Appendix

A result on vanishing cycles

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Let k be a complete discrete valuation field whose residue field $\widetilde{k}$ is algebraically closed, and let l be a prime integer different from $\mathrm{char}(\widetilde{k})$. For a special formal scheme $\mathfrak{X}$ over k° (see [Berk3]), we denote by $\mathfrak{X}(n)$ the scheme $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}}/\mathcal{I}^{n+1})$, where $\mathcal{I}$ is the maximal ideal of definition of $\mathfrak{X}$. We also denote by $\Psi^q_m(\mathfrak{X})$ the vanishing cycles sheaves $R^q\Psi_\eta(\mathbf{Z}/l^m\mathbf{Z})_{\mathfrak{X}_\eta}$ of $\mathfrak{X}$. These are étale sheaves on the closed fibre $\mathfrak{X}_s = \mathfrak{X}(0)$ of $\mathfrak{X}$.

Let $\mathfrak{X}$ be a quasi-compact special formal scheme over k° locally isomorphic to a formal scheme of the form $\widehat{\mathcal{X}}_{/\mathcal{Y}}$, where $\mathcal{X}$ is a scheme of finite type over k° whose generic fibre $\mathcal{X}_\eta$ is smooth over k and $\mathcal{Y}$ is a subscheme of the closed fibre of $\mathcal{X}$. Assume that $\mathfrak{X}$ is endowed with a continuous action of a profinite group G such that the induced action on $\mathfrak{X}_s$ is trivial. We call it the μ-action of G on $\mathfrak{X}$. Let $\{G_n\}_{n\geq 0}$ be a projective system of finite quotients of G such that G acts on $\mathfrak{X}(n)$ via G_n and $G_0 = \{1\}$.

Furthermore, assume we are given a projective system $\{\mathcal{X}_n\}_{n\geq 0}$ of finite étale Galois coverings of the closed fibre $\mathfrak{X}_s$ with the Galois group G such that the Galois group of $\mathcal{X}_n$ over $\mathfrak{X}_s$ is precisely G_n. By [Berk3], Proposition 2.1, it comes from a projective system $\{\mathfrak{X}_n\}_{n\geq 0}$ of finite étale Galois coverings of $\mathfrak{X}$. We call the corresponding action of G_n on $\mathfrak{X}_n$ the ν-action. Since the μ-action of G on $\mathfrak{X}$ induces the trivial action on $\mathfrak{X}_s$, it extends in a unique way to an action on each $\mathfrak{X}_n$ that induces the trivial action on $\mathfrak{X}_{n,s}$. Thus, for each $n \geq 0$, there are two actions of the finite group G_n on the scheme $\mathcal{X}_n(n)$, the one induced by μ and that induced by ν, and therefore one can consider the diagonal action of G_n on $\mathfrak{X}_n(n)$.

We assume that, for each $n \geq 0$, the quotient scheme$\mathfrak{Y}^{(n)}$ of $\mathfrak{X}_n(n)$ by the diagonal action of G_n exists. (For example, it is always true if $\mathfrak{X}_n(n)$ is quasi-projective over k°.) In this case $\{\mathfrak{Y}^{(n)}\}$ is an inductive system of schemes. We also assume that there is a special formal scheme $\mathfrak{Y}$ over k° such that $\mathfrak{Y}(n) = \mathfrak{Y}^{(n)}$ for all $n \geq 0$. Our aim is to describe the vanishing cycles sheaves $\Psi^q_m(\mathfrak{Y})$ of $\mathfrak{Y}$ in terms of the vanishing cycles sheaves $\Psi^q_m(\mathfrak{X})$ of $\mathfrak{X}$. (Notice that the both sheaves are defined on the same scheme $\mathfrak{X}_s = \mathfrak{Y}_s$.)

Recall that the comparison theorem 3.1 from [Berk3] implies that the sheaves $\Psi^q_m(\mathfrak{X})$ are constructible (and equal to zero for $q > \dim(\mathfrak{X}_\eta)$). Furthermore, it follows from [Berk3], Corollary 4.5, that there exists $N \geq 0$ such that any automorphism of $\mathfrak{X}$ trivial on $\mathfrak{X}(N)$ acts trivially on all the sheaves $\Psi^q_m(\mathfrak{X})$ and, in particular, the action of G on the sheaves $\Psi^q_m(\mathfrak{X})$ factors through an action of the finite quotient G_N. Let $n \geq N$. The ν-action of the group G_n on $\mathfrak{X}_n$ induces an action on the sheaves $\Psi^q_m(\mathfrak{X}_n) = \Psi^q_m(\mathfrak{X})\big|_{\mathfrak{X}_{n,s}}$. On the other hand, consider the diagonal action of the group G on $\mathfrak{X}_n$. By *loc. cit.*, the induces action of G on $\Psi^q_m(\mathfrak{X}_n)$ factors through the quotient G_n. We call it the diagonal action of G_n on $\Psi^q_m(\mathfrak{X}_n)$. Since it is compatible with the ν-action of G_n on $\mathfrak{X}_{n,s}$, $\Psi^q_m(\mathfrak{X}_n)$ is the pullback of a unique sheaf Ψ^q_m on $\mathfrak{X}_s = \mathfrak{Y}_s$. Notice that the construction of the sheaf Ψ^q_m does not depend on the choice of the number $n \geq N$.

Theorem. *Assume that the family of formal completions of $\mathfrak{X}$ along a closed point of $\mathfrak{X}_s$ has a finite number of isomorphism classes. Then there is a canonical system of compatible isomorphisms of sheaves $\Psi^q_m \tilde{\to} \Psi^q_m(\mathfrak{Y})$, $m \geq 0$, $q \geq 0$.*

Lemma. *Let $\mathfrak{X}$ and $\mathfrak{Y}$ be special affine formal schemes over k°, and assume that $\mathfrak{X}$ is isomorphic to the formal completion of an affine scheme of finite type over k°, whose generic fibre is smooth over k, along a subscheme of the closed fibre. Furthermore, assume we are given projective systems $\{\mathfrak{X}_n\}_{n\geq 0}$ and $\{\mathfrak{Y}_n\}_{n\geq 0}$ of finite étale coverings of $\mathfrak{X}$ and $\mathfrak{Y}$, respectively, and a compatible system of isomorphisms $\varphi_n : \mathfrak{Y}_n(n) \tilde{\to} \mathfrak{X}_n(n)$. Then for every $r \geq 1$ there exists $n_0 \geq r$ such that for any $n \geq n_0$ there exists an isomorphism of formal schemes $\psi : \mathfrak{Y}_n \tilde{\to} \mathfrak{X}_n$ whose restriction to $\mathfrak{Y}_n(r)$ coincides with that of φ_n.*

Proof. We may assume that all $\mathfrak{X}_n$ and $\mathfrak{Y}_n$ are connected. Let $\mathfrak{X} = \mathrm{Spf}(A)$ and $\mathfrak{Y} = \mathrm{Spf}(B)$, and let I and J be the maximal ideals of definition of A and B, respectively. By the first assumption, A is the $\mathbf{a}$-adic completion of a finitely generated algebra A' over k° with $A' \otimes_{k^\circ} k$ smooth over k, where $\mathbf{a}$ is an ideal of A' with $\mathbf{a}A = I$. By the second assumption, there are inductive systems $\{A_n\}_{n\geq 0}$ and $\{B_n\}_{n\geq 0}$ of finite étale algebras over A and B, respectively, and a compatible system of isomorphisms $\alpha_n : A_n/I^{n+1}A_n \tilde{\to} B_n/J^{n+1}B_n$. We set $B_\infty = \varinjlim B_n$ and $J_\infty = JB_\infty$. Since all of the pairs (B_n, JB_n) are Henselian, the pair (B_∞, J_∞) is also Henselian and, since B_∞ is faithfully flat over each B_n, one has $J_\infty \cap B_n = JB_n$. The above isomorphisms induce compatible systems of homomorphisms $f_n : A' \to B_n/J^{n+1}B_n$ and $f'_n : A' \to B_\infty/J_\infty^{n+1}$. By R. Elkik's result (*Solutions d'équations à coefficients dans un anneau hensélien*, Ann. Scient. Éc.Norm. Sup. **6** (1973), 553-604: Corollary 1 on p. 567 and Remark 2 on p. 587), there exist integers $t \geq 0$ and $m \geq t$ such that for any $n \geq m$ there exists a homomorphism $g_n : A' \to B_\infty$ which is congruent to f'_{n-t} modulo J_∞^{n-t+1}.

Suppose now we are given an integer $r \geq 1$. We may increase it and assume that $r + t \geq m$. The image of the homomorphism g_{r+t} is contained in some B_{n_0}. Increasing n_0, if necessary, we may assume that $n_0 \geq r + t$, and *we claim that it satisfies the property of the lemma*. Indeed, let $n \geq n_0$. By the construction, the homomorphism g_{r+t} induces a homomorphism $h_n : A' \to B_n$ congruent to f_n modulo $J^{r+1}B_n$. The latter implies that h_n uniquely extends to a continuous homomorphism $A \to B_n$ which, in its turn, extends uniquely to the finite étale A-algebra A_n, i.e., to a continuous homomorphism $\beta_n : A_n \to B_n$ congruent to α_n modulo $J^{r+1}B_n$. Since α_n is an isomorphism, it follows that β_n is an isomorphism, and the lemma follows. ∎

Proof of Theorem. By the assumption and Corollary 4.5 from [Berk3], we can find an integer $r \geq 1$ such that, for every closed point $\mathbf{x} \in \mathfrak{X}_s$, each automorphism of $\widehat{\mathfrak{X}}_{/\mathbf{x}}$ trivial on $\widehat{\mathfrak{X}}_{/\mathbf{x}}(r)$ acts trivially on $\Psi^q_m(\widehat{\mathfrak{X}}_{/\mathbf{x}})$. Notice that the comparison theorem 3.1 from [Berk3] implies that $\Psi^q_m(\widehat{\mathfrak{X}}_{/\mathbf{x}}) = \Psi^q_m(\mathfrak{X})_\mathbf{x}$. In particular, since all of the sheaves $\Psi^q_m(\mathfrak{X})$ are constructible, the number N defined before the formulation of the theorem can be taken equal to r. Furthermore, fix a finite covering $\{\mathfrak{X}^i\}$ of $\mathfrak{X}$ by open affine formal subschemes satisfying the first assumption of the lemma. It gives rise to a finite affine covering $\{\mathfrak{Y}^i\}$ of $\mathfrak{Y}$. Let n be the maximum of the numbers n_0 from the lemma taken for r and all of the pairs $(\mathfrak{X}^i, \mathfrak{Y}^i)$, and let ψ^i denote the isomorphisms $\mathfrak{Y}^i_n \tilde{\to} \mathfrak{X}^i_n$ constructed in the lemma. The latter isomorphisms induce isomorphisms of sheaves $\alpha^i = (\psi^i)^* : \Psi^q_m(\mathfrak{X}^i_n) \tilde{\to} \Psi^q_m(\mathfrak{Y}^i_n)$. Notice that $\mathfrak{Y}^i_n$ is an étale Galois covering of $\mathfrak{Y}^i$ with the Galois group G_n and, in particular, there is an action of G_n on the sheaves $\Psi^q_m(\mathfrak{Y}_n)$. To prove the theorem, it suffices to check the following two facts:

(1) the action of G_n on the sheaves $\Psi^q_m(\mathfrak{X}^i_n)$, which is induced via α^i by the canonical action of G_n on $\Psi^q_m(\mathfrak{Y}^i_n)$, coincides with the diagonal action;

(2) the restrictions of α^i and α^j to the intersection $\mathfrak{X}^i_{n,s} \cap \mathfrak{X}^j_{n,s}$ coincide.

(1) In the proof of (1) we can replace $\mathfrak{X}$ by $\mathfrak{X}^i$, and so we may omit the superscript i. For an element $g \in G$, let $\gamma(g)$ denote the automorphism of $\mathfrak{X}_n$ which is induced via $\psi : \mathfrak{Y}_n \tilde{\to} \mathfrak{X}_n$ by the action of g on $\mathfrak{Y}_n$. Since all of the sheaves considered are constructible, it suffices to verify the commutativity of the following diagram for every point $\mathbf{z} \in \mathfrak{X}_{n,s}$

$$\begin{array}{ccc} \Psi^q_m(\mathfrak{X}_{/\mathbf{x}}) & \tilde{\to} & \Psi^q_m(\mathfrak{X}_{n/g\mathbf{z}}) \\ \downarrow \mu(g)^* & & \downarrow \gamma(g)^* \\ \Psi^q_m(\mathfrak{X}_{/\mathbf{x}}) & \tilde{\to} & \Psi^q_m(\mathfrak{X}_{n/\mathbf{z}}) \end{array}$$

where $\mathbf{x}$ is the image of $\mathbf{z}$ in $\mathfrak{X}_s$ and the horizontal arrows are induced by the canonical isomorphisms $\mathfrak{X}_{n/\mathbf{z}} \tilde{\to} \mathfrak{X}_{/\mathbf{x}}$ and $\mathfrak{X}_{n/g\mathbf{z}} \tilde{\to} \mathfrak{X}_{/\mathbf{x}}$. But the construction of ψ gives the commutative diagram

$$\begin{array}{ccc} \mathfrak{X}_{n/\mathbf{z}}(r) & \tilde{\to} & \mathfrak{X}_{/\mathbf{x}}(r) \\ \downarrow \gamma(g) & & \downarrow \mu(g) \\ \mathfrak{X}_{n/g\mathbf{z}}(r) & \tilde{\to} & \mathfrak{X}_{/\mathbf{x}}(r) \end{array}$$

and, therefore, the required fact is true by our choice of r.

(2) Let $\mathbf{z} \in \mathfrak{X}^i_{n,s} \cap \mathfrak{X}^j_{n,s}$. The automorphism $\psi^j \circ (\psi^i)^{-1}$ of $\mathfrak{X}_{n/\mathbf{z}}$ gives rise to an automorphism of $\mathfrak{X}_{/\mathbf{x}}$ whose restriction to $\mathfrak{X}_{/\mathbf{x}}(r)$ is trivial, where $\mathbf{x}$ is the image of $\mathbf{z}$ in $\mathbf{X}_s$. It follows that the automorphism $(\alpha^i)^{-1} \circ \alpha^j$ of $\Psi^q_m(\mathfrak{X}_{n/\mathbf{z}})$ is trivial, and the required fact follows. ∎

Remark. It is very likely that the assumption on $\mathfrak{X}$ from the formulation of the theorem is true for any quasi-compact special formal scheme.

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[Berk3] Berkovich, V. G.: *Vanishing cycles for formal schemes. II*, Invent. Math. **125** (1996), 367-390.