overline{x})$ over $\mathcal{H}(x)$ induces a bijection $F_x \widetilde{\to} F_{\overline{x}}$. One has

$$F_{\overline{x}} = \varinjlim_{C(\overline{x}/X(G))^\circ} F(V(G)) \ .$$

Let $C'(\overline{x}/X)$ denote the full subcategory of $C(\overline{x}/X)$ consisting of the objects for which the morphism $V \to X$ is distinguished étale. (It is a cofinal subcategory of $C(\overline{x}/X)$.) Then any open subgroup $N \subset G_V$ gives rise to an object $V_{G/N}(G)$ of the category $C(\overline{x}/X(G))$, and the family of objects $\{V_{G/N}(G)\}$ is cofinal in $C(\overline{x}/X(G))$. It follows that

$$F_{\overline{x}} = \varinjlim \varinjlim F(V_{G/N}(G) \ ,$$

where the first limit is taken over objects V of $C'(\overline{x}/X)$, and the second limit is taken over open subgroups $N \subset G_V$.

It is easy to see that if we fix a geometric point $\overline{x}$ over each point $x \in X$, then the family $\{\overline{x}\}$ is a conservative family of points of the étale topos of $X(G)$ (see [SGA4], Exp. IV, 6.4.1). In particular, a morphism of étale sheaves $F \to F'$ on $X(G)$ is mono/epi/isomorphism if and only if for all $x \in X$ the induced maps $F_x \to F'_x$ possess the same property.

1.4. G-sheaves. Let X be a G-space. In this subsection we show that the inverse image functor for the morphism $b : X \to X(G)$ identifies the topos $X(G)_{\widetilde{\text{qét}}}$ (resp. $X(G)_{\widetilde{\text{ét}}}$) with the category of G-sheaves on X. The definition of the latter given below is an analog of the usual notion of an equivariant sheaf on a space with operators (see [Gro], Ch. IV).

Consider first the case when the group G is discrete. In this case a quasi-étale (resp. étale) G-sheaf F is a quasi-étale (resp. étale) sheaf on X endowed with an action of G on F, *compatible* with the action of G on X, i.e., endowed with a system of isomorphisms $\tau(g) : F \tilde{\to} g^*F$, $g \in G$, such that $\tau(gh) = h^*(\tau(g)) \circ \tau(h)$. In other words, F is a G-sheaf if for each quasi-étale (resp. étale) morphism $U \to X$ and for each $g \in G$ there is a functorial bijection $F(U) \tilde{\to} F({}^gU) : f \mapsto {}^gf$, where ${}^gU = U \times_{X,g^{-1}} X$, such that ${}^{gh}f = {}^g({}^hf)$. Given a G-sheaf F, then for any quasi-étale (resp. étale) morphism $U(H) \to X(G)$, where H is a subgroup of G, the set $F(U)$ is endowed with a canonical action of the group H. Indeed, for $h \in H$ the morphism $h^{-1} : U \to U$ induces an isomorphism $U \tilde{\to} {}^hU = U \times_{X,h^{-1}} X$ over X. The latter induces a bijection $\sigma(h) : F({}^hU) \tilde{\to} F(U)$, and the action of H on $F(U)$ is defined by $hf = \sigma(h)({}^hf)$.

Consider now the case of an arbitrary topological group G. Recall that the Key Lemma 7.2 from [Ber3] implies that given a quasi-étale morphism $U \to X$ with compact U there exist $\varepsilon \in \mathfrak{E}(X)$, $\delta \in \mathfrak{E}(U)$ and a unique continuous homomorphism $\mathcal{G}_\varepsilon(X) \to \mathcal{G}_\delta(U)$ such that the morphism $U \to X$ commutes with the action of $\mathcal{G}_\varepsilon(X)$. It follows that for any quasi-étale morphism $U \to X$ with compact U the action of G on X extends in a canonical way to a continuous action of some open subgroup $G_U \subset G$ on U. Furthermore, an étale morphism $\mathcal{U} \to X$ is said to be *distinguished* if it can be represented as a composition $\mathcal{U} \overset{j}{\hookrightarrow} U \overset{\varphi}{\to} X$, where U is compact, φ is quasi-étale and j identifies $\mathcal{U}$ with a *distinguished* open subset of U, i.e., with such one whose complement in U is an analytic domain. For such a morphism the action of G on X extends in a canonical way to a continuous action of an open subgroup $G_U \subset G$. Since $U \backslash \mathcal{U}$ is a compact analytic domain in U, then $\mathcal{U}$ is invariant under the action of an open subgroup $G_\mathcal{U} \subset G_U$. If there is another representation of the morphism $\mathcal{U} \to X$ as a composition $\mathcal{U} \overset{j'}{\hookrightarrow} U' \overset{\varphi'}{\to} X$ and, therefore, an extension of the action of G on X to a continuous action of an open subgroup $G'_\mathcal{U} \subset G$ on $\mathcal{U}$, then one can find an open subgroup $N \subset G_\mathcal{U} \cap G'_\mathcal{U}$ such that the two actions of N on $\mathcal{U}$ coincide. (For this it suffices to apply the Key Lemma to the quasi-étale morphism $U \times_X U' \to X$.) We remark that if F is an étale G-sheaf then, for any quasi-étale morphism $U \to X$ with compact U (resp. any distinguished étale morphism $\mathcal{U} \to X$), there is a canonical action of G_U on $F\big|_U$ (resp. $G_\mathcal{U}$ on $F\big|_\mathcal{U}$) compatible with the action of G on X. In particular, the group G_U (resp. $G_\mathcal{U}$) acts on $F(U)$ (resp. $F(\mathcal{U})$).

1.4.1. Definition. A *quasi-étale* (resp. *étale*) *G-sheaf* on X is a quasi-étale (resp. étale) G^d-sheaf F such that for any quasi-étale morphism $U \to X$ with compact U the action of G_U on

$F(U)$ is discrete. The category of quasi-étale (resp. étale) G-sheaves on X will be denoted by $\mathbf{T}_G(X_{\text{qét}})$ (resp. $\mathbf{T}_G(X_{\text{ét}})$).

It is easy to see that an étale G^d-sheaf F is a G-sheaf if and only if for any étale morphism $U \to X$ and any element $f \in F(U)$ each point of U has a distinguished open neighborhood $\mathcal{U}$ such that the stabilizer of $f\big|_{\mathcal{U}}$ is open in $G_{\mathcal{U}}$.

1.4.2. Theorem. *The inverse image functor for the morphism $b : X \to X(G)$ induces an equivalence of categories $\widetilde{X(G)_{\text{qét}}} \widetilde{\to} \mathbf{T}_G(X_{\text{qét}})$ (resp. $\widetilde{X(G)_{\text{ét}}} \widetilde{\to} \mathbf{T}_G(X_{\text{ét}})$).*

Proof. A. Consider first the quasi-étale topology.

1. Let $U \to X$ be a quasi-ètale morphism with compact U. By the Key Lemma from [Ber3], the action of G on X extends in a canonical way to an action of an open subgroup $G_U \subset G$ on U. For an open subgroup $N \subset G_U$ there is a commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X(G) \\ \uparrow & & \uparrow \\ U & \longrightarrow & U_{G/N}(G) \end{array}$$

i.e., an object of the category $C(U/X(G))$. *We claim that the family of objects $\{U_{G/N}(G)\}$ is cofinal in the category $C(U/X(G))$.* (This will imply that $b^p E(U) = \varinjlim E(U_{G/N}(G))$.) Indeed, suppose we are given a morphism $U \to V(G)$ over $X(G)$, where V is quasi-étale over X. Since this morphism is quasi-étale, it follows from the Key Lemma that it is an N-morphism for some open subgroup $N \subset G_U$. By Construction 1.1.2, the morphism $U \to V$ goes through a G-morphism $U_{G/N} \to V$.

2. *If U is compact, then $b^p E(U) \widetilde{\to} b^* E(U)$.* Indeed, for this it suffices to verify that, given a finite quasi-étale covering $\{U_i \to U\}$ by compact analytic spaces, one has

$$b^p E(U) = \text{Ker}(\prod_i b^p E(U_i) \rightrightarrows \prod_{i,j} b^p E(U_i \times_U U_j)) \ .$$

The Key Lemma implies that for a sufficiently small open subgroup N of the intersection $G_U \cap \cap_{i,j}(G_{U_i} \cap G_{U_j})$ the morphisms $U_i \to U$ and $U_i \times_U U_j \to U_i$ are in fact N-morphisms. Thus, $\{U_{i,G/N}(G) \to U_{G/N}(G)\}$ is a covering of $U_{G/N}(G)$. Since $(U_i \times_U U_j)_N \widetilde{\to} U_{i,G/N} \times_{U_{G/N}} U_{j,G/N}$, one has $E(U_{G/N}(G)) = \text{Ker}(\prod_i E(U_{i,G/N}(G)) \rightrightarrows \prod_{i,j} E((U_i \times_U U_j)_{G/N}(G)))$. The inductive limit of the latter over all such N gives the required fact.

3. *$b^* E$ is a G-sheaf.* It suffices to construct for all quasi-étale morphisms $U \to X$ with compact U and all $g \in G$ a compatible system of bijections $b^* E(U) \widetilde{\to} b^* E({}^g U) : f \mapsto {}^g f$ with ${}^{gg'} f = {}^g({}^{g'} f)$. For this we remark that the composition of the projection ${}^g U = U \times_{X, g^{-1}} X \to U$ with the

embedding $U \to U_{G/N}$ and $g : U_{G/N} \to U_{G/N}$ is a (gNg^{-1})-morphism ${}^gU \to U_{G/N}$. It gives rise to an isomorphism $({}^gU)_{G/gNg^{-1}}(G) \widetilde{\to} U_{G/N}(G)$ over $X(G)$. The latter induces the required bijection

$$b^*E(U) = \varinjlim E(U_{G/N}(G)) \widetilde{\to} b^*E({}^gU) = \varinjlim E(({}^gU)_{G/gNg^{-1}}(G)) \ .$$

4. Let now F be a quasi-étale G-sheaf on X. For $(V(G) \to X(G)) \in \mathrm{Q\acute{e}t}(X(G))$ one has $b_*F(V(G)) = F(V)$. The group G acts on $F(V)$, and therefore we can define a sheaf $(b_*F)^G$ on $X(G)^{\sim}_{\mathrm{q\acute{e}t}}$ by $(b_*F)^G(V(G)) = F(V)^G$. *We claim that* $b^*((b_*F)^G) \widetilde{\to} F$. Indeed, if $U \to X$ is a quasi-étale morphism with compact U, then $b^*((b_*F)^G)(U) = \varinjlim F(U_{G/N})^G$, where N runs through open subgroups of G_U. The required isomorphism follows from the facts that the morphism $U \to U_{G/N}$ induces a bijection $F(U_{G/N})^G \widetilde{\to} F(U)^N$ and the action of G_U on the set $F(U)$ is discrete.

5. *For* $E \in X(G)^{\sim}_{\mathrm{q\acute{e}t}}$ *one has* $E \widetilde{\to} (b_*b^*E)^G$. Indeed, since each object of $\mathrm{Q\acute{e}t}(X(G))$ can be covered by objects of the form $U_{G/N}(G)$, where $U \to X$ is a quasi-étale morphism with compact U and N is an open subgroup of G_U, it suffices to verify that $E(U_{G/N}(G)) \widetilde{\to} (b_*b^*E)^G(U_{G/N})$. The right hand side is $b^*E(U_{G/N})^G = b^*E(U)^N = (\varinjlim E(U_{G/N'}(G)))^N$, where N' runs through open subgroups of N. Consider the canonical morphism $\varphi : U(N) \to X(G)$. One has $E(U_{G/N}(G)) = \varphi^*E(U(N))$, and since $U_{G/N'} = (U_{N/N'})_{G/N}$ then $E(U_{G/N'}(G)) = \varphi^*E(U_{N/N'}(N))$, and therefore the claim follows from the fact that $\Gamma_{U(N)}(F) = \Gamma_{U\{N\}}(F)^N$ for all quasi-étale (and étale) sheaves F on $U(N)$.

B. Consider now the étale topology.

1. *For any* $E \in X(G)^{\sim}_{\mathrm{\acute{e}t}}$, b^*E *is a* G*-sheaf.* Indeed, this follows from the fact that $\mu^*(b^*E)$ is a quasi-étale G-sheaf and the functor $\mu^* : X^{\sim}_{\mathrm{\acute{e}t}} \to X^{\sim}_{\mathrm{q\acute{e}t}}$ is fully faithful.

2. As in the quasi-étale case (see A.4), one defines for any $F \in \mathbf{T}_G(X_{\mathrm{\acute{e}t}})$ a sheaf $(b_*F)^G \in X(G)^{\sim}_{\mathrm{\acute{e}t}}$. *We claim that* $b^*(b_*F)^G \widetilde{\to} F$. Indeed, for a geometric point $\overline{x}$ of X one has

$$(b^*(b_*F)^G)_{\overline{x}} = (b_*F)^G_{b(\overline{x})} = \varinjlim \varinjlim F(U_{G/N})^G = \varinjlim \varinjlim F(U)^N = F_{\overline{x}} \ ,$$

where the first limits are taken over $U \in C'(\overline{x}/X)$ and the second ones are taken over open subgroups $N \subset G_U$.

3. Finally, *we claim that for any* $E \in X(G)^{\sim}_{\mathrm{\acute{e}t}}$ *one has* $E \widetilde{\to} (b_*b^*E)^G$. Indeed, for a geometric point $\overline{x}$ of X one has

$$\begin{aligned} E_{b(\overline{x})} &= (b^*E)_{\overline{x}} = \varinjlim \varinjlim (b^*E)(U)^N = \varinjlim \varinjlim (b^*E)(U_{G/N})^G \\ &= \varinjlim \varinjlim (b_*b^*E)^G(U_{G/N}(G)) = (b_*b^*E)^G_{b(\overline{x})} \ , \end{aligned}$$

where the limits are taken over the same systems as in 2. The theorem is proved. ▪

1.4.3. Corollary. *For any* $F \in X(G)^{\sim}_{\text{ét}}$ *one has* $F \tilde{\to} \mu_* \mu^* F$, *where* μ *is the morphism of sites* $X(G)_{\text{qét}} \to X(G)_{\text{ét}}$. *In particular, the functor* $\mu^* : X(G)^{\sim}_{\text{ét}} \to X(G)^{\sim}_{\text{qét}}$ *is fully faithful, and for any quasi-étale morphism* $f : U(G) \to X(G)$ *one has* $(f^*F)(U(G)) \tilde{\to} (\mu^*F)(U(G))$. ▪

1.5. Cohomology with compact support and Verdier Duality. Let X be a G-space. We recall the construction of the Godement resolution from [SGA4], Exp. XVII, §4.2, adopted to the étale site of $X(G)$. First of all, for a topological space I let $\text{Top}(I)$ denote the site on the category of local homeomorphisms $J \to I$ endowed with the evident Grothendieck topology. (The site $\text{Top}(I)$ gives rise to the usual category of sheaves on I.)

Suppose we are given a set I and a surjective map $I \to X : i \mapsto x_i$. We endow I with the discrete topology and fix for each $i \in I$ a geometric point $\overline{x}_i$ over x_i. This gives rise to a morphism of sites $\nu : \text{Top}(I) \to X(G)_{\text{ét}}$. For an étale abelian sheaf F on $X(G)$, let $\mathcal{C}^{\cdot}(F)$ denote the right resolution of F constructed as follows:

(a) $\mathcal{C}^0(F) = \nu_* \nu^*(F)$, and $d^{-1} : F \to \mathcal{C}^0(F)$ is the adjunction morphism;

(b) if $m \geq 0$, then $\mathcal{C}^{m+1}(F) = \mathcal{C}^0(\text{Coker} d^{m-1})$, and d^m is the canonical morphism $\mathcal{C}^m(F) \to \mathcal{C}^{m+1}(F)$.

By *loc. cit.*, 4.2.3, one has:

(i) $\mathcal{C}^m(F)$ is a flabby sheaf;

(ii) the functor $F \mapsto \mathcal{C}^m(F)$ is exact;

(iii) the fibre of the complex $\mathcal{C}^{\cdot}(F)$ at a point $x \in X$ is a canonically split resolution of F_x.

1.5.1. Proposition. *For any* $F \in X(G)^{\sim}_{\text{ét}}$ *and* $m \geq 0$, $b^*(\mathcal{C}^m(F))$ *is a soft sheaf on* $X_{\text{ét}}$.

Proof. It suffices to assume $m = 0$. Let $\mathcal{F} = b^*(\mathcal{C}^m(F))$. We have to verify the following two facts (see [Ber3], §3):

(1) for any $x \in X$, $\mathcal{F}_x$ is a flabby $\text{Gal}_{\mathcal{H}(x)}$-module;

(2) for any paracompact U étale over X, the restriction of $\mathcal{F}$ to the usual topology $|U|$ of U is a soft sheaf, i.e., for any compact subset $\Sigma \subset U$ the map $\mathcal{F}(U) \to \mathcal{F}(\Sigma)$ is surjective.

First of all, we make the following two observations that will simplify the situation.

(a) Let N be an open subgroup of G and let $G = \coprod_{j \in J} g_j N$. Then there is a surjective map $I' = I \times J \to X : (i, j) \mapsto g_j^{-1} x_i$ and a morphism of sites $\nu' : \text{Top}(I') \to X(N)_{\text{ét}}$ that give rise to an isomorphism $(\nu_* \nu^* F)\big|_{X(N)} \tilde{\to} \nu'_* \nu'^* F$. It follows that to verify (1) and (2) we always can replace G by an open subgroup.

(b) For an étale morphism $X'(G) \to X(G)$, let I' be the set of triples (i, x', ψ), where $i \in I$, $x' \in X'$ is over x_i, and ψ is an embedding $\mathcal{H}(x') \hookrightarrow \mathcal{H}(\mathbf{p}_{\overline{x}_i})$ over $\mathcal{H}(x_i)$. then there is a surjective map $I' \to X' : (i, x', \psi) \mapsto x'$ and a morphism of sites $\nu' : \mathrm{Top}(I') \to X'(G)_{\text{ét}}$ that give rise to an isomorphism $(\nu_* \nu^* F)\big|_{X'(G)} \widetilde{\to} \nu'_* \nu'^* (F\big|_{X'(G)})$.

It follows that instead of (1) and (2) it suffices to verify the following two facts for the case when X is paracompact:

(1′) $H^q(\mathrm{Gal}_{\mathcal{H}(x)}, \mathcal{F}_x) = 0$ for all $x \in X$ and $q \geq 1$;

(2′) the restriction of $\mathcal{F}$ to the usual topology $|X|$ of X is a soft sheaf.

(1′) It suffices to verify that for any finite Galois extension K of $\mathcal{H}(x)$ in $\mathcal{H}(x)^{\mathrm{s}}$, one has $H^q(\mathrm{Gal}, \mathcal{F}_x(K)) = 0$, where $\mathrm{Gal} = \mathrm{Gal}(K/\mathcal{H}(x))$. For this we take a distinguished étale morphism $\varphi : X' \to X$ with $\varphi^{-1}(x) = \{x'\}$ and $\mathcal{H}(x') = K$. Using (a), we can shrink X and G so that we may assume that φ is a G-morphism and a finite Galois covering with the Galois group Gal. Then $\mathcal{F}_x(K)$ is the inductive limit of $\mathcal{F}(\mathcal{U}')$ taken over all distinguished open neighborhoods $\mathcal{U}$ of the point x, where $\mathcal{U}' = \varphi^{-1}(\mathcal{U})$. Using (a) again, it suffices to verify that $H^q(\mathrm{Gal}, (\nu_* \nu^* F)(X'(G))) = 0$. By the construction, one has $(\nu_* \nu^* F)(X'(G)) = \prod_{i \in I} \prod_{(i,x',\psi) \in I'} F_{x_i}$, where I' is as in (b), i.e., $(\nu_* \nu^* F)(X'(G))$ is a direct product of coinduced Gal-modules. This implies (1′).

(2′) We have to verify that for any compact subset $\Sigma \subset X$ the map $\mathcal{F}(X) \to \mathcal{F}(\Sigma)$ is surjective. Let $f \in \mathcal{F}(\Sigma)$. Then f can be extended to a section of $\mathcal{F}$ over a distinguished open neighborhood $\mathcal{U}$ of Σ. We can shrink $\mathcal{U}$ and replace G by a sufficiently small open subgroup of $G_{\mathcal{U}}$ so that we may assume that $\mathcal{U}$ is G-invariant and f comes from a section of $\nu_* \nu^*(F)$ over $\mathcal{U}(G)$. But the latter section is induced from a section over $X(G)$ because I is a discrete space. It follows that f is contained in the image of $\mathcal{F}(X)$, i.e., (2′) is also true. ∎

Let F be an étale abelian sheaf on $X(G)$. The *support* of an element $f \in F(X(G))$ is the set $\mathrm{Supp}(f) = \{x \in X \big| f_x \neq 0\}$, where f_x is the image of f in F_x. It is a closed subset of X. The values of the high direct images of the functor $F \mapsto \Gamma_{c,X(G)}(F) := \{f \in F(X(G)) \big| \mathrm{Supp}(f) \text{ is compact}\}$ are called the *cohomology groups with compact support* and denoted by $H^q_c(X(G), F)$. The values of the high direct images of the functor $F \mapsto \Gamma_{c,X\{G\}}(F) := \varinjlim \Gamma_{c,X(N)}(F)$, where N runs through open subgroups of G, are denoted by $H^q_c(X\{G\}, F)$. One evidently has $H^q_c(X\{G\}, F) = \varinjlim H^q_c(X(N), F)$.

1.5.2. Corollary. *(i) For any étale abelian sheaf F on $X(G)$ there are canonical isomorphisms $H^q_c(X\{G\}, F) \widetilde{\to} H^q_c(X, b^* F)$, $g \geq 0$. In particular, the canonical action of G on the groups*

$H^q_c(X, b^*F)$ is discrete and there is a spectral sequence

$$E_2^{p,q} = H^p(G, H^q_c(X, b^*F)) \Longrightarrow H^{p+q}_c(X(G), F) \ .$$

(ii) Given a ringed space $(X(G), \mathcal{O})$, the values of the high derived functors of the functor $F \mapsto \Gamma_{c,X(G)}$ (resp. $\Gamma_{c,X\{G\}}$) on the category $\mathbf{S}(X(G), \mathcal{O})$ are the groups $H^q_c(X(G), F)$ (resp. $H^q_c(X\{G\}, F)$).

Proof. The case $q = 0$ in (i) (resp. (ii)) follows from the fact that every element of $H^0_c(X, b^*F)$ is fixed by an open subgroup of G (resp. is trivial), and therefore the general case follows from Proposition 1.5.1. ∎

Suppose now we are given a G-morphism of G-spaces $\varphi : Y \to X$. For an étale abelian sheaf F on $Y(G)$ and an étale morphism $U(G) \to X(G)$, let $(\varphi_! F)(U(G))$ denote the subgroup of $F((Y \times_X U)(G))$ that consists of the elements f for which the map $\mathrm{Supp}(f) \to U$ is proper. The correspondence $U(G) \mapsto (\varphi_! F)(U(G))$ is an étale abelian sheaf on $X(G)$.

1.5.3. Corollary. *(i) For any étale abelian sheaf F on $Y(G)$ there are canonical isomorphisms $b^*_X(R^q\varphi_! F) \widetilde{\to} R^q\varphi_!(b^*_Y F)$, $q \geq 0$.*

(ii) If $\mathcal{O}$ is a G-sheaf of rings on Y, then the values of the high derived functors of the functor $F \mapsto \varphi_! F$ on $\mathbf{S}(Y(G), \mathcal{O})$ are the sheaves $R^q\varphi_! F$.

Proof. (i) One easily verifies that the homomorphism considered induces an isomorphism of stalks for $q = 0$, and therefore in the general case it is an isomorphism, by Proposition 1.5.1. (ii) follows from the same proposition. ∎

Corollary 1.5.3 implies that the results on cohomological dimension and base change for the functors $R^q\varphi_!$ established in [Ber2] are applicable to the G-morphisms of G-spaces. In particular, if $\varphi : Y \to X$ is a G-morphism of dimension d between k-analytic G-spaces, then for any abelian torsion sheaf F on Y and any $q > 2d$ one has $R^q\varphi_!(F) = 0$.

1.5.4. Corollary *(Verdier Duality). Let $\varphi : Y \to X$ be a G-morphism of finite dimension between k-analytic G-spaces, and let $\mathcal{O}$ be a G-sheaf of torsion rings on $X(G)$. Then*

(i) there is an exact functor $R\varphi^! : D^+(X(G), \mathcal{O}) \to D^+(Y(G), \varphi^*\mathcal{O})$ and, for any $E^\cdot \in D^-(Y(G), \varphi^*\mathcal{O})$ and $F^\cdot \in D^+(X(G), \mathcal{O})$, a functorial isomorphism

$$R\varphi_*(\underline{\mathcal{H}om}(E^\cdot, R\varphi^! F^\cdot)) \widetilde{\to} \underline{\mathcal{H}om}(R\varphi_! E^\cdot, F^\cdot) \ ;$$

(ii) for any $F^\cdot \in D^+(X(G), \mathcal{O})$ there is a functorial isomorphism $b^*_Y(R\varphi^! F^\cdot) \widetilde{\to} R\varphi^!(b^*_X F)$. ∎

1.6. Comparison of étale and quasi-étale cohomology groups.

1.6.1. Theorem. *Let X be a G-space. Then*

(i) if X is paracompact, then the values of the high direct images of the functor $F \mapsto (b^*F)(X)$ on the category of quasi-étale (resp. étale) abelian sheaves on $X(G)$ are the cohomology groups $H^q(X, b^*F)$;

(ii) if X is compact, then for any quasi-étale (resp. étale) abelian sheaf F on $X(G)$ one has $H^q(X\{G\}, F) \widetilde{\rightarrow} H^q(X, b^*F)$ and, in particular, there is a spectral sequence

$$E_2^{p,q} = H^p(G, H^q(X, b^*F)) \Longrightarrow H^{p+q}(X(G), F) \ ;$$

(iii) for any étale abelian sheaf F on $X(G)$ there are canonical isomorphisms

$$H^q_{\text{ét}}(X(G), F) \widetilde{\rightarrow} H^q_{\text{qét}}(X(G), \mu^*F) \text{ and } H^q_{\text{ét}}(X\{G\}, F) \widetilde{\rightarrow} H^q_{\text{qét}}(X\{G\}, \mu^*F) \ .$$

Proof. First of all we remark that (ii) trivially follows (i). Furthermore, (i) for étale sheaves follows from Proposition 1.5.1 and [Ber3], Lemma 3.2(i). *We claim that if X is compact and F is a quasi-étale injective sheaf on $X(G)$, then $H^q(X, b^*F) = 0$ for $q \geq 1$.* Indeed, for this it suffices to verify that given a finite quasi-étale covering $\mathcal{U} = \{U_i \stackrel{f_i}{\rightarrow} X\}$ by compact U_i the Čech cohomology groups $\check{H}^q(\mathcal{U}, b^*F)$ are trivial for $q \geq 1$. For a sufficiently small open subgroup $N \subset G$ each f_i can be considered as an N-morphism, and therefore $\mathcal{U}$ gives rise to a quasi-étale covering $\mathcal{U}(N) = \{U_i(N) \rightarrow X(N)\}$ of $X(N)$. Since the pullback of F on $X(N)$ is also injective, then $\check{H}^q(\mathcal{U}(N), F) = 0$ for $q \geq 1$. It remains to note that $\check{H}^q(\mathcal{U}, b^*F)$ is an inductive limit of the latter groups taken over sufficiently small open subgroups N of G. Thus, (i) for compact X and (ii) are true for quasi-étale sheaves.

We now consider for an arbitrary X the commutative diagram of morphisms of sites

$$\begin{array}{ccc} X(G)_{\text{qét}} & \stackrel{\mu_G}{\longrightarrow} & X(G)_{\text{ét}} \\ \uparrow b_{\text{q}} & & \uparrow b \\ X_{\text{qét}} & \stackrel{\mu}{\longrightarrow} & X_{\text{ét}} \end{array}$$

We claim that for any quasi-étale abelian sheaf F on $X(G)$ there are a canonical isomorphisms $b^*(R^q\mu_{G*}F) \widetilde{\rightarrow} R^q\mu_*(b^*_{\text{q}}F)$, $q \geq 0$. Indeed, the stalk of $b^*(R^q\mu_{G*}F)$ at a geometric point $\overline{x}$ is

$$\lim_{\overrightarrow{U}} \lim_{\overrightarrow{N}} H^q(U(N), F) = \lim_{\overrightarrow{U}} H^q(U\{G_U\}, F) \ ,$$

where the limit is taken over all compact $U \in C(\overline{x}/X)$ such that the image of $\overline{x}$ in U is contained in the relative interior of U over X and over all open subgroups $N \subset G_U$. But we already know that

$H^q(U\{G_U\}, F) = H^q(U, b_{\rm q}^* F)$, and therefore the inductive limit is exactly the stalk of the sheaf $R^q\mu_*(b_{\rm q}^* F)$ at $\overline{x}$.

To prove (i), it remains to verify that if X is paracompact then for any quasi-étale injective sheaf F on $X(G)$ and any $q \ge 1$ one has $H^q_{\rm q\acute{e}t}(X, b_{\rm q}^F) = 0$. Consider the spectral sequence $E_2^{p,q} = H^p_{\rm \acute{e}t}(X, R^q\mu_*(b_{\rm q}^* F)) \Longrightarrow H^{p+q}_{\rm q\acute{e}t}(X, b_{\rm q}^* F)$. One has $R^q\mu_*(b_{\rm q}^* F) = b^*(R^q\mu_{G*}F) = 0$ for $q \ge 1$, and therefore $H^q_{\rm q\acute{e}t}(X, b_{\rm q}^F) = H^q_{\rm \acute{e}t}(X, b^*(\mu_{G*}F)) = 0$. Furthermore, since the étale sheaf $\mu_{G*}F$ is injective, then the latter group is zero.

To prove (iii), it suffices to verify that for any étale abelian sheaf F on $X(G)$ and any $q \ge 1$ one has $R^q\mu_{G*}(\mu_G^* F) = 0$. But $b^*(R^q\mu_{G*}\mu_G^* F) = R^q\mu_*(b_{\rm q}^*\mu_G^* F) = R^q\mu_*\mu^*(b^* F)$. The latter is zero, by [Ber3], Theorem 3.3(ii). ∎

Due to Theorem 1.6.1, in the notations of cohomology groups it is not necessary to specify the topology, quasi-étale or étale, with respect to which those groups are considered. Furthermore, given a morphism $\varphi : X'(G') \to X(G)$, we will use, for brevity, the notation $H^q(X'(G'), F)$ instead of $H^q(X'(G'), \varphi^* F)$.

1.6.2. Corollary. *Let $\varphi : Y \to X$ be a compact G-morphism of G-spaces that gives rise to the commutative diagram of morphisms of sites*

$$\begin{array}{ccc} Y(G)_{\rm \acute{e}t} & \stackrel{\varphi}{\longrightarrow} & X(G)_{\rm \acute{e}t} \\ \uparrow \mu_Y & & \uparrow \mu_X \\ Y(G)_{\rm q\acute{e}t} & \stackrel{\varphi_{\rm q}}{\longrightarrow} & X(G)_{\rm q\acute{e}t} \end{array}$$

Then for any étale (resp. étale abelian) sheaf F on $Y(G)$ there is a canonical isomorphism $\mu_X^*(\varphi_* F) \widetilde{\to} \varphi_{{\rm q}*}(\mu_Y^* F)$ (resp. $\mu_X^*(R^q\varphi_* F) \widetilde{\to} R^q\varphi_{{\rm q}*}(\mu_Y^* F)$, $q \ge 0$). ∎

1.7. Étale fundamental groups of a G-space. Recall ([Ber2], 6.3.4(iii); [deJ], 2.1) that an étale covering space of an analytic space X is a morphism $f : Y \to X$ with the property that every point $x \in X$ has an open neighborhood $\mathcal{U}$ such that $f^{-1}(\mathcal{U})$ is a disjoint union of analytic spaces finite étale over X. We will call such an Y an *étale covering space in the strong sense*, and we say that a morphism $f : Y \to X$ is an *étale covering space* if every connected component of Y is an étale covering space of X in the strong sense.

Let X be a G-space. We denote by $\underline{\rm Cov}_{X(G)}$ the category of morphisms of G-spaces $Y(G) \to X(G)$ that are étale covering spaces of $X(G)$. (It is clear that every G-connected component of such $Y(G)$ is an étale covering spaces in the strong sense.) For a geometric point $\overline{x}$ of $X(G)$, let $\Phi_{\overline{x}} : \underline{\rm Cov}_{X(G)} \to \mathcal{S}ets$ be the functor defined by

$$\Phi_{\overline{x}}(Y(G)) = \{\overline{y} : {\bf p}_{\overline{x}} \to Y(G) \big| f(\overline{y}) = \overline{x}\} \ .$$

1.7.1. Proposition. *Suppose that $X(G)$ is connected. Then for any pair of geometric points $\overline{x}$ and $\overline{x}'$ of $X(G)$ there exists an isomorphism of functors $\Phi_{\overline{x}} \widetilde{\to} \Phi_{\overline{x}'}$.*

Proof. Since any element $g \in G$ defines an isomorphism of functors $\Phi_{\overline{x}} \widetilde{\to} \Phi_{g(\overline{x})}$ and $X(G)$ is connected, we may assume that the images of $\overline{x}$ and $\overline{x}'$ are contained in one connected component of X. Furthermore, consider the morphism $b : X \to X(G)$. Then $\overline{x}$ and $\overline{x}'$ define geometric points $\overline{y}$ and $\overline{y}'$ of X with $b(\overline{y}) = \overline{x}$ and $b(\overline{y}') = \overline{x}'$. It follows from de Jong's Theorem ([deJ], 2.9) that there exists an isomorphism of functors $\Phi_{\overline{y}} \widetilde{\to} \Phi_{\overline{y}'}$. It gives rise to the required isomorphism of functors $\Phi_{\overline{x}} \widetilde{\to} \Phi_{\overline{x}'}$. ∎

The *étale fundamental group of $X(G)$ with base point $\overline{x}$* is the endomorphism group of the functor $\Phi_{\overline{x}}$, i.e., $\pi_1(X(G), \overline{x}) = \mathrm{Aut}(\Phi_{\overline{x}})$. For a pair $(Y(G), \overline{y})$ with $Y(G) \in \underline{\mathrm{Cov}}_{X(G)}$ and $\overline{y} \in \Phi_{\overline{x}}(Y(G))$, let $H(Y(G), \overline{y})$ denote the stabilizer of $\overline{y}$ in $\pi_1(X(G), \overline{x})$. The system of subgroups $H(Y(G), \overline{y})$ define a topological group structure on $\pi_1(X(G), \overline{x})$. It is easy to see (see [deJ], Lemma 2.7) that there is a topological isomorphism

$$\pi_1(X(G), \overline{x}) \widetilde{\to} \varprojlim \pi_1(X(G), \overline{x}) / H(Y(G), \overline{y}) \ .$$

In particular, the group $\pi_1(X(G), \overline{x})$ is Hausdorff and prodiscrete.

References

[Ber1] Berkovich, V. G.: *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, American Mathematical Society, Providence, R.I., 1990.

[Ber2] Berkovich, V. G.: *Étale cohomology for non-Archimedean analytic spaces*, Publ. Math. IHES **78** (1993), 5-161.

[Ber3] Berkovich, V. G.: *Vanishing cycles for formal schemes*, Invent. Math. **115** (1994), 539-571.

[Ber4] Berkovich, V. G.: *Vanishing cycles for formal schemes. II*, Invent. Math. **125** (1996), 367-390.

[Ber5] Berkovich, V. G.: *On the comparison theorem for étale cohomology of non-Archimedean analytic spaces*, Israel J. Math. **92** (1995), 45-60.

[deJ] de Jong, A. J.: *Étale fundamental group of non-Archimedean analytic spaces*, Compositio Math. **97** (1995), 89-118.