VANISHING CYCLES
FOR NON-ARCHIMEDEAN ANALYTIC SPACES

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INTRODUCTION

In this work we develop a formalism of vanishing cycles for non-Archimedean analytic spaces which is an analog of that for complex analytic spaces from [SGA7], Exp. XIV. As an application we prove that in the equicharacteristic case the stalks of the vanishing cycles sheaves of a scheme $\mathcal{X}$ at a closed point $\mathbf{x} \in \mathcal{X}_s$ depend only on the formal completion $\mathrm{Spf}(\widehat{\mathcal{O}}_{\mathcal{X},\mathbf{x}})$ of $\mathcal{X}$ at $\mathbf{x}$. In particular, any continuous homomorphism $\widehat{\mathcal{O}}_{\mathcal{X},\mathbf{x}} \to \widehat{\mathcal{O}}_{\mathcal{Y},\mathbf{y}}$ induces a homorphism from the stalks of the vanishing cycles sheaves of $\mathcal{X}$ at $\mathbf{x}$ to those of $\mathcal{Y}$ at $\mathbf{y}$. Furthermore, we prove that, given $\widehat{\mathcal{O}}_{\mathcal{X},\mathbf{x}}$ and $\widehat{\mathcal{O}}_{\mathcal{Y},\mathbf{y}}$, there exists $n \geq 1$ such that, for any pair of continuous homomorphisms $\widehat{\mathcal{O}}_{\mathcal{X},\mathbf{x}} \to \widehat{\mathcal{O}}_{\mathcal{Y},\mathbf{y}}$ that coincide modulo the n-th power of the maximal ideal of $\widehat{\mathcal{O}}_{\mathcal{Y},\mathbf{y}}$, the induced homomorphisms between the stalks of the vanishing cycles sheaves coincide. These facts generalize a result of G. Laumon from [Lau] (see Remark 7.6).

Throughout the paper we fix a non-Archimedean field k (whose valuation is not assumed to be nontrivial). In §1 we study étale Galois sheaves on k-analytic spaces. To define the vanishing cycles functor and to work with it, we use the language of pro-analytic spaces, i.e., pro-objects of the category of analytic spaces ([SGA4], Exp. I). Examples of such objects are the germs of analytic spaces as in [Ber2], §3.4. Another example is considered in §3. In §4 we define the vanishing cycles functor and establish its basic properties. In §5 we show that the vanishing cycles sheaves are trivial for smooth morphisms. In §6 we prove a comparison theorem for vanishing cycles. This theorem is more general than its analog over $\mathbf{C}$ from [SGA7], Exp. XIV, and its proof does not use Hironaka's theorem on resolution of singularities. In §7 we apply the comparison theorem to prove the properties of the vanishing cycles sheaves of schemes formulated above. It is worthwhile to note that this application is obtained by considering non-Archimedean analytic geometry over fields with trivial valuation.

Like [Ber3], this work arose from a suggestion of P. Deligne to apply the étale cohomology theory from [Ber2] to the study of the vanishing cycles sheaves of schemes. I am very grateful to him for useful discussions on the subject. I also

gratefully appreciate the hospitality and support of the Institute for Advanced Study, where a part of this work was done.

§1. Galois sheaves

For a k-analytic space X and an algebraic extension K over k we set $X_K = X \widehat{\otimes} \widehat{K}$ and denote by $\pi = \pi_X$ the canonical morphism $X_K \to X$.

1.1. Lemma. *Any K-analytic space V étale over X_K admits an étale covering $\{U^i_K\}_{i \in I}$ for some U^i étale over X.*

Proof. We have to verify that each point $v \in V$ has an étale neighborhood U_K for some U étale over X. Let x' be the image of v in X_K and set $x = \pi(x')$. Then the field $\mathcal{H}(v)$ is a finite separable extension of $\mathcal{H}(x')$. Since the compositum $\mathcal{H}(x)K$ is everywhere dense in $\mathcal{H}(x')$, it follows that $\mathcal{H}(v) = \mathcal{H}(x')L$ for some finite separable extension L of $\mathcal{H}(x)$. By Theorem 3.4.1 from [Ber2], the field L gives rise to an étale morphism of k-germs $(U, u) \to (X, x)$ with $\mathcal{H}(u) = L$. By construction, there exists a point $u' \in U_K$ with $\mathcal{H}(u') = \mathcal{H}(x')L = \mathcal{H}(v)$, and therefore, by the same theorem, there is an isomorphism of K-germs $(U_K, u') \xrightarrow{\sim} (V, v)$ over (X_K, x'). Thus, replacing X by U, we may assume that V is an open neighborhood of x' in X_K. It suffices to know that there exist a finite separable extension k' of k in K and an open neighborhood U of the image of x' in $X_{k'}$ such that the preimage of U in X_K is contained in V. But this is established in the proof of Lemma 5.3.4 from [Ber2]. □

Suppose that K is a normal extension of k. Then the Galois group $G(K/k)$ acts on X_K (considered as an analytic space over k). Let $\nu : G \to G(K/k)$ be a continuous homomorphism from a profinite group G to $G(K/k)$. The group G acts on X_K via ν. An action of G on a sheaf of sets F on X_K, *compatible* with the action of G on X_K, is a system of isomorphisms $\sigma(g) : \nu(g)_* F \to F$, $g \in G$, such that $\sigma(gh) = \sigma(g) \circ \nu(g)_*(\sigma(h))$. If G acts on F, then for any U étale over X the group G acts on the set $F(U_K)$.

1.2. Lemma. *The following properties of an action of G on F are equivalent:*

(a) *for any $f \in F(U_K)$, where U is étale over X, there exists an open covering $\{\mathcal{U}^i\}_{i \in I}$ of U such that for any $i \in I$ the stabilizer of $f|_{\mathcal{U}^i_K}$ is open in G;*

(b) *for any $f \in F(U_K)$, where U is étale over X, there exists an étale covering $\{U^i\}_{i \in I}$ of U such that for any $i \in I$ the stabilizer of $f|_{U^i_K}$ is open in G.*

Proof. Let $f \in F(U_K)$, where U is étale over X, and suppose that there is an étale covering $\{U_i \xrightarrow{\varphi_i} U\}_{i \in I}$ such that for any $i \in I$ the stabilizer of $f|_{U^i_K}$ is open in G. Since étale morphisms are quasifinite (see [Ber2], §3), we can replace the covering by a refinement so that all of the induced morphisms $U^i \to \mathcal{U}^i = \varphi_i(U^i)$ are finite. In this case the morphisms $U^i_K \to \mathcal{U}^i_K$ are finite and étale, and therefore the maps $F(\mathcal{U}^i_K) \to F(U^i_K)$ are injective. We get an open covering $\{\mathcal{U}^i\}_{i \in I}$ of U such that for any $i \in I$ the stabilizer of $f|_{\mathcal{U}^i_K}$ is open in G. □

An action of G on F, which possesses the equivalent properties of Lemma 1.2, is said to be *continuous*. Let $\mathbf{T}_G(X_K)$ (resp. $\mathbf{S}_G(X_K)$) denote the category of sheaves of sets (resp. abelian groups) endowed with a continuous action of G (*G-sheaves*). The category $\mathbf{S}_G(X_K)$ has injectives. Its derived category will be

denoted by $D_G(X_K)$. If I is a subgroup of G which is contained in the kernel of $\nu : G \to G(K/k)$, then ν induces a continuous homomorphism $\nu' : G/I \to G(K/k)$, and one has a left exact functor $\mathbf{S}_G(X_K) \to \mathbf{S}_{G/I}(X_K) : F \to F^I$. The values of its right derived functors are denoted by $\mathcal{H}^q(I, F)$. For $x' \in X_K$ one has $\mathcal{H}^q(I, F)_{x'} = H^q(I, F_{x'})$.

Suppose that k' is another non-Archimedean field, K' is a normal extension of k', and $\nu' : G' \to G(K'/k')$ is a continuous homomorphism of profinite groups. Furthermore, suppose we are given a commutative diagram of isometric embeddings

$$
\begin{array}{ccc}
K & \hookrightarrow & K' \\
\uparrow & & \uparrow \\
k & \hookrightarrow & k'
\end{array}
$$

and a commutative diagram of continuous homomorphisms

$$
\begin{array}{ccc}
G & \xrightarrow{\nu} & G(K/k) \\
\uparrow & & \uparrow{\scriptstyle\mu} \\
G' & \xrightarrow{\nu'} & G(K'/k')
\end{array}
$$

where μ is induced by the above embeddings. Finally, let X' be a k'-analytic space, and let $\varphi : X' \to X$ be a morphism over the embedding $k \hookrightarrow k'$. It induces a morphism $\overline{\varphi} : X'_{K'} \to X_K$ over the embedding $\widehat{K} \hookrightarrow \widehat{K'}$.

1.3. Lemma. *The inverse image functor for the morphism $\overline{\varphi}$ induces a well defined functor $\overline{\varphi}^* : \mathbf{T}_G(X_K) \to \mathbf{T}_{G'}(X'_{K'})$.*

Proof. Let $F \in \mathbf{T}_G(X_K)$ and $f \in (\overline{\varphi}^* F)(V_{K'})$, where V is étale over X'. Then there is an étale covering $\{W^i \to V_{K'}\}_{i \in I}$ and, for each $i \in I$, a commutative diagram

$$
\begin{array}{ccc}
V_{K'} & \longrightarrow & X_K \\
\uparrow & & \uparrow \\
W^i & \longrightarrow & U^i
\end{array}
$$

where U^i is étale over X_K, such that $f\big|_{W^i}$ is the image of some element $g_i \in F(U^i)$. By Lemma 1.1, we can replace the covering $\{W^i\}_{i \in I}$ by a refinement and assume that $W^i = V^i_{K'}$ for some étale covering $\{V^i\}_{i \in I}$ of V. Furthermore, by the same lemma, we can find for each $i \in I$ an étale covering $\{U^{ij}_K\}_{j \in J_i}$ of U^i, where U^{ij} are étale over X. Since the action of G on $\mathcal{G}$ is continuous, we can replace the latter covering by a refinement and assume that the stabilizer of $g_i\big|_{U^{ij}_K}$ is open in G. We get an étale covering $\{V^{ij} = V^i \times_X U^{ij}\}_{i,j}$ of V such that $f\big|_{V^{ij}_{K'}}$ is the image of $g_i\big|_{U^{ij}_K}$. It follows that the stabilizer of $f\big|_{V^{ij}_{K'}}$ is open in G'. $\qquad\square$

From Lemma 1.3 it follows that the inverse image functor for the morphism $\pi : X_K \to X$ induces a well defined functor $\pi^* : \mathbf{T}(X) \to \mathbf{T}_G(X_K)$. In the case $G \xrightarrow{\sim} G(K/k)$ one can easily show that there is an equivalence of categories $\pi^* : \mathbf{T}(X) \xrightarrow{\sim} \mathbf{T}_{G(K/k)}(X_K)$.

§2. Pro-analytic spaces

Recall that the category of pro-objects of a category C, $\mathrm{Pro}(C)$, is defined as follows (see [SGA4], Exp. I). Its objects are functors $I \to C : i \mapsto X_i$, where I is

a small cofiltered category. Such an object is denoted by $"\varprojlim_I" X_i$. Morphisms are defined by

$$\mathrm{Hom}("\varprojlim_J "Y_j, "\varprojlim_I "X_i) = \varprojlim_I \varinjlim_{J^\circ} \mathrm{Hom}(Y_j, X_i).$$

The category $\mathrm{Pro}(C)$ admits cofiltered projective limits, and if C admits fiber products, then so is $\mathrm{Pro}(C)$. The canonical fully faithful functor $L : C \to \mathrm{Pro}(C)$ commutes with fiber products, but does not commute, in general, with cofiltered projective limits. One has $"\varprojlim_I" X_i = \varprojlim_I L(X_i)$.

A *pro-k-analytic space* is an object of the category $\mathrm{Pro}(k\text{-}\mathcal{An})$. The category $\mathrm{Pro}(k\text{-}\mathcal{An})$ admits fiber products and cofiltered projective limits, and for a non-Archimedean field K over k there is the ground field extension functor $\mathrm{Pro}(k\text{-}\mathcal{An}) \to \mathrm{Pro}(K\text{-}\mathcal{An}) : \mathbf{X} = "\varprojlim_I" X_i \mapsto \mathbf{X}\widehat{\otimes}K = "\varprojlim_I"(X_i\widehat{\otimes}K)$. A *pro-analytic space over k* is a pair $(K, \mathbf{X})$, where K is a non-Archimedean field over k and $\mathbf{X} \in \mathrm{Pro}(K\text{-}\mathcal{An})$. A morphism $(L, \mathbf{Y}) \to (K, \mathbf{X})$ is a pair consisting of an isometric embedding $K \hookrightarrow L$ and a morphism $\mathbf{Y} \to \mathbf{X}\widehat{\otimes}_K L$.

Let $\mathbf{X} = "\varprojlim_I" X_i$ be a pro-k-analytic space. It gives rise to a pro-object of the category of locally ringed spaces. Since the latter category admits cofiltered projective limits, we get the *underlying locally ringed space* $|\mathbf{X}|$ of $\mathbf{X}$. (We remark that the space $|\mathbf{X}|$ may be empty even when $\mathbf{X}$ is nontrivial.) If $x \in \mathbf{X}$ (i.e., $x \in |\mathbf{X}|$), then $\mathcal{O}_{\mathbf{X},x} = \varinjlim_{I^\circ} \mathcal{O}_{X_i,x_i}$, where x_i is the image of x in X_i. The residue field of the local ring $\mathcal{O}_{\mathbf{X},x}$ is denoted by $\kappa(x)$. Furthermore, let $\mathcal{H}(x)$ denote the completion of the field $\varinjlim \mathcal{H}(x_i)$. If, for each i, the point x_i has an affinoid neighborhood in X_i, then the field $\kappa(x)$ is quasicomplete and its completion coincides with $\mathcal{H}(x)$.

For a pro-k-analytic space $\mathbf{X} = "\varprojlim_I" X_i$ we define the category of "étale sheaves of sets" $\mathbf{T}(\mathbf{X})$ as the inductive limit $\varinjlim_{I^\circ} \mathbf{T}(X_i)$ (see [SGA4], Exp. VI). Namely, objects of $\mathbf{T}(\mathbf{X})$ are pairs (i, F), where $i \in I$ and $F \in \mathbf{T}(X_i)$. A representative of a morphism $(i, F) \to (j, G)$ is a triple (α, β, u), where $\alpha : l \to i$ and $\beta : l \to j$ are arrows in I and $u : \nu_\alpha^*(F) \to \nu_\beta^*(G)$ is a morphism of sheaves on X_l (ν_α is the morphism $X_l \to X_i$ that corresponds to α). Two representatives (α, β, u) and (α', β', u') of a morphism $(i, F) \to (j, G)$ are said to be equivalent if there exist arrows $\gamma : q \to l$ and $\gamma' : q \to l'$ such that $\alpha \circ \gamma = \alpha' \circ \gamma'$, $\beta \circ \gamma = \beta' \circ \gamma'$ and $\gamma^*(u) = \gamma'^*(u')$. A morphism is an equivalence class of representatives. We remark that if $\mathbf{X} = \varprojlim_I \mathbf{X}_i$ is a cofiltered projective limit in the category $\mathrm{Pro}(k\text{-}\mathcal{An})$, then there is an equivalence of categories $\varinjlim_{I^\circ} \mathbf{T}(\mathbf{X}_i) \xrightarrow{\sim} \mathbf{T}(\mathbf{X})$.

One also has the abelian categories of "abelian sheaves" $\mathbf{S}(\mathbf{X})$ and of "sheaves of Λ-modules" $\mathbf{S}(\mathbf{X}, \Lambda)$. There is a left exact functor $\mathbf{S}(\mathbf{X}) \to \mathcal{Ab} : F \mapsto F(\mathbf{X}) = \varinjlim_{I^\circ} F(X_i)$. Suppose that all of the morphisms $\nu_\alpha : X_i \to X_j$ are étale. Then the functors $\nu_\alpha^* : \mathbf{S}(X_j) \to \mathbf{S}(X_i)$ take injectives into injectives, and therefore the category $\mathbf{S}(\mathbf{X})$ has injectives. The values of the right derived functors of the functor $F \mapsto F(\mathbf{X})$ are $H^q(\mathbf{X}, F) = \varinjlim_{I^\circ} H^q(X_i, F)$.

Let $\varphi : \mathbf{Y} = "\varprojlim_{J}" Y_j \to \mathbf{X} = "\varprojlim_{I}" X_i$ be a morphism of pro-analytic spaces over k. Then one can define the inverse image functor $\varphi^* : \mathbf{T}(\mathbf{X}) \to \mathbf{T}(\mathbf{Y})$ and, in a situation we really need, the direct image functor $\varphi_* : \mathbf{T}(\mathbf{Y}) \to \mathbf{T}(\mathbf{X})$ as follows. If for each $i \in I$ we fix $\sigma(i) \in J$ and a morphism $\varphi_i : Y_{\sigma(i)} \to X_i$ determined by φ, then $\varphi^*(i, F) = (\sigma(i), \varphi_i^*(F))$. Furthermore, suppose that there are a full cofinal subcategory $I' \subset I$, a cofinal functor $\sigma : I' \to J$, and a system of morphisms $\{\varphi_i : Y_{\sigma(i)} \to X_i\}_{i \in I'}$ which defines φ and such that for any arrow $\alpha : l \to i$ in I' the diagram

$$
\begin{array}{ccc}
Y_{\sigma(i)} & \xrightarrow{\varphi_i} & X_i \\
\uparrow{\nu_\alpha} & & \uparrow{\nu_\alpha} \\
Y_{\sigma(l)} & \xrightarrow{\varphi_l} & X_l
\end{array}
$$

is commutative and cartesian. Suppose also that all of the morphisms $\nu_\alpha : X_l \to X_i$ for arrows α in I are étale. Then for any $F \in \mathbf{T}(Y_{\sigma(i)})$ there is a canonical isomorphism $\nu_\alpha^* \varphi_{i*}(F) \xrightarrow{\sim} \varphi_{l*} \nu_\alpha^*(F)$, and therefore the correspondence $(\sigma(i), F) \mapsto (i, \varphi_{i*}(F))$, $i \in I'$, gives the required functor φ_* which is right adjoint to φ^*. In this situation the category $\mathbf{S}(\mathbf{Y})$ has injectives, and there are the right derived functors $R^q \varphi_* : \mathbf{S}(\mathbf{Y}) \to \mathbf{S}(\mathbf{X})$.

For a k-analytic space X we denote by $X\text{-}\mathcal{A}n$ the category of morphisms of k-analytic spaces $Y \to X$. Such an Y is said to be an X-*analytic space*. If $\mathbf{X} = "\varprojlim_{I}" X_i$ is a pro-k-analytic space, then an $\mathbf{X}$-*analytic space* is an object of the category $\mathbf{X}\text{-}\mathcal{A}n := \varinjlim_{I^\circ} X_i\text{-}\mathcal{A}n$. If P is a class of morphisms between k-analytic spaces which is preserved under any base change, then one can extend in the evident way the class P to morphisms between $\mathbf{X}$-analytic spaces. Furthermore, if all of the morphisms $\nu_\alpha : X_i \to X_j$ are étale, then for any morphism of $\mathbf{X}$-analytic spaces $\varphi : \mathbf{Z} \to \mathbf{Y}$ the direct image functor $\varphi_* : \mathbf{S}(\mathbf{Z}) \to \mathbf{S}(\mathbf{Y})$ as well as the right derived functors $R^q \varphi_*$ are well defined.

Germs of analytic spaces (see [Ber2], §3.4) are examples of pro-analytic spaces, namely, there is a fully faithful functor

$$
k\text{-}Germs \to \mathrm{Pro}(k\text{-}\mathcal{A}n) : (X, \Sigma) \mapsto X(\Sigma) = "\varprojlim_{\mathcal{U} \supset \Sigma}" \mathcal{U},
$$

where $\mathcal{U}$ runs through open neighborhoods of Σ. The functor commutes with extensions of the ground field, but does not commute with fiber products. For example, let $\varphi : Y \to X$ be a morphism of k-analytic spaces and $x \in X$. Then the fiber product $Y \times_X (X, x)$ in the category $k\text{-}Germs$ is the k-germ $(Y, \varphi^{-1}(x))$, i.e., it gives rise to $Y(\varphi^{-1}(x)) = "\varprojlim" \mathcal{V}$, where $\mathcal{V}$ runs through *all* open neighborhoods of the fiber $\varphi^{-1}(x)$. The corresponding fiber product $Y(x) := Y \times_X X(x)$ in the category $\mathrm{Pro}(k\text{-}\mathcal{A}n)$ is $"\varprojlim" \varphi^{-1}(\mathcal{U})$, where $\mathcal{U}$ runs through open neighborhoods of the point x. We remark that the canonical morphism $Y(\varphi^{-1}(x)) \to Y(x)$ induces an isomorphism between the underlying locally ringed spaces, and there is a morphism $Y_x \to Y(\varphi^{-1}(x))$ which induces a homeomorphism between the underlying topological spaces. The space $Y(x)$ is an example of an $X(x)$-analytic space. And in fact any $X(x)$-analytic space $\mathbf{Y}$ is isomorphic to $Y(x)$ for some $Y \to X$. The fiber of $\mathbf{Y}$ over x is, by definition, the $\mathcal{H}(x)$-analytic space $\mathbf{Y}_x := Y_x$.

For a k-germ (X, Σ) there is an exact functor $\mathbf{T}(X(\Sigma)) \to \mathbf{T}(X, \Sigma) : F \mapsto F_{(X,\Sigma)}$ which associates with a sheaf $F \in \mathbf{T}(\mathcal{U})$, $\mathcal{U} \supset \Sigma$, its pullback on (X, Σ). The continuity theorem ([Ber2], 4.3.5) tells that if a k-germ (X, Σ) is paracompact, then for all $F \in \mathbf{S}(X(\Sigma))$ and $q \geq 0$ there is a canonical isomorphism $H^q(X(\Sigma), F) \xrightarrow{\sim} H^q((X, \Sigma), F_{(X,\Sigma)})$.

§3. GAGA over the local ring of a point

Let (S, s) be a k-germ such that S is *good at* s, i.e., the point s has an affinoid neighborhood in S, and let $A = \mathcal{O}_{S,s}$ and $\mathcal{S} = \mathrm{Spec}(A)$. We recall that $A = \varinjlim \mathcal{A}_W$, where W runs through affinoid neighborhoods of the point s, and that the rings A and $\mathcal{A}_W$ are Noetherian. For an affinoid domain $V = \mathcal{M}(\mathcal{A}_V)$ we denote by $\mathcal{V}$ the affine scheme $\mathrm{Spec}(\mathcal{A}_V)$. By [EGA4], 8.8.2, for any scheme $\mathcal{X}$ of finite type over $\mathcal{S}$ there exist an affinoid neighborhood V of s and a scheme $\mathcal{X}_V$ of finite type over $\mathcal{V}$ such that $\mathcal{X} \xrightarrow{\sim} \mathcal{X}_V \otimes_\mathcal{V} \mathcal{S}$ over $\mathcal{S}$. Furthermore, a projective limit of the projective system $\{\mathcal{X}_W = \mathcal{X}_V \otimes_\mathcal{V} \mathcal{W}\}_{V \supset W \ni s}$ exists in the category of schemes over $\mathcal{S}$, and one has $\mathcal{X} \xrightarrow{\sim} \varprojlim \mathcal{X}_W$. Finally, if $\mathcal{Y}$ is another scheme of finite type over $\mathcal{S}$ with $\mathcal{Y} \xrightarrow{\sim} \mathcal{Y}_V \otimes_\mathcal{V} \mathcal{S}$ for some scheme $\mathcal{Y}_V$ of finite type over $\mathcal{V}$, then there is a canonical bijection $\varinjlim \mathrm{Hom}_\mathcal{W}(\mathcal{Y}_W, \mathcal{X}_W) \xrightarrow{\sim} \mathrm{Hom}_\mathcal{S}(\mathcal{Y}, \mathcal{X})$.

By [Ber2], §2.6, one can associate with the scheme $\mathcal{X}_V$ a k-analytic space $\mathcal{X}_V^{\mathrm{an}} = (\mathcal{X}_V)^{\mathrm{an}}$ closed over V. The V-analytic space $\mathcal{X}_V^{\mathrm{an}}$ gives rise to an $S(s)$-analytic space $\mathcal{X}_V^{\mathrm{an}}(s)$. From the above EGA-facts it follows that $\mathcal{X}_V^{\mathrm{an}}(s)$ does not depend on the choice of V and $\mathcal{X}_V$ up to a canonical isomorphism, and therefore we can set $\mathcal{X}^{\mathrm{an}} = \mathcal{X}_V^{\mathrm{an}}(s)$. (For example, $\mathcal{S}^{\mathrm{an}} = S(s)$.) It follows also that the correspondence $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{an}}$ is a functor which commutes with fiber products. Moreover, if $(S', s') \to (S, s)$ is a morphism of germs over k, then there is a canonical isomorphism $(\mathcal{X} \times_\mathcal{S} \mathcal{S}')^{\mathrm{an}} \xrightarrow{\sim} \mathcal{X}^{\mathrm{an}} \times_{S(s)} S'(s')$. One has $|\mathcal{X}^{\mathrm{an}}| = f_V^{-1}(s)$, where f_V is the morphism $\mathcal{X}_V^{\mathrm{an}} \to V$, and, for $x \in \mathcal{X}^{\mathrm{an}}$, $\mathcal{O}_{\mathcal{X}^{\mathrm{an}}, x} = \mathcal{O}_{\mathcal{X}_V^{\mathrm{an}}, x}$. From [Ber2], 2.6.2, it follows that the canonical morphism of locally ringed spaces $\pi : \mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ is flat and its image coincides with the closed fiber of $\mathcal{X}$, $\mathcal{X}_s = \mathcal{X} \otimes_A \kappa(s)$. For a point $x \in \mathcal{X}^{\mathrm{an}}$ (i.e., $x \in |\mathcal{X}^{\mathrm{an}}|$) we denote by $\mathbf{x}$ its image in $\mathcal{X}$.

By the analytification of the closed fiber we mean the $\mathcal{H}(s)$-analytic space $\mathcal{X}_s^{\mathrm{an}} = (\mathcal{X}^{\mathrm{an}})_s = (\mathcal{X}_s \otimes_{\kappa(s)} \mathcal{H}(s))^{\mathrm{an}}$ (and not the $S(s)$-analytic space $(\mathcal{X}_s)^{\mathrm{an}}$). The canonical morphism $\mathcal{X}_s^{\mathrm{an}} \to \mathcal{X}^{\mathrm{an}}$ induces a homeomorphism between the underlying topological spaces. Similarly, by the analytification of the geometric closed fiber of f, $\mathcal{X}_{\overline{s}} = \mathcal{X} \otimes_A \kappa(s)^{\mathrm{a}}$, we mean the $\widehat{\mathcal{H}(s)^{\mathrm{a}}}$-analytic space $\mathcal{X}_{\overline{s}}^{\mathrm{an}} = (\mathcal{X}^{\mathrm{an}})_{\overline{s}} = (\mathcal{X}_{\overline{s}} \otimes_{\kappa(s)^{\mathrm{s}}} \widehat{\mathcal{H}(s)^{\mathrm{a}}})^{\mathrm{an}}$. There are the evident functors $\mathbf{T}(\mathcal{X}_s) \to \mathbf{T}(\mathcal{X}_s^{\mathrm{an}})$ and $\mathbf{T}(\mathcal{X}_{\overline{s}}) \to \mathbf{T}(\mathcal{X}_{\overline{s}}^{\mathrm{an}}) : \mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$. We remark that since all of the schemes $\mathcal{X}_W$ are quasicompact and quasiseparated, then the isomorphism $\mathcal{X} \xrightarrow{\sim} \varprojlim \mathcal{X}_W$ induces an equivalence of categories $\varinjlim \mathbf{T}(\mathcal{X}_W) \xrightarrow{\sim} \mathbf{T}(\mathcal{X})$. In this way one gets a functor $\mathbf{T}(\mathcal{X}) \to \mathbf{T}(\mathcal{X}^{\mathrm{an}}) : \mathcal{F} \mapsto \mathcal{F}^{\mathrm{an}}$.

3.1. Proposition. *Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism of schemes of finite type over $\mathcal{S}$. Then*

  (i) *φ is separated (proper, finite, quasifinite, a closed immersion) if and only if φ^{an} possesses the same property;*

  (ii) *φ is flat quasifinite (unramified, étale, smooth) if and only if φ^{an} possesses the same property.*

Proof. (i) follows from [EGA4], 8.10.5, and [Ber2], 2.6.9, 3.1.7.

(ii) is proved in the same way as the corresponding statements [Ber2], 3.2.10, 3.3.11 and 3.5.8, using the following fact which is an easy consequence of [Ber2], 2.6.10.   □

3.2. Lemma. *Suppose that φ is finite. Let $\mathbf{y} \in \mathcal{Y}$ and $x \in \mathcal{X}^{\mathrm{an}}$ be points with $\varphi(\mathbf{y}) = \mathbf{x}$, and let $\varphi^{\mathrm{an}^{-1}}(x) = \{y_1, \ldots, y_n\}$ and $\varphi^{\mathrm{an}^{-1}}(x) \cap \pi^{-1}(\mathbf{y}) = \{y_1, \ldots, y_m\}$, $m \leq n$. Then there is an isomorphism of rings*

$$\mathcal{O}_{\mathcal{X}^{\mathrm{an}}, x} \otimes_{\mathcal{O}_{\mathcal{X}, \mathbf{x}}} \mathcal{O}_{\mathcal{Y}, \mathbf{y}} \xrightarrow{\sim} \prod_{i=1}^{m} \mathcal{O}_{\mathcal{Y}^{\mathrm{an}}, y_i} \times \prod_{i=m+1}^{n} (\mathcal{O}_{\mathcal{Y}^{\mathrm{an}}, y_i})_{\mathbf{m_y}} \ ,$$

where $(\mathcal{O}_{\mathcal{Y}^{\mathrm{an}}, y_i})_{\mathbf{m_y}}$ is the localization with respect to the complement of the maximal ideal $\mathbf{m_y}$ of $\mathcal{O}_{\mathcal{Y}, \mathbf{y}}$.   □

The following is a direct consequence of the comparison theorem for cohomology with compact support [Ber2], (§7.1).

3.3. Proposition. *Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a proper morphism between schemes of finite type over $\mathcal{S}$, and let $\mathcal{F}$ be an abelian torsion sheaf on $\mathcal{Y}$. Then for any $q \geq 0$ there is a canonical isomorphism*

$$(R^q \varphi_* \mathcal{F})^{\mathrm{an}} \xrightarrow{\sim} R^q \varphi_*^{\mathrm{an}}(\mathcal{F}^{\mathrm{an}}).$$

                                                                        □

The following fact is an essential ingredient in the proof of the comparison theorem for vanishing cycles 6.1.

3.4. Theorem. *Let x be a point of $\mathcal{X}^{\mathrm{an}}$ and $\mathbf{x}$ its image in $\mathcal{X}$. Then the field $\kappa(x)$ is separable over $k(\mathbf{x})$.*

Recall that an extension of fields L/K is called separable if the tensor product $L \otimes_K K^{p^{-1}}$ is a field, where $p = \mathrm{char}(K)$. In this case the tensor product $L \otimes_K K^{p^{-\infty}}$ is also a field, and L^{s} is separable over K^{s}. Therefore the tensor product $L^{\mathrm{s}} \otimes_{K^{\mathrm{s}}} K^{\mathrm{a}}$ is a field and, in particular, if K' is a finite extension of K^{s}, then $[L^{\mathrm{s}} K' : L^{\mathrm{s}}] = [K' : K^{\mathrm{s}}]$.

First of all we want to show that the theorem follows from the following fact.

3.5. Proposition. *Suppose that $\mathcal{O}_{S,s}$ is a field, i.e., $\mathcal{O}_{S,s} = \kappa(s)$. If $\mathcal{X}$ is reduced at $\mathbf{x}$, then $\mathcal{X}^{\mathrm{an}}$ is reduced at x.*

Indeed, to prove the theorem, it suffices to assume that $\mathcal{O}_{S,s} = \kappa(s)$, $\mathcal{X}$ is reduced and irreducible, and $\mathbf{x}$ is the generic point of $\mathcal{X}$. Let K be a finite purely inseparable extension of the field $k(\mathbf{x})$. Then we can shrink $\mathcal{X}$ and find a finite radicial morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ such that $\mathcal{Y}$ is reduced and, for the generic point $\mathbf{y}$ of $\mathcal{Y}$, one has $k(\mathbf{y}) = K$. By Proposition 3.5, the $S(s)$-analytic space $\mathcal{Y}^{\mathrm{an}}$ is also reduced. Since the morphism φ is radicial, there is a unique point $y \in \mathcal{Y}^{\mathrm{an}}$ whose images in $\mathcal{Y}$ and $\mathcal{X}^{\mathrm{an}}$ are $\mathbf{y}$ and x, respectively. By Lemma 3.2, we get $k(\mathbf{y}) \otimes_{k(\mathbf{x})} \kappa(x) \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}^{\mathrm{an}}, y}$. Since the local ring $\mathcal{O}_{\mathcal{Y}^{\mathrm{an}}, y}$ is reduced and finite over the field $\kappa(x)$, it follows that that it is a field (i.e., it coincides with $\kappa(y)$), and the theorem follows.

To prove Proposition 3.5, we need the following two lemmas (one does not assume in them that $\mathcal{O}_{S,s}$ is a field). Let $\mathcal{X}_0$ denote the set of closed points of the closed fiber of $\mathcal{X}$. Furthermore, for an $S(s)$-analytic space $\mathbf{X}$ let $\mathbf{X}_0$ denote the set of points $x \in \mathbf{X}$ with $[\kappa(x) : \kappa(s)] < \infty$. We set $\mathcal{X}_0^{\mathrm{an}} = (\mathcal{X}^{\mathrm{an}})_0$.

3.6. Lemma. *The map $\mathcal{X}^{\mathrm{an}} \to \mathcal{X}$ induces a bijection $\mathcal{X}_0^{\mathrm{an}} \xrightarrow{\sim} \mathcal{X}_0$. Furthermore, if $x \in \mathcal{X}_0^{\mathrm{an}}$, then there is an isomorphism of completions $\widehat{\mathcal{O}}_{\mathcal{X},\mathbf{x}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathcal{X}^{\mathrm{an}},x}$.*

Proof. Let $\mathbf{x} \in \mathcal{X}_0$. For $n \geq 1$ we set $\mathcal{Y} = \mathrm{Spec}(\mathcal{O}_{\mathcal{X},\mathbf{x}}/\mathbf{m}_{\mathbf{x}}^n)$. The scheme $\mathcal{Y}$ consists of one point $\mathbf{y}$ and is finite over $\mathcal{S}$. Therefore $\mathcal{Y}^{\mathrm{an}}$ consists of one point y and, by Lemma 3.2, one has $\mathcal{O}_{\mathcal{Y},\mathbf{y}} = \mathcal{O}_{\mathcal{X},\mathbf{x}}/\mathbf{m}_{\mathbf{x}}^n \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}^{\mathrm{an}},y}$. Furthermore, there is a canonical closed immersion $\mathcal{Y} \to \mathcal{X}$ that takes $\mathbf{y}$ to $\mathbf{x}$. From Lemma 3.2 it follows that $\mathcal{Y}^{\mathrm{an}} \to \mathcal{X}^{\mathrm{an}}$ is also a closed immersion, and the point x is the only preimage of $\mathbf{x}$ in $\mathcal{X}^{\mathrm{an}}$. (In particular, $\mathcal{X}_0^{\mathrm{an}} \xrightarrow{\sim} \mathcal{X}_0$.) Moreover, one has $\mathcal{O}_{\mathcal{Y}^{\mathrm{an}},y} = \mathcal{O}_{\mathcal{X}^{\mathrm{an}},x}/\mathbf{m}_{\mathbf{x}}^n \mathcal{O}_{\mathcal{X}^{\mathrm{an}},x}$. It follows that $\mathcal{O}_{\mathcal{X},\mathbf{x}}/\mathbf{m}_{\mathbf{x}}^n \xrightarrow{\sim} \mathcal{O}_{\mathcal{Y}^{\mathrm{an}},y}$. If $n = 1$ we get $\mathbf{m}_x = \mathbf{m}_{\mathbf{x}}\mathcal{O}_{\mathcal{X}^{\mathrm{an}},x}$ and $k(\mathbf{x}) \xrightarrow{\sim} \kappa(x)$. Hence $\widehat{\mathcal{O}}_{\mathcal{X},\mathbf{x}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathcal{X}^{\mathrm{an}},x}$. $\square$

3.7. Lemma. *Suppose that the valuation on $\kappa(s)$ is nontrivial. Then for any $S(s)$-analytic space $\mathbf{X}$ closed over $S(s)$ the set $\mathbf{X}_0$ is everywhere dense in $\mathbf{X}$.*

Proof. For a closed morphism $f : X \to S$, we set

$$X(s)_0 = \{x \in f^{-1}(s) \big| [\kappa(x) : \kappa(s)] < \infty\}$$

and

$$(X_s)_0 = \{x \in X_s \big| [\mathcal{H}(x) : \mathcal{H}(s)] < \infty\}.$$

(Note that there is a homeomorphism $X_s \xrightarrow{\sim} f^{-1}(s)$.) First, we claim that if f is the projection $X = S \times E^d \to S$, where E^d is the closed unit polydisc in $\mathbf{A}^d$ with center at zero, then $X(s)_0$ is everywhere dense in $f^{-1}(s)$. Indeed, one has

$$X(s)_0 = \{\alpha \in \kappa(s)^{\mathrm{a}} \big| |\alpha| \leq 1\}^d/G$$

and

$$(X_s)_0 = \{\alpha \in \mathcal{H}(s)^{\mathrm{a}} \big| |\alpha| \leq 1\}^d/G,$$

where $G = G(\kappa(s)^{\mathrm{s}}/\kappa(s)) = G(\mathcal{H}(s)^{\mathrm{s}}/\mathcal{H}(s))$. Since the field $\kappa(s)^{\mathrm{s}}$ is everywhere dense in $\mathcal{H}(s)^{\mathrm{a}}$ and the set $X(s)_0$ is everywhere dense in $f^{-1}(s)$, the claim follows.

Now let $f : X \to S$ be a closed morphism and $x \in f^{-1}(s)$. We have to show that, for any affinoid neighborhood $V = \mathcal{M}(\mathcal{B})$ of x, there exists a point $x' \in V$ over s with $[\kappa(x') : \kappa(s)] < \infty$. Shrinking V, we may assume that $\dim_x(V_s) = \dim(V_s)$. We may also assume that $S = \mathcal{M}(\mathcal{A})$ is k-affinoid. Since $x \in \mathrm{Int}(V/S)$, there is an admissible epimorphism $\mathcal{A}\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to \mathcal{B}$ with $|f_i(X)| < r_i$, $1 \leq i \leq n$, where f_i is the image of T_i in $\mathcal{B}$. Replacing V by the Weierstrass domain $V(r_i'^{-1}f_i)$ for some $|f_i(x)| < r_i' < r_i$ with $r_i' \in \sqrt{|\mathcal{H}(s)^*|}$, we may assume that the $\mathcal{H}(s)$-affinoid algebra $\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{H}(s)$ is strictly $\mathcal{H}(s)$-affinoid. By the Noether normalization lemma, we can find elements $g_1, \ldots, g_d \in \mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{H}(s)$ such that there is an admissible, injective and finite homomorphism $\mathcal{H}(s)\{T_1, \ldots, T_d\} \to \mathcal{B} : T_i \mapsto g_i$. It defines a finite surjective morphism $V_s \to E^d_{\mathcal{H}(s)}$. (One has $d = \dim(V_s)$.) Since the image of $\mathcal{B} \otimes_{\mathcal{A}} \kappa(s)$ in $\mathcal{B}\widehat{\otimes}_{\mathcal{A}}\mathcal{H}(s)$ is everywhere dense, we may assume that $g_1, \ldots, g_d$ come from $\mathcal{B}\otimes_{\mathcal{A}}\kappa(s)$. Shrinking S, we may assume that they come from $\mathcal{B}$. Furthermore, consider the morphism $g : V \to S \times \mathbf{A}^d$ defined by the functions $g_1, \ldots, g_d$. Since $x \in \mathrm{Int}(V/S)$, it follows that g is finite at x ([Ber2], 3.1.4). Let W be the preimage of $S \times E^d$ in V. The induced morphism $h : W \to S \times E^d$ is also finite at x. This means that there are open neighborhoods $\mathcal{W}$ of x in W and $\mathcal{U}$ of $h(x)$ in $S \times E^d$ such that h induces a finite morphism $\mathcal{W} \to \mathcal{U}$. Since $\dim_x(\mathcal{W}_s) = \dim(\mathcal{W}_s) = d$, from [Ber1], 3.2.4, it follows that we can shrink $\mathcal{U}$ and assume that the morphism $\mathcal{W}_s \to \mathcal{U}_s$ is surjective. We can find a point $y' \in \mathcal{U}$ over s with $[\kappa(y') : \kappa(s)] < \infty$. If x' is a point of $\mathcal{W}$ over y', then $[\kappa(x') : \kappa(s)] < \infty$. $\square$

Proof of Proposition 3.5. We can shrink $\mathcal{X}$ and assume that it is reduced. Let V be an affinoid neighborhood of s such that $\mathcal{X}$ comes from a scheme $\mathcal{X}_V$ of finite type over $\mathcal{V}$, and let Y be the set of points $y \in \mathcal{X}_V^{\mathrm{an}}$ such that $\mathcal{X}_V^{\mathrm{an}}$ is not reduced at y. By [Ber2], 2.2.1, the set Y is Zariski closed in $\mathcal{X}_V^{\mathrm{an}}$, and therefore it can be considered as a k-analytic space closed over V. It gives rise to an $S(s)$-analytic space $\mathbf{Y}$ closed over $S(s)$. Assume that $\mathbf{Y}$ is nonempty. We claim that then the set $\mathbf{Y}_0$ is nonempty. Indeed, if the valuation on $\kappa(s)$ is nontrivial, this follows from Lemma 3.7. Suppose that the valuation on $\kappa(s)$ is trivial. The closed fiber $\mathcal{X}_s$ is an $\mathcal{H}(s)$-analytic space, and $\mathbf{Y}_s$ is a Zariski closed subset of $\mathcal{X}_s^{\mathrm{an}}$. By [Ber1], §3.5, one has $\mathbf{Y}_s = \mathcal{Z}^{\mathrm{an}}$, where $\mathcal{Z}$ is a Zariski closed subset of $\mathcal{X}_s$. It follows that there exists a point $\mathbf{y} \in \mathcal{Z}$ with $[k(\mathbf{y}) : \kappa(s)] < \infty$, and therefore $\mathbf{Y}_0$ is nonempty. Now let $y \in \mathbf{Y}_0$. By Lemma 3.6, one has $\widehat{\mathcal{O}}_{\mathcal{X},\mathbf{y}} \xrightarrow{\sim} \widehat{\mathcal{O}}_{\mathcal{X}^{\mathrm{an}},y}$. Since $\mathcal{O}_{S,s} = \kappa(s)$, it follows that $\mathcal{O}_{\mathcal{X},\mathbf{y}}$ is a localization of a ring which is finitely generated over the field $\kappa(s)$. Such a ring is reduced if and only if its completion is reduced (see [EGA4], 7.8.3). It follows that $\mathcal{O}_{\mathcal{X}^{\mathrm{an}},y}$ is reduced, which is impossible because $y \in \mathbf{Y}$. $\square$

3.8. Remark. One can construct the functor $\mathcal{X} \mapsto \mathcal{X}^{\mathrm{an}}$ in another way so that the construction works also over $\mathbf{C}$ and $\mathbf{R}$. Namely, one has $A = \varinjlim A_i$, where A_i runs through subrings of A which are finitely generated over k. For any scheme $\mathcal{X}$ of finite type over $\mathcal{S}$ there exist i and a scheme $\mathcal{X}_i$ of finite type over $\mathcal{S}_i = \mathrm{Spec}(A_i)$ such that $\mathcal{X} \xrightarrow{\sim} \mathcal{X}_i \otimes_{\mathcal{S}_i} \mathcal{S}$ over $\mathcal{S}$. By GAGA over the field k, one can associate with $\mathcal{S}_i$ and $\mathcal{X}_i$ k-analytic spaces $S_i = \mathcal{S}_i^{\mathrm{an}}$ and $X_i = \mathcal{X}_i^{\mathrm{an}}$. The canonical homomorphism $A_i \to A = \mathcal{O}_{S,s}$ defines a point $s_i \in S_i$ and a morphism of k germs $(S,s) \to (S_i, s_i)$. The latter is induced by a morphism $\mathcal{U} \to S_i$ from an open neighborhood $\mathcal{U}$ of the point s. One has $\mathcal{X}^{\mathrm{an}} = (X_i \times_{S_i} \mathcal{U})(s)$. The essential difference of the Archimedean situation from the non-Archimedean one is that in this situation the map $|\mathcal{X}^{\mathrm{an}}| \to \mathcal{X}_s$ is injective and its image coincides with the set of closed points of the closed fiber $\mathcal{X}_s$. (In particular, for a point $x \in \mathcal{X}^{\mathrm{an}}$, one has $\kappa(x) = k(\mathbf{x})$.)

§4. The vanishing cycles functor

Beginning with this section, we assume that, for the k-germ (S,s) from §3, $A = \mathcal{O}_{S,s}$ is a discrete valuation ring. In this case the scheme $\mathcal{S} = \mathrm{Spec}(A)$ consists of the closed point $s = \mathrm{Spec}(\kappa(s))$ and the generic point $\eta = \mathrm{Spec}(\mathcal{K})$, where $\mathcal{K}$ is the fraction field of A. We denote by $\mathbf{S}$ the pro-k-analytic space $\mathcal{S}^{\mathrm{an}} = S(s)$. The scheme η is of finite type over $\mathcal{S}$, and therefore one can associate with it a pro-k-analytic space η^{an}. One has $\eta^{\mathrm{an}} = "\varprojlim"(\mathcal{U}\backslash T)$, where $\mathcal{U}$ runs through open neighborhoods of s and $T \subset S$ is a Zariski closed subset that goes through the point s. For an $\mathbf{S}$-analytic space $\mathbf{X}$ we set $\mathbf{X}_\eta = \mathbf{X} \times_{\mathbf{S}} \eta^{\mathrm{an}}$. (For example, $\mathbf{S}_\eta = \eta^{\mathrm{an}}$.)

Since $\mathcal{K}$ is the fraction field of the Henselian discrete valuation ring A, its valuation extends uniquely to a valuation on the separable closure $\mathcal{K}^{\mathrm{s}}$, and the integral closure $\overline{A}$ of A in $\mathcal{K}^{\mathrm{s}}$ is a local ring. The residue field $\widetilde{\mathcal{K}^{\mathrm{s}}}$ is an algebraic closure $\kappa(s)^{\mathrm{a}}$ of $\kappa(s)$. Let $\overline{\mathcal{S}} = \mathrm{Spec}(\overline{A})$, $\overline{\eta} = \mathrm{Spec}(\mathcal{K}^{\mathrm{s}})$ and $\overline{s} = \mathrm{Spec}(\kappa(s)^{\mathrm{a}})$, and let ν be the canonical homomorphism $G_\eta = G(\mathcal{K}^{\mathrm{s}}/\mathcal{K}) \to G_s = G(\kappa(s)^{\mathrm{s}}/\kappa(s))$. Recall the definition of the vanishing cycles functor $\Psi_\eta : \mathbf{T}(\mathcal{X}_\eta) \to \mathbf{T}_{G_\eta}(\mathcal{X}_{\overline{s}})$ for a scheme $\mathcal{X}$ over $\mathcal{S}$.

One sets $\overline{\mathcal{X}} = \mathcal{X} \times_{\mathcal{S}} \overline{\mathcal{S}}$, $\mathcal{X}_{\overline{\eta}} = \mathcal{X} \times_{\mathcal{S}} \overline{\eta}$ and $\mathcal{X}_{\overline{s}} = \mathcal{X} \times_{\mathcal{S}} \overline{s}$. Furthermore, for a finite extension $\mathcal{L}$ of $\mathcal{K}$ in $\mathcal{K}^{\mathrm{s}}$, one denotes by $A_{\mathcal{L}}$ the integral closure of A in $\mathcal{L}$ and set $\mathcal{S}_{\mathcal{L}} = \mathrm{Spec}(A_{\mathcal{L}})$. The scheme $\mathcal{S}_{\mathcal{L}}$ consists of the closed point $s_{\mathcal{L}} = \mathrm{Spec}(\widetilde{\mathcal{L}})$ and

the generic point $\eta_{\mathcal{L}} = \mathrm{Spec}(\mathcal{L})$. One sets $\mathcal{X}_{\mathcal{L}} = \mathcal{X} \times_{\mathcal{S}} \mathcal{S}_{\mathcal{L}}$, $\mathcal{X}_{\eta_{\mathcal{L}}} = \mathcal{X} \times_{\mathcal{S}} \eta_{\mathcal{L}}$ and $\mathcal{X}_{s_{\mathcal{L}}} = \mathcal{X} \times_{\mathcal{S}} s_{\mathcal{L}}$. There is a commutative diagram

$$
\begin{array}{ccccc}
\mathcal{X}_{\eta_{\mathcal{L}}} & \xrightarrow{\ j_{\mathcal{L}}\ } & \mathcal{X}_{\mathcal{L}} & \xleftarrow{\ i_{\mathcal{L}}\ } & \mathcal{X}_{s_{\mathcal{L}}} \\
\uparrow & & \uparrow & \nwarrow{\bar{i}_{\mathcal{L}}} & \uparrow \\
\mathcal{X}_{\overline{\eta}} & \xhookrightarrow{\ \overline{j}\ } & \overline{\mathcal{X}} & \xleftarrow{\ \overline{i}\ } & \mathcal{X}_{\overline{s}}
\end{array}
$$

For $\mathcal{F} \in \mathbf{T}(\mathbf{X}_\eta)$, let $\mathcal{F}_{\mathcal{L}}$ and $\overline{\mathcal{F}}$ denote the pullbacks of $\mathcal{F}$ on $\mathcal{X}_{\eta_{\mathcal{L}}}$ and $\mathcal{X}_{\overline{\eta}}$. Then

$$
\Psi_\eta(\mathcal{F}) = \varinjlim \overline{i}^{\,*}_{\mathcal{L}} j_{\mathcal{L}*}(\mathcal{F}_{\mathcal{L}}) \xrightarrow{\sim} \overline{i}^{\,*} \overline{j}_*(\overline{\mathcal{F}}) \ ,
$$

where $\mathcal{L}$ runs through finite extensions of $\mathcal{K}$ in $\mathcal{K}^{\mathrm{s}}$.

We now return to analytic geometry. For a finite extension $\mathcal{L}$ of $\mathcal{K}$ in $\mathcal{K}^{\mathrm{s}}$, we set $\mathbf{S}_{\mathcal{L}} = \mathcal{S}_{\mathcal{L}}^{\mathrm{an}}$ (this space consists of one point $s_{\mathcal{L}}$) and $\overline{\mathbf{S}} = \text{"}\varprojlim\text{"} \mathbf{S}_{\mathcal{L}}$. We also set $\mathbf{S}_{\eta_{\mathcal{L}}} = (\eta_{\mathcal{L}})^{\mathrm{an}}$ and $\mathbf{S}_{\overline{\eta}} = \text{"}\varprojlim\text{"} \mathbf{S}_{\eta_{\mathcal{L}}}$. Furthermore, for an $\mathbf{S}$-analytic space $\mathbf{X}$, we set $\mathbf{X}_{\mathcal{L}} = \mathbf{X} \times_{\mathbf{S}} \mathbf{S}_{\mathcal{L}}$, $\mathbf{X}_{\eta_{\mathcal{L}}} = \mathbf{X} \times_{\mathbf{S}} \mathbf{S}_{\eta_{\mathcal{L}}}$, $\overline{\mathbf{X}} = \mathbf{X} \times_{\mathbf{S}} \overline{\mathbf{S}}$, and $\mathbf{X}_{\overline{\eta}} = \mathbf{X} \times_{\mathbf{S}} \mathbf{S}_{\overline{\eta}}$. There is a commutative diagram of morphisms of pro-analytic spaces

$$
\begin{array}{ccccc}
\mathbf{X}_{\eta_{\mathcal{L}}} & \xrightarrow{\ j_{\mathcal{L}}\ } & \mathbf{X}_{\mathcal{L}} & \xleftarrow{\ i_{\mathcal{L}}\ } & \mathbf{X}_{s_{\mathcal{L}}} \\
\uparrow & & \uparrow & \nwarrow{\bar{i}_{\mathcal{L}}} & \uparrow \\
\mathbf{X}_{\overline{\eta}} & \xhookrightarrow{\ \overline{j}\ } & \overline{\mathbf{X}} & \xleftarrow{\ \overline{i}\ } & \mathbf{X}_{\overline{s}}
\end{array}
$$

For $F \in \mathbf{T}(\mathbf{X}_\eta)$, we denote by $F_{\mathcal{L}}$ the pullback of F on $\mathbf{X}_{\eta_{\mathcal{L}}}$ and define the *vanishing cycles functor* $\Psi_\eta : \mathbf{T}(\mathbf{X}_\eta) \to \mathbf{T}_{G_\eta}(\mathbf{X}_{\overline{s}})$ by

$$
\Psi_\eta(F) = \varinjlim \overline{i}^{\,*}_{\mathcal{L}} j_{\mathcal{L}*}(F_{\mathcal{L}}) \ ,
$$

where $\mathcal{L}$ runs through finite extensions of $\mathcal{K}$ in $\mathcal{K}^{\mathrm{s}}$. It follows from the definition that, for a scheme $\mathcal{X}$ of finite type over $\mathcal{S}$ and a sheaf of sets (resp. abelian groups) $\mathcal{F}$ on $\mathcal{X}_\eta$, there is a canonical morphism of sheaves $\mathcal{X}_{\overline{s}}^{\mathrm{an}}$, $(\Psi_\eta(\mathcal{F}))^{\mathrm{an}} \to \Psi_\eta(\mathcal{F}^{\mathrm{an}})$ (resp. $(R^q\Psi_\eta(\mathcal{F}))^{\mathrm{an}} \to R^q\Psi_\eta(\mathcal{F}^{\mathrm{an}})$, $q \geq 0$).

For a morphism of $\mathbf{S}$-analytic spaces $\varphi : \mathbf{Y} \to \mathbf{X}$ we denote by φ_s, $\varphi_{\overline{s}}$, φ_η and $\varphi_{\overline{\eta}}$ the induced morphisms $\mathbf{Y}_s \to \mathbf{X}_s$, $\mathbf{Y}_{\overline{s}} \to \mathbf{X}_{\overline{s}}$, $\mathbf{Y}_\eta \to \mathbf{X}_\eta$ and $\mathbf{Y}_{\overline{\eta}} \to \mathbf{X}_{\overline{\eta}}$, respectively. The following statements follow straightforwardly from the weak base change theorem 5.3.6 and the smooth base change theorem 7.8.1 from [Ber2].

4.1. Proposition. *Let $\varphi : \mathbf{Y} \to \mathbf{X}$ be a morphism of $\mathbf{S}$-analytic spaces.*

(i) *If φ is compact, then for any $F^{\cdot} \in D^+(\mathbf{Y}_\eta)$ there is a canonical isomorphism $R\Psi_\eta(R\varphi_{\eta_*} F^{\cdot}) \xrightarrow{\sim} R\varphi_{\overline{s}*}(R\Psi_\eta F^{\cdot})$. In particular, if $\mathbf{X}$ is an $\mathbf{S}$-analytic space compact over $\mathbf{S}$, then for any abelian sheaf F on $\mathbf{X}_\eta$ there is a spectral sequence*

$$
E_2^{p,q} = H^p(\mathbf{X}_{\overline{s}}, R^q\Psi_\eta(F)) \Longrightarrow H^{p+q}(\mathbf{X}_{\overline{\eta}}, F).
$$

(ii) *If φ is smooth, then for any abelian torsion sheaf on $\mathbf{X}_\eta$ with torsion orders prime to $\mathrm{char}(\widetilde{k})$ and any $q \geq 0$ there is a canonical isomorphism*

$$
\varphi_{\overline{s}}^*(R^q\Psi_\eta(F)) \xrightarrow{\sim} R^q\Psi_\eta(\varphi_\eta^* F).
$$

$\square$

Let $\mathcal{L}$ be a Galois extension of $\mathcal{K}$ in $\mathcal{K}^{\mathrm{s}}$ that contains $\mathcal{K}^{\mathrm{nr}}$, the maximal unramified extension of $\mathcal{K}$. The residue field $\widetilde{\mathcal{L}}$ is an algebraic extension of $\kappa(s)$ that contains $\kappa(s)^{\mathrm{s}}$, and therefore its completion coincides with $\widehat{\mathcal{H}(s)^{\mathrm{a}}}$. One has, for an $\mathbf{S}$-analytic space $\mathbf{X}$, a left exact functor $\Psi_{\eta,\mathcal{L}} : \mathbf{T}(\mathbf{X}_\eta) \to \mathbf{T}_{G(\mathcal{L}/\mathcal{K})}(\mathbf{X}_{\overline{s}})$ defined by $\Psi_{\eta,\mathcal{L}}(F) = \varinjlim \overline{i}_{\mathcal{N}}^{\,*} j_{\mathcal{N}*}(F_{\mathcal{N}})$, where $\mathcal{N}$ runs through finite extensions of $\mathcal{K}$ in $\mathcal{L}$. For example, if $\mathcal{L} = \mathcal{K}^{\mathrm{nr}}$, then $R^q \Psi_{\eta,\mathcal{K}^{\mathrm{nr}}}(F) = \overline{i}^{\,*}(R^q j_*(F))$, where $\overline{i} = \overline{i}_{\mathcal{K}} : \mathbf{X}_{\overline{s}} \to \mathbf{X}$ and $j = j_{\mathcal{K}} : \mathbf{X}_\eta \to \mathbf{X}$. One has the following simple fact.

4.2. Proposition. *In the above situation, for any $F \in \mathbf{S}(\mathbf{X}_\eta)$ there is a spectral sequence*
$$E_2^{p,q} = \mathcal{H}^p\left(G(\mathcal{K}^{\mathrm{s}}/\mathcal{L}), R^q \Psi_\eta(F)\right) \Longrightarrow R^{p+q} \Psi_{\eta,\mathcal{L}}(F).$$
$\square$

Let $S' \to S$ be a morphism of good k-analytic spaces, and let $s \in S$ and $s' \in S'$ be points such that $\mathcal{O}_{S,s}$ and $\mathcal{O}_{S',s'}$ are discrete valuation rings, the image of s' in S is s, and the induced homomorphism $\mathcal{O}_{S,s} \to \mathcal{O}_{S',s'}$ is injective. One has morphisms of affine schemes $\mathcal{S}' = \mathrm{Spec}(\mathcal{O}_{S',s'}) \to \mathcal{S} = \mathrm{Spec}(\mathcal{O}_{S,s})$ and of pro-k-analytic spaces $\mathbf{S}' = S'(s') \to \mathbf{S} = S(s)$. Furthermore, let $X \to S'$ be a morphism of k-analytic spaces. It gives rise to an $\mathbf{S}'$-analytic space $\mathbf{X}'$ (with $\mathbf{X}'_{s'} = X_{s'}$). The induced morphism $X \to S$ gives rise to an $\mathbf{S}$-analytic space $\mathbf{X}$ (with $\mathbf{X}_s = X_s$). Thus, there is a commutative diagram of morphisms of pro-k-analytic spaces

$$\begin{array}{ccc} \mathbf{X}' & \longrightarrow & \mathbf{S}' \\ \downarrow & & \downarrow \\ \mathbf{X} & \longrightarrow & \mathbf{S} \end{array}$$

The morphism $\mathbf{X}' \to \mathbf{X}$ induces the evident inverse image functor $\mathbf{T}(\mathbf{X}_\eta) \to \mathbf{T}(\mathbf{X}'_{\eta'}) : F \mapsto F'$. Let $\mathcal{K}$ and $\mathcal{K}'$ be the fraction fields of $\mathcal{O}_{S,s}$ and $\mathcal{O}_{S',s'}$, respectively. We fix an embedding of fields $\mathcal{K}^{\mathrm{s}} \hookrightarrow \mathcal{K}'^{\mathrm{s}}$ over the canonical embedding $\mathcal{K} \hookrightarrow \mathcal{K}'$. It induces a homomorphism of Galois groups $G_{\eta'} \to G_\eta$. It induces also an embedding $\kappa(s)^{\mathrm{a}} \hookrightarrow \kappa(s')^{\mathrm{a}}$ and, therefore, a morphism $\lambda : \mathbf{X}'_{\overline{s'}} \to \mathbf{X}_{\overline{s}}$ of analytic spaces over k.

4.3. Proposition. *In the above situation, assume that the morphism $S' \to S$ is smooth, and let $\mathcal{L} = \mathcal{K}'^{\mathrm{nr}} \mathcal{K}^{\mathrm{s}}$. Then for any $F \in \mathbf{S}(\mathbf{X}_\eta)$ and any $q \geq 0$ there is a canonical isomorphism $\lambda^*(R^q \Psi_\eta(F)) \xrightarrow{\sim} R^q \Psi_{\eta',\mathcal{L}}(F')$ compatible with the action of Galois groups.*

Proof. Let $\mathcal{N}$ be a finite Galois extension of $\mathcal{K}$ in $\mathcal{K}^{\mathrm{s}}$ and $\mathcal{N}' = \mathcal{K}'\mathcal{N}$. We can shrink S and assume that $\mathbf{S}_\eta$ comes from $S \backslash T$, where T is a Zariski closed subset of S, the morphism $\mathbf{S}_{\mathcal{N}} \to \mathbf{S}$ comes from a flat finite morphism $S_{\mathcal{N}} \to S$, and the sheaf F comes from $X \backslash Y$, where Y is the preimage of T in X. Furthermore, we can shrink S' and assume that the morphism $\mathbf{S}'_{\mathcal{N}'} \to \mathbf{S}'$ comes from a flat finite morphism $S'_{\mathcal{N}'} \to S'$. We may assume also that S and $S_{\mathcal{N}}$ are regular. By hypothesis, the morphism $S' \times_S S_{\mathcal{N}} \to S_{\mathcal{N}}$ is smooth, and therefore $S' \times_S S_{\mathcal{N}}$ is regular. It follows that the ring $\mathcal{O}_{S',s'} \otimes_{\mathcal{O}_{S,s}} \mathcal{O}_{S_{\mathcal{N}},s_{\mathcal{N}}}$ is the integral closure of $\mathcal{O}_{S',s'}$ in $\mathcal{K}' \otimes_{\mathcal{K}} \mathcal{N}$. The latter is a direct product of finite separable extensions of $\mathcal{K}'$. One of these factors is $\mathcal{N}'$. We get a point $t \in S' \times_S S_{\mathcal{N}}$ over s' with the local ring $\mathcal{O}_{S'_{\mathcal{N}'},s'_{\mathcal{N}'}}$, and therefore we can shrink S' and find a morphism $S'_{\mathcal{N}'} \to S' \times_S S_{\mathcal{N}}$ over S' such that it takes the point $s'_{\mathcal{N}'}$ to t and is a local isomorphism at $s'_{\mathcal{N}'}$. We get a morphism

$$\lambda_{\mathcal{N}} : X'_{\mathcal{N}'} = X \times_{S'} S'_{\mathcal{N}'} \to X \times_{S'} (S' \times_S S_{\mathcal{N}}) = X_{\mathcal{N}}$$

that induces an open immersion of the preimage of an open neighborhood of the point $s'_{\mathcal{N}'}$ in $X'_{\mathcal{N}'}$ to $X_{\mathcal{N}}$. This implies that there is an isomorphism of sheaves

$$\lambda^*_{s_{\mathcal{N}}}(i^*_{\mathcal{N}} j_{\mathcal{N}*}(F)) \xrightarrow{\sim} i'^*_{\mathcal{N}'} j'_{\mathcal{N}'*}(F'),$$

where $X_{\mathcal{N}} \backslash Y_{\mathcal{N}} \overset{j_{\mathcal{N}}}{\hookrightarrow} X_{\mathcal{N}} \overset{i_{\mathcal{N}}}{\hookleftarrow} X_{s_{\mathcal{N}}}$, $X'_{\mathcal{N}'} \backslash Y'_{\mathcal{N}'} \overset{j'_{\mathcal{N}'}}{\hookrightarrow} X'_{\mathcal{N}'} \overset{i'_{\mathcal{N}'}}{\hookleftarrow} X'_{s'_{\mathcal{N}'}}$, $Y_{\mathcal{N}}$ and $Y'_{\mathcal{N}'}$ are the preimages of T in $X_{\mathcal{N}}$ and $X'_{\mathcal{N}'}$, and $\lambda_{s_{\mathcal{N}}}$ is the morphism $X'_{s'_{\mathcal{N}'}} \to X_{s_{\mathcal{N}}}$. The proposition follows.   □

4.4. Corollary. *The Galois group $P = G(\mathcal{K}'^{\mathrm{s}}/\mathcal{K}'^{\mathrm{nr}}\mathcal{K}^{\mathrm{s}})$ is a pro-p-group, where $p = \mathrm{char}(k)$, and if F is torsion with torsion orders prime to p, then there is a canonical isomorphism $\lambda^*(R^q \Psi_\eta(F)) \xrightarrow{\sim} R^q \Psi_{\eta'}(F')^P$.*

Proof. To prove the first statement, it suffices to verify that the field $\mathcal{K}'^{\mathrm{nr}}\mathcal{K}^{\mathrm{s}}$ contains $\mathcal{K}'^{\mathrm{mr}}$, the maximal moderately ramified extension of $\mathcal{K}'$. For this it suffices to show that $\mathcal{K}'^{\mathrm{mr}} = \mathcal{K}'^{\mathrm{nr}}\mathcal{K}^{\mathrm{mr}}$, or, equivalently, that the homomorphism of Galois groups $G(\mathcal{K}'^{\mathrm{mr}}/\mathcal{K}'^{\mathrm{nr}}) \to G(\mathcal{K}^{\mathrm{mr}}/\mathcal{K}^{\mathrm{nr}})$ is injective. By [Ber2], 2.4.4, the first (resp. second) group is canonically isomorphic to $\mathrm{Hom}(\sqrt{|\mathcal{K}^*|}/|\mathcal{K}^*|, \widetilde{\mathcal{K}}^{s*})$ (resp. $\mathrm{Hom}(\sqrt{|\mathcal{K}'^*|}/|\mathcal{K}'^*|, \widetilde{\mathcal{K}'}^{s*})$). Since $|\mathcal{K}^*|$ is a subgroup of finite index in the cyclic group $|\mathcal{K}'^*|$, then $\sqrt{|\mathcal{K}^*|} = \sqrt{|\mathcal{K}'^*|}$, and the required fact follows. The second statement is an easy consequence of the first one and Propositions 4.2 and 4.3.   □

§5. Smooth analytic spaces

Let (S, s) be a k-germ such that s is contained in the interior $\mathrm{Int}(S)$ of S (in particular, S is good at s) and $\mathcal{O}_{S,s}$ is a discrete valuation ring.

5.1. Theorem. *Suppose that the field k is perfect, and let n be an integer prime to $\mathrm{char}(k)$. Then for $\mathbf{X} = \mathbf{S}$ one has $\Psi_\eta(\mathbf{Z}/n\mathbf{Z})_\eta = (\mathbf{Z}/n\mathbf{Z})_{\overline{s}}$ and $R^q \Psi_\eta(\mathbf{Z}/n\mathbf{Z})_\eta = 0$ for $q \geq 1$.*

To prove the theorem, we show that the assumptions guarantee the smoothness of the k-analytic spaces $S_{\mathcal{L}}$ and $T_{\mathcal{L}}$ at the point $s_{\mathcal{L}}$, and, after that, we apply the cohomological purity theorems from [Ber2] and [Ber4].

5.2. Theorem. *Let X be a k-analytic space and K a perfect non-Archimedean field over k. Then, for a point $x \in X$, the following are equivalent:*

(a) *X is smooth at x;*
(b) *$x \in \mathrm{Int}(X)$ and there exists an open neighborhood $\mathcal{U}$ of x such that the K-analytic space $\mathcal{U} \widehat{\otimes} K$ is regular;*
(c) *$x \in \mathrm{Int}(X)$ and the K-analytic space $X \widehat{\otimes} K$ is regular at some point over x.*

Proof. The implications (a) $\Longrightarrow$ (b) $\Longrightarrow$ (c) are trivial. To prove the implication (c) $\Longrightarrow$ (a), we need the following fact.

5.3. Lemma. *The following properties of a k-analytic space X at a point $x \in X$ with $d = \dim_x(X)$ are equivalent:*

(a) *X is smooth at x;*
(b) *$x \in \mathrm{Int}(X)$ and $\Omega_{X,x}$ is a free $\mathcal{O}_{X,x}$-module of rank d;*
(c) *$x \in \mathrm{Int}(X)$ and $\Omega_{X,x}$ is generated over $\mathcal{O}_{X,x}$ by at most d elements.*

Proof. The implication (a) $\Longrightarrow$ (b) follows from [Ber2], 3.5.4. The implication (b) $\Longrightarrow$ (c) is trivial. Suppose that (c) is true. Shrinking X, we may assume that X is closed and there are functions $f_1, \ldots, f_d \in \mathcal{O}(X)$ such that the sheaf Ω_X is generated by $df_1, \ldots, df_d$. We claim that the morphism $f : X \to Y = \mathbf{A}^d$, defined by $f_1, \ldots, f_d$, is étale. Indeed, the exact sequence $f^*(\Omega_Y) \longrightarrow \Omega_X \longrightarrow \Omega_{X/Y} \longrightarrow 0$ (see [Ber2], 3.3.2(i)) implies that $\Omega_{X/Y} = 0$. It follows that f has discrete fibers. Since f is closed, f is quasifinite, by [Ber2], 3.1.4, and therefore f is unramified. We have to verify that f is flat. Let $y = f(y)$. Since $\mathcal{O}_{X,x}$ is a finite unramified $\mathcal{O}_{Y,y}$-algebra, and the ring $\mathcal{O}_{Y,y}$ is normal, by [SGA1], Exp. I, 9.5(ii), it suffices to show that the canonical homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ is injective. But this is clear because $\dim_y(Y) = d = \dim_x(X)$. $\square$

5.4. Corollary. *Let x be an inner point of a k-analytic space X such that the local ring $\mathcal{O}_{X,x}$ is regular and $\mathcal{H}(x) = k'$ or $k' \widehat{\otimes} K_r$, where k' is a finite separable extension of k and $r \notin \sqrt{|k^*|}$. Then X is smooth at x.*

Proof. Suppose first that $\mathcal{H}(x) = k'$. Then the set $\{x\}$ is Zariski closed in X. By [Ber2], 3.3.2(ii), there is an exact sequence $\mathbf{m}_x/\mathbf{m}_x^2 \longrightarrow \Omega_X \otimes_{\mathcal{O}_X} k' \longrightarrow \Omega_{k'/k} \longrightarrow 0$. It follows that $\Omega_{X,x}$ is generated over $\mathcal{O}_{X,x}$ by at most $\dim_{k'}(\mathbf{m}_x/\mathbf{m}_x^2) = \dim(\mathcal{O}_{X,x}) = \dim_x(X)$ elements. Suppose now that $\mathcal{H}(x) = k' \widehat{\otimes} K_r$. We can replace k by k' and assume that $\mathcal{H}(x) = K_r$. We may assume also that X is k-affinoid. Consider the canonical morphism $X' = X \widehat{\otimes} K_r \to X$. From [Ber2], 2.2.1 and 2.2.5, it follows that there exists a point $x' \in \pi^{-1}(x)$ which is contained in the Zariski open set $\mathcal{V}$ of regular points of X'. Furthermore, the fiber $\pi^{-1}(x)$ is isomorphic to the annulus $A(r,r)_{K_r} \subset \mathbf{A}^1_{K_r}$. It follows that there exists a point $x'' \in \mathcal{V} \cap \pi^{-1}(x)$ for which $\mathcal{H}(x'')$ is a finite separable extension of K_r. By the first case, $\Omega_{X',x''}$ is generated over $\mathcal{O}_{X',x''}$ by at most $\dim_{x''}(X')$ elements. Since $\Omega_{X',x''} = \Omega_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x''}$, it follows that $\Omega_{X,x}$ is generated over $\mathcal{O}_{X,x}$ by at most $\dim_{x''}(X') \le \dim_x(X)$ elements. $\square$

Suppose that $x \in \mathrm{Int}(X)$ and some point $x' \in X \widehat{\otimes} K$ over x is regular. By Lemma 5.3, it suffices to verify that $\Omega_{X,x}$ is generated over $\mathcal{O}_{X,x}$ by at most $\dim_x(X)$ elements. Since $\Omega_{X',x'} = \Omega_{X,x} \otimes_{\mathcal{O}_{X,x}} \mathcal{O}_{X',x'}$, it suffices to verify that $\Omega_{X',x'}$ is generated over $\mathcal{O}_{X',x'}$ by at most $\dim_{x'}(X') \le \dim_x(X)$ elements. The situation is reduced to the following. Suppose that the field k is perfect. Then, for any point $x \in \mathrm{Int}(X)$ such that the local ring $\mathcal{O}_{X,x}$ is regular, $\Omega_{X,x}$ is generated over $\mathcal{O}_{X,x}$ by at most $d = \dim_x(X)$ elements. For this we need the following fact.

5.5. Lemma. *Suppose that k is perfect, and let X be a closed k-analytic space. Then the set of points $x \in X$ with $\mathcal{H}(x) = k'$ or $k' \widehat{\otimes} K_r$ (as in Corollary 5.4) is everywhere dense in X.*

Proof. If the valuation on k is nontrivial, then the the space X is strictly k-analytic (see [Ber1], 3.1.2), and therefore the set $X_0 = \{x \in X \,|\, [\mathcal{H}(x) : k] < \infty\}$ is everywhere dense in X. Suppose that the valuation on k is trivial. It suffices to show that any affinoid neighborhood U of a point $x \in X$ contains a point y with $\mathcal{H}(y) = k'$ or $k' \widehat{\otimes} K_r$, where k' is a finite extension of k and $0 < r < 1$. For this we take a closed immersion $U \to E(0; r_1, \ldots, r_n) \subset \mathbf{A}^n$ with $|T_i(x)| < r_i$. Furthermore, we take a number $0 < r < 1$ with $|T_i(x)| < r^{l_i} < r_i$ for some integers l_i, and set $V = U \cap E(0; r^{l_1}, \ldots, r^{l_n})$. Then $V' = V \widehat{\otimes} K_r$ is a strictly K_r-affinoid space, and therefore the set V'_0 is everywhere dense in V'. Let y be the image in V of some

point $y' \in V_0'$. We claim that the field $\mathcal{H}(y)$ is of the required form. Suppose that $\mathcal{H}(y)$ is infinite over k. One has $k \subset \mathcal{H}(y) \subset \mathcal{H}(y')$ and $[\mathcal{H}(y') : K_r] < \infty$. Since $\widetilde{K}_r = k$, k is perfect, and the ring $\mathcal{H}(y)^\circ$ is Henselian, there exists a finite extension k' of k in $\mathcal{H}(y)$ with $k' \xrightarrow{\sim} \widetilde{\mathcal{H}(y)}$. Furthermore, the group $|\mathcal{H}(y)^*|$ is generated by a number $0 < r' < 1$. Let t be an element of $\mathcal{H}(y)$ with $|t| = r'$. Then the series $\sum_{i=-\infty}^{\infty} a_i t^i$, $a_i \in k'$, are convergent in $\mathcal{H}(y)$, and any element of $\mathcal{H}(y)$ can be represented in a unique way as such a series. It follows that $\mathcal{H}(y) = k' \widehat{\otimes} K_{r'}$. $\square$

Shrinking X, we may assume that X is closed, regular, and of pure dimension d. Let Y be the set of points $x \in X$ such that $\Omega_{X,x}$ is generated over $\mathcal{O}_{X,x}$ by at least $d+1$ elements, and assume that Y is nonempty. Then Y is a Zariski closed subset of X, and therefore it can be considered as a closed k-analytic space. By Lemma 5.5, there exists a point $y \in Y$ with $\mathcal{H}(y) = k'$ or $k' \widehat{\otimes} K_r$. By Corollary 5.4, $\Omega_{X,y}$ is generated over $\mathcal{O}_{X,y}$ by at most d elements. The latter is impossible, and therefore Theorem 5.2 is proved. $\square$

5.6. Corollary. *The set of smooth points in a closed k-analytic space X is Zariski open. Furthermore, if the field k is perfect, this set coincides with $\mathrm{Reg}(X)$, the set of regular points of X.* $\square$

Proof of Theorem 5.1. We can shrink S and assume that $\mathbf{S}_\eta$ comes from $S \backslash T$, where T is a Zariski closed subset of S, and that S and T are smooth. Furthermore, for a fixed finite separable extension $\mathcal{L}$ of $\mathcal{K}$, we can shrink S and assume that the morphism $\mathbf{S}_\mathcal{L} \to \mathbf{S}$ comes from a flat finite morphism $S_\mathcal{L} \to S$. Since $S_\mathcal{L}$ is regular at the point $s_\mathcal{L}$, from Theorem 5.2 it follows that we can shrink S and assume that $S_\mathcal{L}$ and $T_\mathcal{L}$, the preimage of T in $S_\mathcal{L}$, are smooth. By the cohomological purity theorem ([Ber2], 7.4.5, and [Ber4], 2.1), applied to the smooth pair of codimension one $S_\mathcal{L} \backslash T_\mathcal{L} \overset{j_\mathcal{L}}{\hookrightarrow} S_\mathcal{L} \overset{i_\mathcal{L}}{\hookleftarrow} T_\mathcal{L}$, one has $j_{\mathcal{L}*}(\mathbf{Z}/n\mathbf{Z})_{S_\mathcal{L} \backslash T_\mathcal{L}} = (\mathbf{Z}/n\mathbf{Z})_{S_\mathcal{L}}$, $R^1 j_{\mathcal{L}*}(\mu_{n,S_\mathcal{L} \backslash T_\mathcal{L}}) \xrightarrow{\sim} i_{\mathcal{L}*}(\mathbf{Z}/n\mathbf{Z})_{T_\mathcal{L}}$ and $R^q j_{\mathcal{L}*}(\mathbf{Z}/n\mathbf{Z})_{S_\mathcal{L} \backslash T_\mathcal{L}} = 0$ for $q \geq 2$. Theorem 5.1 easily follows from this. $\square$

5.7. Corollary. *Suppose that the field k is perfect, and let $\mathbf{X}$ be an $\mathbf{S}$-analytic space smooth over $\mathbf{S}$. Then, for any finite locally constant abelian sheaf F on $\mathbf{X}$ with torsion orders prime to $\mathrm{char}(k)$, one has $\Psi_\eta(F_\eta) = F_{\overline{s}}$ and $R^q \Psi_\eta(F_\eta) = 0$ for $q \geq 1$.* $\square$

§6. The comparison theorem for vanishing cycles

In this section we assume that the field k is perfect and (S, s) is a k-germ such that $s \in \mathrm{Int}(S)$ and $A = \mathcal{O}_{S,s}$ is a discrete valuation ring.

6.1. Theorem. *Let $\mathcal{X}$ be a scheme of finite type over $\mathcal{S} = \mathrm{Spec}(A)$, and let $\mathcal{F}$ be an abelian constructible sheaf on $\mathcal{X}_\eta$ with torsion orders prime to $\mathrm{char}(k)$. Then for any $q \geq 0$ there is a canonical isomorphism*

$$(R^q \Psi_\eta(\mathcal{F}))^{\mathrm{an}} \xrightarrow{\sim} R^q \Psi_\eta(\mathcal{F}^{\mathrm{an}}) .$$

6.2. Remark. Recall that, by Deligne's theorem 3.2 from [SGA4$\frac{1}{2}$], Th. Finitude, the sheaves $R^q \Psi_\eta(\mathcal{F})$ are constructible. The proof of Theorem 6.1 uses the induction reasoning from the proof of Deligne's theorem and does not work in the classical situation over $\mathbf{C}$. But in the case covered by the corresponding theorem

from [SGA7], Exp. XIV, the statement is easily deduced as follows from the comparison theorem for étale cohomology ([Ber2], 7.5.1, [Ber4], 3.1). (Moreover, in this case one does not need the assumption that the field k is perfect.) Namely, this is the case when $\mathcal{X} = \mathcal{Y} \times_\mathcal{R} \mathcal{S}$, where $\mathcal{R}$ is an algebraic curve over k, $\mathcal{Y}$ is a scheme of finite type over $\mathcal{R}$, and $\mathcal{S} \to \mathcal{R}$ is a morphism which induces an isomorphism of k-germs $(S, s) \xrightarrow{\sim} (\mathcal{R}^{\mathrm{an}}, s)$, and $\mathcal{F}$ comes from a constructible sheaf $\mathcal{G}$ on $\mathcal{Y}_\eta$, where $\mathcal{Y}_\eta$ is the preimage of $\mathcal{R} \backslash \{s\}$ in $\mathcal{Y}$. We may assume that $\mathcal{R}$ is regular and connected. Let K be the field of rational functions on $\mathcal{R}$, and fix an embedding $K^{\mathrm{s}} \hookrightarrow \mathcal{K}^{\mathrm{s}}$. Since K is everywhere dense in $\mathcal{K}$ and $\mathcal{K}$ is quasicomplete, one has $\mathcal{K}^{\mathrm{s}} = K^{\mathrm{s}}\mathcal{K}$. For a finite extension L of K, let $\mathcal{R}_L$ denote the normalization of $\mathcal{R}$ in L, and let $\mathcal{Y}_L = \mathcal{Y} \times_\mathcal{R} \mathcal{R}_L$. The embedding $L \hookrightarrow \mathcal{K}^{\mathrm{s}}$ defines a point $s_L \in \mathcal{R}_L$. There are morphisms $\mathcal{Y}_{\eta_L} = \mathcal{Y}_L \backslash (\mathcal{Y}_L)_{s_L} \xrightarrow{j_L} \mathcal{Y}_L \xleftarrow{\bar{i}_L} \mathcal{Y}_{\overline{s}}$, and one has $R^q \Psi_\eta(\mathcal{F}) = \varinjlim \bar{i}_L^*(R^q j_{L*} \mathcal{G})$, where L runs through finite extensions of K in K^{s}. On the other hand, since $\mathcal{K}^{\mathrm{s}} = K^{\mathrm{s}}\mathcal{K}$, one has

$$R^q \Psi_\eta(\mathcal{F}^{\mathrm{an}}) = \varinjlim \bar{i}_L^{\mathrm{an}*}(R^q j_{L*}^{\mathrm{an}} \mathcal{G}^{\mathrm{an}}).$$

The comparison theorem for étale cohomology implies that

$$(R^q \Psi_\eta(\mathcal{F}))^{\mathrm{an}} \xrightarrow{\sim} R^q \Psi_\eta(\mathcal{F}^{\mathrm{an}}).$$

Proof of Theorem 6.1. First of all we remark that it suffices to assume that $\mathcal{X}_\eta$ is everywhere dense in $\mathcal{X}$. We prove the theorem by induction on $d = \dim(\mathcal{X}_\eta)$.

Step 1. *The theorem is true for $d = 0$.*

We may assume that $\mathcal{X}$ is reduced. Using Proposition 4.1(i), we can replace $\mathcal{X}$ by its normalization, and therefore we may assume that $\mathcal{X}$ is the normalization of $\mathcal{S}$ in a finite extension $\mathcal{N}$ of $\mathcal{K}$. Since $\mathcal{F}$ has a resolution $0 \to \mathcal{F} \to \mathcal{F}^0 \to \mathcal{F}^1 \to \dots$ with $\mathcal{F}^i$ of the form $\varphi_{\eta*}(\mathbf{Z}/n\mathbf{Z})_{\mathcal{X}'_\eta}$, where $\varphi : \mathcal{X}' \to \mathcal{X}$ is the normalization of $\mathcal{X}$ in a finite extension of $\mathcal{N}$, we again can apply Proposition 4.1(i) and reduce the situation to the case $\mathcal{F} = \Lambda_{\mathcal{X}_\eta}$, where $\Lambda = \mathbf{Z}/n\mathbf{Z}$. Furthermore, since the scheme $\mathcal{X}_{\overline{s}}$ is zero-dimensional, it suffices to verify that $H^0(\mathcal{X}_{\overline{s}}, R^q \Psi_\eta(\Lambda_{\mathcal{X}_\eta})) \xrightarrow{\sim} H^0(\mathcal{X}_{\overline{s}}^{\mathrm{an}}, R^q \Psi_\eta(\Lambda_{\mathcal{X}_\eta^{\mathrm{an}}}))$ or, equivalently, that $H^q(\mathcal{X}_{\overline{\eta}}, \Lambda) \xrightarrow{\sim} H^q(\mathcal{X}_{\overline{\eta}}^{\mathrm{an}}, \Lambda)$. For the latter, we may assume that $\mathcal{N}$ is separable over $\mathcal{K}$, but then it suffices to consider the case $\mathcal{N} = \mathcal{K}$, i.e., $\mathcal{X} = \mathcal{S}$. In this case $\Psi_\eta(\Lambda_\eta) = \Lambda_{\overline{s}}$ and $R^q \Psi_\eta(\Lambda_\eta) = 0$ for $q > 0$ and, by Theorem 5.1, the similar facts are true for $\mathbf{S}$.

Suppose now that $d \geq 1$ and the theorem is true for the schemes whose generic fiber has dimension at most $d - 1$.

Step 2. *The homomorphism $(R^q \Psi_\eta(\mathcal{F}))^{\mathrm{an}} \to R^q \Psi_\eta(\mathcal{F}^{\mathrm{an}})$ is an isomorphism at any point of $\mathcal{X}_{\overline{s}}^{\mathrm{an}}$ whose image in the scheme $\mathcal{X}_s$ is not a closed point.*

Let $y \in \mathcal{X}_{\overline{s}}^{\mathrm{an}}$ be such a point, and let x and $\mathbf{x}$ be its images in $\mathcal{X}_s^{\mathrm{an}}$ and $\mathcal{X}_s$, respectively. Since our statement is local, we may assume that $\mathcal{X}$ is affine and, moreover, that $\mathcal{X}$ is a closed subscheme of the affine space $A_\mathcal{S}^m$. By hypothesis, there exists a projection $\varphi : \mathcal{X} \to A_\mathcal{S}^1$ such that $\mathbf{s}' = \varphi(\mathbf{x})$ is the generic point of the closed fiber of $A_\mathcal{S}^1$. Then we may shrink $\mathcal{X}$ and assume that the generic fiber of φ has dimension $d - 1$. Let s' be the image of x in the closed fiber of $\mathbf{A}_\mathcal{S}^1 = (A_\mathcal{S}^1)^{\mathrm{an}}$, i.e., $s' = \varphi^{\mathrm{an}}(x)$, and let (S', s') be the k-germ $(\mathbf{A}_\mathcal{S}^1, s')$. To prove the statement, it suffices to show that the inverse images of the sheaves $R^q \Psi_\eta(\mathcal{F})^{\mathrm{an}}$ and $R^q \Psi_\eta(\mathcal{F}^{\mathrm{an}})$ on $\mathcal{X}_{\overline{s}'}^{\mathrm{an}}$ are isomorphic.

We set $\mathcal{S}' = \mathrm{Spec}(\mathcal{O}_{S', s'})$ and denote by $\mathcal{S}''$ the spectrum of the Henselization of the local ring $\mathcal{O}_{A_\mathcal{S}^1, \mathbf{s}'}$. Since the ring $\mathcal{O}_{S', s'}$ is Henselian, there is a canonical

morphism of schemes

$$\mathcal{S}' = \{s', \eta'\} \to \mathcal{S}'' = \{s'', \eta''\}.$$

Consider the following commutative diagram with Cartesian squares:

$$
\begin{array}{ccccc}
\mathcal{X} & \longrightarrow & A^1_{\mathcal{S}} & \longrightarrow & \mathcal{S} \\
\uparrow & & \uparrow & & \\
\mathcal{X}'' & \longrightarrow & \mathcal{S}'' & & \\
\uparrow & & \uparrow & & \\
\mathcal{X}' & \longrightarrow & \mathcal{S}' & &
\end{array}
$$

Let $\mathcal{F}'$ (resp. $\mathcal{F}''$) be the inverse image of $\mathcal{F}$ on $\mathcal{X}'_{\eta'}$ (resp. $\mathcal{X}''_{\eta''}$), and let $\mathcal{K}'$ (resp. $\mathcal{K}''$) be the fraction field of $\mathcal{O}_{\mathcal{S}',s'}$ (resp. $\mathcal{O}_{\mathcal{S}'',s''}$). (We note that the field $\mathcal{K}''$ is quasicomplete because its ring of integers is Henselian.) Let us fix embeddings of fields $\mathcal{K}^{\mathrm{s}} \hookrightarrow \mathcal{K}''^{\mathrm{s}} \hookrightarrow \mathcal{K}'^{\mathrm{s}}$ over the canonical embeddings $\mathcal{K} \hookrightarrow \mathcal{K}'' \hookrightarrow \mathcal{K}'$. We get a homomorphism of Galois groups $G_{\eta'} \to G_{\eta''} \to G_{\eta}$ and morphisms

$$
\begin{array}{ccccc}
\mathcal{X}'_{s'} & \xrightarrow{\beta} & \mathcal{X}''_{s''} & \xrightarrow{\alpha} & \mathcal{X}_{\bar{s}} \\
\uparrow & & & & \uparrow \\
\mathcal{X}'^{\mathrm{an}}_{s'} & \xrightarrow{\lambda} & & & \mathcal{X}^{\mathrm{an}}_{\bar{s}}
\end{array}
$$

By Lemma 3.4 from [SGA4$\frac{1}{2}$], Th. Finitude, there is an isomorphism of sheaves on $\mathcal{X}''_{s''}$, $\alpha^*\left(R^q\Psi_{\eta}(\mathcal{F})\right) \xrightarrow{\sim} R^q\Psi_{\eta''}(\mathcal{F}'')^P$, where $P = G(\mathcal{K}''^{\mathrm{s}}/\mathcal{K}''^{\mathrm{nr}}\mathcal{K}^{\mathrm{s}})$ (it is a pro-p-group for $p = \mathrm{char}(k)$). Furthermore, since the formation of vanishing cycles is compatible with any base change (*loc. cit.*, Proposition 3.7), there is an isomorphism of sheaves on $\mathcal{X}'_{s'}$, $\beta^*\left(R^q\Psi_{\eta''}(\mathcal{F}'')\right) \xrightarrow{\sim} R^q\Psi_{\eta'}(\mathcal{F}')$, compatible with the action of Galois groups. Therefore, there is an isomorphism of sheaves on $\mathcal{X}'_{s'}$, $(\alpha\beta)^*\left(R^q\Psi_{\eta}(\mathcal{F})\right) \xrightarrow{\sim} R^q\Psi_{\eta'}(\mathcal{F}')^P$. Applying the induction hypothesis to the morphism $\mathcal{X}' \to \mathcal{S}'$, we get an isomorphism of sheaves on $\mathcal{X}'^{\mathrm{an}}_{s'}$

$$\lambda^*\left(R^q\Psi_{\eta}(\mathcal{F})^{\mathrm{an}}\right) \xrightarrow{\sim} R^q\Psi_{\eta'}(\mathcal{F}'^{\mathrm{an}})^P \ .$$

On the other hand, Corollary 4.4 gives an isomorphism

$$\lambda^*\left(R^q\Psi_{\eta}(\mathcal{F}^{\mathrm{an}})\right) \xrightarrow{\sim} R^q\Psi_{\eta'}(\mathcal{F}'^{\mathrm{an}})^Q \ ,$$

where $Q = G(\mathcal{K}'^{\mathrm{s}}/\mathcal{K}'^{\mathrm{nr}}\mathcal{K}^{\mathrm{s}})$. It is clear that the image of Q under the homomorphism $G_{\eta'} \to G_{\eta''}$ is contained in P. Thus, Step 2 follows from the following fact.

6.2. Lemma. *The homomorphism $Q \to P$ is surjective.*

Proof. Consider the diagram of embeddings of fields

$$
\begin{array}{ccccccccc}
\mathcal{K}' & \longrightarrow & \mathcal{K}'^{\mathrm{nr}} & \longrightarrow & \mathcal{K}'^{\mathrm{nr}}\mathcal{K}^{\mathrm{s}} & \longrightarrow & \mathcal{K}'^{\mathrm{nr}}\mathcal{K}''^{\mathrm{s}} & \longrightarrow & \mathcal{K}'^{\mathrm{s}} \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
\mathcal{K}'' & \longrightarrow & \mathcal{K}''^{\mathrm{nr}} & \longrightarrow & \mathcal{K}''^{\mathrm{nr}}\mathcal{K}^{\mathrm{s}} & \longrightarrow & \mathcal{K}''^{\mathrm{s}} & & \\
\uparrow & & \uparrow & & \uparrow & & & & \\
\mathcal{K} & \longrightarrow & \mathcal{K}^{\mathrm{nr}} & \longrightarrow & \mathcal{K}^{\mathrm{s}} & & & &
\end{array}
$$

It induces a commutative diagram of homomorphisms of Galois groups

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & Q & \longrightarrow & G(\mathcal{K}'^{\mathrm{s}}/\mathcal{K}'^{\mathrm{nr}}) & \longrightarrow & G(\mathcal{K}'^{\mathrm{nr}}\mathcal{K}^{\mathrm{s}}/\mathcal{K}'^{\mathrm{nr}}) & \longrightarrow & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & P & \longrightarrow & G(\mathcal{K}''^{\mathrm{s}}/\mathcal{K}''^{\mathrm{nr}}) & \longrightarrow & G(\mathcal{K}''^{\mathrm{nr}}\mathcal{K}^{\mathrm{s}}/\mathcal{K}''^{\mathrm{nr}}) & \longrightarrow & 0
\end{array}
$$

Since the third vertical homomorphism is injective (both groups are subgroups of G_η), it suffices to show that the second one is surjective.

Let $\mathcal{L}$ be a finite extension of $\mathcal{K}''^{\mathrm{nr}}$ in $\mathcal{K}''^{\mathrm{s}}$, and let $\mathcal{L}' = \mathcal{K}'^{\mathrm{nr}}\mathcal{L}$. It suffices to verify that

$$
[\mathcal{L} : \mathcal{K}''^{\mathrm{nr}}] = [\mathcal{L}' : \mathcal{K}'^{\mathrm{nr}}] \ .
$$

First of all, we remark that since $\mathcal{K}'^{\mathrm{nr}}$ and $\mathcal{K}''^{\mathrm{nr}}$ are quasicomplete discrete valuation fields, then the number on the left hand side is equal to $[|\mathcal{L}^*| : |\mathcal{K}''^{\mathrm{nr}*}|][\widetilde{\mathcal{L}} : \widetilde{\mathcal{K}''^{\mathrm{nr}}}]$, and that on the right hand side is equal to $[|\mathcal{L}'^*| : |\mathcal{K}'^{\mathrm{nr}*}|][\widetilde{\mathcal{L}'} : \widetilde{\mathcal{K}'^{\mathrm{nr}}}]$. One has $|\mathcal{K}'^{\mathrm{nr}*}| = |\mathcal{K}''^{\mathrm{nr}*}| = |\mathcal{K}^*|$. Since $|\mathcal{L}^*| \subset |\mathcal{L}'^*|$, it suffices to verify that

$$
[\widetilde{\mathcal{L}} : \widetilde{\mathcal{K}''^{\mathrm{nr}}}] \le [\widetilde{\mathcal{L}'} : \widetilde{\mathcal{K}'^{\mathrm{nr}}}] \ .
$$

One has $\widetilde{\mathcal{K}'^{\mathrm{nr}}} = \kappa(s')^{\mathrm{s}}$ and $\widetilde{\mathcal{K}''^{\mathrm{nr}}} = k(\mathbf{s}')^{\mathrm{s}}$. By Theorem 3.4, the field $\kappa(s')$ is separable over $k(\mathbf{s}')$. It follows that $[\kappa(s')^{\mathrm{s}}\widetilde{\mathcal{L}} : \kappa(s')^{\mathrm{s}}] = [\widetilde{\mathcal{L}} : k(\mathbf{s}')^{\mathrm{s}}]$. Since $\kappa(s')^{\mathrm{s}}\widetilde{\mathcal{L}} \subset \widetilde{\mathcal{L}'}$, the inequality follows. $\qquad\square$

Step 3. *The homomorphism considered is an isomorphism over all points of $\mathcal{X}_{\bar{s}}^{\mathrm{an}}$.*

Since our statement is local, we may assume that $\mathcal{X}$ is affine. After that we may assume that $\mathcal{X}$ is projective over $\mathcal{S}$. Let φ denote the canonical morphism $\mathcal{X} \to \mathcal{S}$. Define a complex $\Delta^{\cdot}$ by the exact triangle in $D_{G_\eta}^+(\mathcal{X}_{\bar{s}}^{\mathrm{an}})$,

$$
\longrightarrow R\Psi_\eta(\mathcal{F})^{\mathrm{an}} \longrightarrow R\Psi_\eta(\mathcal{F}^{\mathrm{an}}) \longrightarrow \Delta^{\cdot} \longrightarrow \ .
$$

We have to show that the cohomology sheaves $H^q(\Delta^{\cdot})$ of the complex $\Delta^{\cdot}$ are trivial. By Step 2, we know, at least, that they are concentrated at the points of $\mathcal{X}_{\bar{s}}^{\mathrm{an}}$ whose images in $\mathcal{X}_s$ are closed points. In particular, these are $\widehat{\mathcal{H}(s)^{\mathrm{a}}}$-points of the $\widehat{\mathcal{H}(s)^{\mathrm{a}}}$-analytic space $\mathcal{X}_{\bar{s}}^{\mathrm{an}}$. To show that $H^q(\Delta^{\cdot}) = 0$, we need the following fact which is a purely non-Archimedean phenomenon.

6.3. Proposition. *Suppose that k is algebraically closed. Let X be a Hausdorff k-analytic space, and let F be an abelian sheaf on X such that either (1) for any point $x \in X$ with $F_x \ne 0$, one has $x \in X(k)$ (as in our situation), or (2) F is torsion with torsion orders prime to $\mathrm{char}(\widetilde{k})$ and, for any point $x \in X$ with $F_x \ne 0$, one has $\widetilde{\mathcal{H}(x)} = \widetilde{k}$ and $|\mathcal{H}(x)^*| = |k^*|$ (i.e., $d(x) = 0$ in the notation of [Ber1], §9). Then $H_c^q(X, F) = 0$ for all $q \ge 1$. Furthermore, if $H_c^0(X, F) = 0$, then $F = 0$.*

Proof. Consider the morphism of sites $\pi : X_{\mathrm{ét}} \to |X|$. By [Ber2], 4.2.4, one has $(R^q\pi_* F)_x = H^q(G_{\mathcal{H}(x)}, F_x)$, $q \ge 0$. If $x \in X(k)$, then the group $G_{\mathcal{H}(x)}$ is trivial. If $\widetilde{\mathcal{H}(x)} = \widetilde{k}$ and $|\mathcal{H}(x)^*| = |k^*|$, then $G_{\mathcal{H}(x)}$ is a p-group, where $p = \mathrm{char}(\widetilde{k})$, by [Ber2], 2.4.4. In both cases one has $R^q\pi_* F = 0$ for all $q \ge 1$. Therefore the Leray spectral sequence of the morphism π gives an isomorphism $H_c^q(|X|, \pi_* F) \xrightarrow{\sim} H_c^q(X, F)$, $q \ge 0$. Thus, to prove the statement, it suffices to show that the restriction of F to the usual topology of X satisfies the following condition: for any compact subset $\Sigma \subset X$, the canonical homomorphism $F(X) \to F(\Sigma)$ is surjective (see [God], Ch. II, §3.5).

Let $f \in F(\Sigma)$. Since the set Σ has a basis of paracompact neighborhoods, then, by *loc. cit.*, 3.3.1, f extends to an open neighborhood $\mathcal{U}$ of Σ. Furthermore, since Σ is compact, we can find a finite family of affinoid domains $V_1, \ldots, V_n$ such that $V = \cup_{i=1}^n V_i \subset \mathcal{U}$ and $\Sigma \subset \mathcal{V} = \cup_{i=1}^n \mathrm{Int}(V_i/X)$. We remark that it suffices to show that

$$\mathrm{Supp}(f|_{\mathcal{V}}) = \mathrm{Supp}(f|_{\mathcal{U}}) \cap V.$$

Indeed, from the equality it follows that the set $\mathrm{Supp}(f|_{\mathcal{V}})$ is compact, and therefore there exists an element $g \in F(X)$ which is zero outside $\mathrm{Supp}(f|_{\mathcal{V}})$ and coincides with f on $\mathcal{V}$. The equality follows from the following lemma (the field k in it is not assumed to be algebraically closed).

6.4. Lemma. *Any point x of a k-analytic space X, for which the extension $\widetilde{\mathcal{H}(x)}/\widetilde{k}$ is algebraic and the group $|\mathcal{H}(x)^*|/|k^*|$ is torsion, is contained in the interior of X.*

Proof. We may assume that $X = \mathcal{M}(\mathcal{A})$ is k-affinoid. By [Ber1], 2.5.2, we have to verify that, for any bounded homomorphism $k\{r^{-1}T\} \to \mathcal{A} : T \mapsto f$, there exists a polynomial $P = T^n + a_1 T^{n-1} + \cdots + a_n \in k[T]$ such that $|a_i| \leq r^i$, $1 \leq i \leq n$, and $|P(f)(x)| < r^n$. This is evident if $|f(x)| < r$, and therefore we assume that $|f(x)| = r$. One has $r^m = |a|$ for some integer $m \geq 1$ and an element $a \in k^*$. In particular, $|\frac{f^m}{a}(x)| = 1$. Furthermore, we can find a polynomial $Q = T^l + a_1 T^{l-1} + \cdots + a_l \in k^{\circ}[T]$ with $|Q(\frac{f^m}{a})(x)| < 1$. It follows that the polynomial $P(T) = a^l Q\left(\frac{T^m}{a}\right)$ is the required one. $\square$

By Proposition 6.3, to show that $H^q(\Delta^{\cdot}) = 0$, it suffices to show that the co-homology of the complex $R\varphi_{\bar{s}}^{\mathrm{an}}{}_*(\Delta^{\cdot})$ is trivial or, equivalently, that the canonical morphism $R\varphi_{\bar{s}}^{\mathrm{an}}{}_*(R\Psi_\eta(\mathcal{F})^{\mathrm{an}}) \longrightarrow R\varphi_{\bar{s}}^{\mathrm{an}}{}_*(R\Psi_\eta(\mathcal{F}^{\mathrm{an}}))$ is an isomorphism. By Proposition 4.1(i), the complex on the left hand side is $R\Psi_\eta\left(R\varphi_\eta^{\mathrm{an}}{}_*(\mathcal{F}^{\mathrm{an}})\right)$ and, by the similar fact from algebraic geometry and the comparison theorem for cohomology with compact support, the complex on the right hand side is $(R\Psi_\eta(R\varphi_{\eta_*}\mathcal{F}))^{\mathrm{an}}$. Since $(R\varphi_{\eta_*}\mathcal{F})^{\mathrm{an}} \xrightarrow{\sim} R\varphi_\eta^{\mathrm{an}}{}_*(\mathcal{F}^{\mathrm{an}})$ (Proposition 3.3), the required statement follows from the fact that the theorem is true for $\mathcal{S}$ (Step 1). The theorem is proved. $\square$

The following statement is deduced from the comparison theorem [Ber2], 7.5.1, and theorem 6.1 in the same way as the theorem [SGA4$\frac{1}{2}$], 1.1, is deduced from the corresponding Theorems [SGA4$\frac{1}{2}$], 1.9 and 3.2.

6.5. Corollary. *Let $\varphi : \mathcal{Y} \to \mathcal{X}$ be a morphism of finite type between schemes of locally finite type over $\mathcal{S}$ or over $\mathrm{Spec}(\mathcal{A})$, where $\mathcal{M}(\mathcal{A})$ is a one-dimensional regular k-affinoid space. Then, for any constructible sheaf $\mathcal{F}$ on $\mathcal{Y}$ with torsion orders prime to $\mathrm{char}(\widetilde{k})$ and any $q \geq 0$, there is a canonical isomorphism*

$$(R^q \varphi_* \mathcal{F})^{\mathrm{an}} \xrightarrow{\sim} R^q \varphi_*^{\mathrm{an}}(\mathcal{F}^{\mathrm{an}}) . \qquad \square$$

§7. An application

Let A be a Henselian discrete valuation ring with fraction field $\mathcal{K}$ and algebraically closed residue field k, and let $\mathcal{S} = \mathrm{Spec}(A) = \{s, \eta\}$. A finite discrete $G_\eta = G_{\mathcal{K}}$-module Λ defines on every scheme $\mathcal{Z}$ over $\mathcal{K}$ a finite locally constant sheaf $\Lambda_{\mathcal{Z}}$. We note that any morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ between schemes of finite type over $\mathcal{S}$ and any closed point $\mathbf{y} \in \mathcal{Y}_s$ give rise to homomorphisms of finite abelian

groups $\theta^q(\varphi, \Lambda) : (R^q j_* \Lambda_{\mathcal{X}_\eta})_{\mathbf{x}} \to (R^q j_* \Lambda_{\mathcal{Y}_\eta})_{\mathbf{y}}$ and $R^q \Psi_\eta (\Lambda_{\mathcal{X}_\eta})_{\mathbf{x}} \to R^q \Psi_\eta (\Lambda_{\mathcal{Y}_\eta})_{\mathbf{y}}$, where $\mathbf{x} = \varphi(\mathbf{y})$ and $q \geq 0$. Furthermore, for a scheme $\mathcal{X}$ of finite type over $\mathcal{S}$ and a closed point $\mathbf{x} \in \mathcal{X}_s$, let $\widehat{\mathcal{X}}_{/\mathbf{x}}$ denote the formal completion of $\mathcal{X}$ at $\mathbf{x}$, i.e., $\widehat{\mathcal{X}}_{/\mathbf{x}} = \mathrm{Spf}(\widehat{\mathcal{O}}_{\mathcal{X}, \mathbf{x}})$. It is a formal scheme over $\widehat{\mathcal{S}} = \mathrm{Spf}(\widehat{A})$. Formal schemes considered here are only of this type. The purpose of this section is to prove the following two theorems.

Suppose that A is equicharacteristic, i.e., $\mathrm{char}(\mathcal{K}) = \mathrm{char}(k)$.

7.1. Theorem. *One can associate homomorphisms*

$$\theta^q(\alpha, \Lambda) : (R^q j_* \Lambda_{\mathcal{X}_\eta})_{\mathbf{x}} \to (R^q j_* \Lambda_{\mathcal{Y}_\eta})_{\mathbf{y}} \quad and \quad R^q \Psi_\eta (\Lambda_{\mathcal{X}_\eta})_{\mathbf{x}} \to R^q \Psi_\eta (\Lambda_{\mathcal{Y}_\eta})_{\mathbf{y}}$$

with any morphism of formal schemes over $\widehat{\mathcal{S}}$, $\alpha : \widehat{\mathcal{Y}}_{/\mathbf{y}} \to \widehat{\mathcal{X}}_{/\mathbf{x}}$, and any finite discrete G_η-module Λ of order prime to $\mathrm{char}(k)$, so that they possess the following properties:

(a) *if α is induced by a morphism $\varphi : \mathcal{Y} \to \mathcal{X}$ over $\mathcal{S}$ with $\varphi(\mathbf{y}) = \mathbf{x}$, then $\theta^q(\alpha, \Lambda) = \theta^q(\varphi, \Lambda)$;*
(b) *if $\beta : \widehat{\mathcal{Z}}_{/\mathbf{z}} \to \widehat{\mathcal{Y}}_{/\mathbf{y}}$ is a similar morphism, then $\theta^q(\alpha\beta, \Lambda) = \theta^q(\beta, \Lambda) \circ \theta^q(\alpha, \Lambda)$;*
(c) *$\theta^q(\alpha, \Lambda)$ is functorial on Λ.*

The second theorem is proved for the schemes $\mathcal{X}$ of finite type over $\mathcal{S}$ satisfying one of the following assumptions:

(1) the generic fiber $\mathcal{X}_\eta$ of $\mathcal{X}$ is smooth;
(2) $\mathcal{X} = \mathcal{Z} \times_{\mathcal{R}} \mathcal{S}$, where $\mathcal{R}$ is an algebraic curve over k, $\mathcal{Z} \to \mathcal{R}$ is a morphism of finite type, and $\mathcal{S} \to \mathcal{R}$ is a morphism induced by a homomorphism $\mathcal{O}_{\mathcal{R},s} \to A$ for which $\widehat{\mathcal{O}}_{\mathcal{R},s} \xrightarrow{\sim} \widehat{A}$.

The assumptions (1) and (2) are necessary to apply a result from [Ber3] on the finiteness of the cohomology groups of certain compact k-analytic spaces. (Of course, the latter fact should be true for arbitrary compact k-analytic spaces, and therefore the assumptions are superfluous.)

7.2. Theorem. *Given $\widehat{\mathcal{X}}_{/\mathbf{x}}$ and $\widehat{\mathcal{Y}}_{/\mathbf{y}}$, where each of $\mathcal{X}$ and $\mathcal{Y}$ satisfies (1) or (2), and a finite discrete G_η-module Λ of order prime to $\mathrm{char}(k)$, there exists $n \geq 1$ such that, for any pair of morphisms $\alpha, \beta : \widehat{\mathcal{Y}}_{/\mathbf{y}} \to \widehat{\mathcal{X}}_{/\mathbf{x}}$ over $\widehat{\mathcal{S}}$ that coincide modulo the n-th power of the maximal ideal of $\widehat{\mathcal{O}}_{\mathcal{Y},\mathbf{y}}$, one has $\theta^q(\alpha, \Lambda) = \theta^q(\beta, \Lambda)$ for all $q \geq 0$.*

Proof. By a result of Deligne ([SGA4$\frac{1}{2}$], Th. Finitude, 3.7), the formation of vanishing cycles is compatible with any base change. Furthermore, since A is Henselian, then the Galois groups of $\mathcal{K}$ and $\widehat{\mathcal{K}}$ coincide (see [Ber2], §2.4), and therefore the canonical morphism from the spectral sequence $\mathcal{H}^p(I, R^q \Psi_\eta(\Lambda_{\mathcal{X}_\eta})) \implies \bar{i}^*(R^{p+q} j_* \Lambda_{\mathcal{X}_\eta})$ over A to the similar spectral sequence over $\widehat{A}$ shows that the sheaves $R^q j_*(\Lambda_{\mathcal{X}_\eta})$ do not change if we replace A by its completion. Thus, we may assume that A is complete.

Since A is equicharacteristic, it is isomorphic to $k[[T]]$. We endow the field k with the trivial valuation. The ring $k[[T]]$ is topologically isomorphic to the k-affinoid algebra $k\{r^{-1}T\}$ for any $0 < r < 1$. In particular, $A = \mathcal{O}(S) = \mathcal{O}_{S,s}$, where S is the open unit disc $D(0;1)$ in $\mathbf{A}^1$ with center s at zero. The scheme $\mathcal{X}$ over A gives rise, for each $0 < r < 1$, to a k-analytic space closed over $E(0;r) = \mathcal{M}(k\{r^{-1}T\})$. These k-analytic spaces are glued together to a k-analytic space closed over $S = D(0;1)$

which will be denoted by $\mathcal{X}^{\mathrm{an}}$. Shrinking $\mathcal{X}$, we can find regular functions $f_1, \ldots, f_d$ on $\mathcal{X}$ such that $T, f_1, \ldots, f_d$ generate the maximal ideal $\mathbf{m_x}$ of $\mathcal{O}_{\mathcal{X},\mathbf{x}}$. We set

$$\mathcal{X}^{\mathrm{an}}_{(x)} = \{y \in \mathcal{X}^{\mathrm{an}} \big| |f_i(y)| < 1, 1 \le i \le d\} \ .$$

It is an open subset of $\mathcal{X}^{\mathrm{an}}$, and it is clear that it does not change if we shrink $\mathcal{X}$ or replace $f_1, \ldots, f_d$ by a similar system of elements.

7.3. Lemma. (i) $\mathcal{O}(\mathcal{X}^{\mathrm{an}}_{(x)}) = \mathcal{O}_{\mathcal{X}^{\mathrm{an}},x} = \widehat{\mathcal{O}}_{\mathcal{X},\mathbf{x}}$;

(ii) *if $\mathcal{Y}$ is a scheme of finite type over A and $\mathbf{y}$ is a closed point of $\mathcal{Y}_s$, then there is a canonical bijection*

$$\mathrm{Hom}_S\left(\mathcal{Y}^{\mathrm{an}}_{(y)}, \mathcal{X}^{\mathrm{an}}_{(x)}\right) \xrightarrow{\sim} \mathrm{Hom}_A\left(\widehat{\mathcal{O}}_{\mathcal{X},\mathbf{x}}, \widehat{\mathcal{O}}_{\mathcal{Y},\mathbf{y}}\right) \ .$$

Proof. If $\mathcal{X}$ is the d-dimensional affine space over A, then $\mathcal{X}^{\mathrm{an}}_{(x)}$ is the $(d+1)$-dimensional open unit disc in $\mathbf{A}^{d+1}$ with center at zero, and one has

$$\mathcal{O}(\mathcal{X}^{\mathrm{an}}_{(x)}) = \mathcal{O}_{\mathcal{X}^{\mathrm{an}},x} = \widehat{\mathcal{O}}_{\mathcal{X},\mathbf{x}} = k[[T, T_1, \ldots, T_d]].$$

In the general case we can shrink $\mathcal{X}$ and assume that there is a closed immersion $i : \mathcal{X} \to \mathcal{Z} = \mathbf{A}^d_{\mathrm{Spec}(A)}$. Then $\mathcal{X}^{\mathrm{an}}_{(x)} = \mathcal{Z}^{\mathrm{an}}_{(z)} \cap \mathcal{X}^{\mathrm{an}}$, where $\mathbf{z} = i(\mathbf{x})$. Since $\mathcal{Z}^{\mathrm{an}}_{(z)}$ is a Stein space, then the canonical homomorphism $\mathcal{O}(\mathcal{Z}^{\mathrm{an}}_{(z)}) \to \mathcal{O}(\mathcal{X}^{\mathrm{an}}_{(x)})$ is surjective, and (i) follows. To prove (ii), it suffices to verify that it is true when $\mathcal{X}$ is the d-dimensional affine space over A. In this case, the left hand side is $\mathcal{O}(\mathcal{Y}^{\mathrm{an}}_{(y)})^d$ and the right hand side is $(\widehat{\mathcal{O}}_{\mathcal{Y},\mathbf{y}})^d$, and therefore the required fact follows from (i). $\quad\square$

Theorem 7.1 follows directly from Lemma 7.3, Theorem 6.1 and Corollary 6.5.

Furthermore, since the groups $(R^q j_* \Lambda_{\mathcal{X}_\eta})_\mathbf{x}$ and $R^q \Psi_\eta(\Lambda_{\mathcal{X}_\eta})_\mathbf{x}$ are finite and the latter is an inductive limit of $\overline{i}^*_{\mathcal{L}}(R^q j_{\mathcal{L}*}\Lambda_{\mathcal{X}_\eta})$, where $\mathcal{L}$ runs through finite extensions of $\mathcal{K}$ in $\mathcal{K}^{\mathrm{s}}$, it suffices to prove Theorem 7.2 only for the groups $(R^q j_* \Lambda_{\mathcal{X}_\eta})_\mathbf{x}$. Since only finite number among these groups are non-zero, it suffices to prove Theorem 7.2 for each q separately. We set

$$\mathcal{X}^{\mathrm{an}}_{(x)\eta} = \{y \in \mathcal{X}^{\mathrm{an}}_{(x)} \big| T(y) \ne 0\} \ .$$

7.4. Lemma. *If $\mathcal{X}$ satisfies (1) or (2), then there is a canonical isomorphism*

$$H^q(\mathcal{X}^{\mathrm{an}}_{(x)\eta}, \Lambda) \xrightarrow{\sim} (R^q j_* \Lambda_{\mathcal{X}_\eta})_\mathbf{x} \ .$$

Proof. For $0 < r < 1$, we set

$$E(x; r) = \{y \in \mathcal{X}^{\mathrm{an}}_{(x)} \big| |T(y)| \le r, |f_i(y)| \le r, 1 \le i \le d\}.$$

It is an affinoid neighborhood of the point x in $\mathcal{X}^{\mathrm{an}}_{(x)}$. We also set $E_\eta(x; r) = \{y \in E(x; r) \big| T(y) \ne 0\}$. By Theorem 6.1, there is a canonical isomorphism

$$\varinjlim_{r \to 0} H^q(E_\eta(x; r), \Lambda) \xrightarrow{\sim} (R^q j_* \Lambda_{\mathcal{X}_\eta})_\mathbf{x}.$$

We will prove that for any $0 < r < 1$ there exists $0 < r' \le r$ (resp. $r \le r'' < 1$) such that the image of $H^q(E_\eta(x; r), \Lambda)$ in $H^q(E_\eta(x; r'), \Lambda)$ (resp. $H^q(E_\eta(x; r''), \Lambda)$ in $H^q(E_\eta(x; r), \Lambda))$ maps isomorphically onto $(R^q j_* \Lambda_{\mathcal{X}_\eta})_\mathbf{x}$. This will imply the required fact because, by [Ber2], 6.3.12, this will give an isomorphism

$$H^q(\mathcal{X}^{\mathrm{an}}_{(x)\eta}, \Lambda) \xrightarrow{\sim} \varprojlim_{r \to 1} H^q(E_\eta(x; r), \Lambda) \xrightarrow{\sim} (R^q j_* \Lambda_{\mathcal{X}_\eta})_\mathbf{x}.$$

If $\mathcal{X}$ satisfies (1), then $\mathcal{X}^{\mathrm{an}}_{(x)_\eta}$ is a smooth k-analytic space. If $\mathcal{X}$ satisfies (2), then $\mathcal{X}^{\mathrm{an}}_{(x)}$ is an open analytic domain in the analytification of the scheme $\mathcal{Z}$ over k. In both cases, Corollary 5.6 from [Ber3] implies that for any compact analytic domain $V \subset \mathcal{X}^{\mathrm{an}}_{(x)_\eta}$ the group $H^q(V, \Lambda)$ is finite. For $0 < r' \leq r$, we set $A(x; r', r) = \{y \in E(x; r) \big| |T(y)| \geq r'\}$ (it is an affinoid domain in $\mathcal{X}^{\mathrm{an}}_{(x)_\eta}$). Since $E_\eta(x; r)$ is a union of $A(x; r', r)$ over all $0 < r' \leq r$, then, by [Ber2], 6.3.12, the group $H^q(E_\eta(x; r), \Lambda)$ is a projective limit of the finite groups $H^q(A(x; r', r), \Lambda)$, and, in particular, it can be endowed with the structure of a profinite group.

Let r_0 be such that the homomorphism $H^q(E_\eta(x; r_0), \Lambda) \to (R^q j_* \Lambda_{\mathcal{X}_\eta})_{\mathbf{x}}$ is surjective and denote by P its kernel. For $0 < t \leq r_0$, let P_t denote the kernel of the continuous homomorphism $H^q(E_\eta(x; r_0), \Lambda) \to H^q(E_\eta(x; t), \Lambda)$. We claim that there exists $0 < t \leq r_0$ such that the subgroup P_t has finite index. Indeed, assume that this is not true. The closure $\overline{P}$ of P is an open subgroup of $H^q(E_\eta(x; r_0), \Lambda)$, and the subgroups P_t are nowhere dense in $\overline{P}$. Let $g_1, \ldots, g_m$ be representatives of the cosets of P in $\overline{P}$, and let $t_1, t_2, \ldots$ be an arbitrary sequence of positive numbers with $t_j \leq r_0$ and $t_j \to 0$ as $j \to \infty$. Then the compact space $\overline{P}$ is a union of the countable family of the nowhere dense subsets $g_i + P_{t_j}$. By the classical Baire Theorem (see [Kel], Ch. 6, Theorem 34), this is impossible.

Thus, we can find a number $0 < r'_0 \leq r_0$ such that the image of $H^q(E_\eta(x; r_0), \Lambda)$ in $H^q(E_\eta(x; r'_0), \Lambda)$ is finite. We can even decrease r'_0 and assume that this image maps isomorphically onto $(R^q j_* \Lambda_{\mathcal{X}_\eta})_{\mathbf{x}}$. Let t_0 be the number with $r'_0 = r_0^{t_0}$. We claim that any $0 < r < 1$ possesses the above property with $r' = r^{t_0}$. (This will give the required fact for $r'' = r^{t_0^{-1}}$.)

Let t be a positive number. Then for any k-affinoid algebra $\mathcal{A}$ the Banach k-algebra $\mathcal{A}^t$, which coincides with $\mathcal{A}$ and is endowed with the t-th power of the norm on $\mathcal{A}$, is also k-affinoid. This gives rise to a functor $\Phi_t : X = \mathcal{M}(\mathcal{A}) \mapsto X^t = \mathcal{M}(\mathcal{A}^t)$ from the category of k-affinoid spaces to itself, and one has $\Phi_t \circ \Phi_{t'} = \Phi_{tt'}$. The functors $\Phi_t : X \mapsto X^t$ extend in the evident way to the whole category of k-analytic spaces, and one has $\Phi_t \circ \Phi_{t'} \xrightarrow{\sim} \Phi_{tt'}$. If $Y \to X$ is an étale morphism, then the induced morphism $Y^t \to X^t$ is also étale. In this way we get an isomorphism of sites $X_{\mathrm{ét}} \xrightarrow{\sim} X^t_{\mathrm{ét}}$ that induces an isomorphism $F \mapsto F^t$ between the corresponding topoi. In particular, for any étale abelian sheaf F on X there is a canonical isomorphism $H^q(X, F) \xrightarrow{\sim} H^q(X^t, F^t)$.

For example, in our situation one has $E_\eta(x; r)^t \xrightarrow{\sim} E_\eta(x; r^t)$, and therefore if the image of $H^q(E_\eta(x; r_0), \Lambda)$ in $H^q(E_\eta(x; r_0^{t_0}), \Lambda)$ maps isomorphically onto $(R^q j_* \Lambda)_{\mathbf{x}}$, then the same is true for r_0^t instead of r_0. Since $\{r_0^t \big| t > 0\} = \{0 < r < 1\}$, we get our claim and the lemma. $\qquad\square$

Fix a number $0 < r < 1$. Given $\widehat{\mathcal{X}}_{/\mathbf{x}}$ and $\widehat{\mathcal{Y}}_{/\mathbf{y}}$ as in Theorem 7.2, we can find, by Lemma 7.4 and its proof, a number $0 < r' \leq r$ such that the canonical homomorphisms $H^q(\mathcal{X}^{\mathrm{an}}_{(x)_\eta}, \Lambda) \to H^q(A(x; r', r), \Lambda)$ and $H^q(\mathcal{Y}^{\mathrm{an}}_{(y)_\eta}, \Lambda) \to H^q(A(y; r', r), \Lambda)$ are injective. We now apply Theorem 7.1 from [Ber3] to the k-affinoid space $A(x; r', r)$. Since the functions $f_1, \ldots, f_d, \frac{1}{T}$ form a k-affinoid generating system for the k-affinoid algebra $\mathcal{O}(A(x; r', r))$, it follows that there exist $t_1, \ldots, t_d > 0$ such that for any pair of morphisms of Hausdorff k-analytic spaces $\varphi, \psi : Y \to A(x; r', r)$ over S with $\rho(\varphi^* f_i - \psi^* f_i) \leq t_i$, $1 \leq i \leq d$, the homomorphisms from the image of $H^q(\mathcal{X}^{\mathrm{an}}_{(x)_\eta}, \Lambda)$ in $H^q(A(x; r', r), \Lambda)$ to $H^q(Y, \Lambda)$ induced by φ and ψ coincide. Let

$n \geq 1$ be an integer with $r^n \leq t_i$, $1 \leq i \leq d$. Then, for any pair of morphisms $\alpha, \beta : \widehat{\mathcal{Y}}_{/\mathbf{y}} \to \widehat{\mathcal{X}}_{/\mathbf{x}}$ over $\widehat{\mathcal{S}}$ that coincide modulo the n-th power of the maximal ideal of $\widehat{\mathcal{O}}_{\mathcal{Y},\mathbf{y}}$, one has $\rho(\alpha^* f_i - \beta^* f_i) \leq t_i$ on $A(y; r', r)$. It follows that the homomorphisms from $H^q(\mathcal{X}^{\mathrm{an}}_{(x)_\eta}, \Lambda) \to H^q(\mathcal{Y}^{\mathrm{an}}_{(y)_\eta}, \Lambda)$ induced by α and β coincide, and therefore $\theta^q(\alpha, \Lambda) = \theta^q(\beta, \Lambda)$. Theorem 7.2 is proved. $\qquad\square$

Given $\widehat{\mathcal{X}}_{/\mathbf{x}}$, let $\mathcal{G}(\widehat{\mathcal{X}}_{/\mathbf{x}})$ denote the group of automorphisms of the formal scheme $\widehat{\mathcal{X}}_{/\mathbf{x}}$ over $\widehat{\mathcal{S}}$. By Theorem 7.1, the group $\mathcal{G}(\widehat{\mathcal{X}}_{/\mathbf{x}})$ acts on the finite groups $(R^q j_* \Lambda_{\mathcal{X}_\eta})_{\mathbf{x}}$ and $R^q \Psi_\eta(\Lambda_{\mathcal{X}_\eta})_{\mathbf{x}}$. Furthermore, for $n \leq 1$, let $\mathcal{G}_n(\widehat{\mathcal{X}}_{/\mathbf{x}})$ denote the subgroup of $\mathcal{G}(\widehat{\mathcal{X}}_{/\mathbf{x}})$ consisting of the automorphisms that are trivial modulo the n-th power of the maximal ideal of $\widehat{\mathcal{O}}_{\widehat{\mathcal{X}},\mathbf{x}}$. By Lemma 8.7 from [Ber3] the groups $\mathcal{G}_n(\mathcal{X}_{/\mathbf{x}})$, $n \geq 2$, are uniquely l-divisible for any prime $l \neq \mathrm{char}(k)$. The following statement easily follows from this fact and Theorem 7.1.

7.5. Corollary. *Given $\widehat{\mathcal{X}}_{/\mathbf{x}}$, where $\mathcal{X}$ satisfies (1) or (2), and a finitely generated $\mathbf{Z}_l$-module Λ, $l \neq \mathrm{char}(k)$, endowed with a continuous action of G_η, there exists $n \geq 1$ such that $\mathcal{G}_n(\widehat{\mathcal{X}}_{/\mathbf{x}})$ acts trivially on all of the groups $(R^q j_*(\Lambda/l^m\Lambda)_{\mathcal{X}_\eta})_{\mathbf{x}}$ and $(R^q \Psi_\eta(\Lambda/l^m\Lambda)_{\mathcal{X}_\eta})_{\mathbf{x}}$, $q \geq 0$, $m \geq 0$.* $\qquad\square$

7.6. Remark. Laumon proved a statement similar to that of Corollary 7.5 for the action of the automorphism group of the Henselization of $\mathcal{X}$ at $\mathbf{x}$ on $R^q \Psi_\eta(\Lambda_{\mathcal{X}_\eta})_{\mathbf{x}}$ under the assumption that the morphism $\mathcal{X} \to \mathcal{S}$ is smooth outside $\mathbf{x}$ (see [Lau], p. 34, 6.3.1). In the case when A is of mixed characteristic, Brylinski proved a similar statement under the assumptions that the morphism $\mathcal{X} \to \mathcal{S}$ is of relative dimension one and $\mathcal{X}_\eta$ is smooth (see [Bry]).

References

[Ber1]   V. G. Berkovich, *Spectral theory and analytic geometry over non-Archimedean fields*, Mathematical Surveys and Monographs, vol. 33, Amer. Math. Soc., Providence, RI, 1990. MR **91k:**32038

[Ber2]   ———, *Étale cohomology for non-Archimedean analytic spaces*, Publ. Math. IHES **78** (1993), 5–161. MR **95c:**14017

[Ber3]   ———, *Vanishing cycles for formal schemes*, Invent. Math. **115** (1994), 539–571. MR **95f:**14034

[Ber4]   ———, *On the comparison theorem for étale cohomology of non-Archimedean analytic spaces*, Israel J. Math. **92** (1995), 45–60. CMP 96:03

[Bry]   J. L. Brylinski, *Un lemme sur les cycles évanescents en dimension relative 1*, Ann. Scient. Éc. Norm. Sup. **19** (1986), 460–467.

[God]   R. Godement, *Topologie Algébrique et Théorie des Faisceaux*, Hermann, Paris, 1958. MR **21:**1583

[EGA1]   A. Grothendieck and J. Dieudonné, *Eléments de Géométrie Algébrique. I. Le langage des schémas*, Springer, Berlin-Heidelberg-New York, 1971. MR **36:**177a; MR **29:**1207

[EGA4]   ———, *Eléments de Géométrie Algébrique. IV. Étude local des schémas et des morphismes de schémas*, Publ. Math. IHES, **20** (1964), **24** (1965), **28** (1966), **32** (1967). MR **30:**3885; MR **33:**7330; MR **36:**178; MR **39:**220

[Kel]   J. L. Kelley, *General Topology*, D. Van Nostrand Company, Toronto-London-New York, 1955. MR **16:**1136c

[Lau]   G. Laumon, *Charactéristique d'Euler-Poincaré et sommes exponentielles*, Thése, Université de Paris-Sud, Orsay, 1983.

[SGA1]   A. Grothendieck, *Revêtements Étales et Groupe Fondamental*, Lecture Notes in Math. **224**, Springer, Berlin-Heidelberg-New York, 1971. MR **50:**7129

[SGA4]   M. Artin, A. Grothendieck, and J.-L. Verdier, *Théorie des Topos et Cohomologie Étale des Schémas*, Lecture Notes in Math. **269, 270, 305**, Springer, Berlin-Heidelberg-New York, 1972–1973. MR **50:**7130; MR **50:**7131; MR **50:**7132

[SGA4$\frac{1}{2}$]   P. Deligne et al., *Cohomologie Étale*, Lecture Notes in Math. **569**, Springer, Berlin-Heidelberg-New York, 1977. MR **57:**3132

[SGA7]   A. Grothendieck, P. Deligne, and N. Katz, *Groupes de Monodromie en Géométrie Algéb-rique*, Lecture Notes in Math. **288, 340**, Springer, Berlin-Heidelberg-New York, 1972–1973. MR **50:**7134; MR **50:**7135

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