Analytic geometry over $\mathbf{F}_1$

Vladimir Berkovich

Weizmann Institute of Science

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From p-adic differential equations to arithmetic algebraic geometry
on the occasion of Francesco Baldassarri's 60th birthday

Where did I come across them?

I came across them in searching for appropriate framework for so called skeletons of non-Archimedean analytic spaces and formal schemes.

What do they represent?

They represent schemes and non-Archimedean and complex analytic spaces which are defined independently on the ground field or ring (e.g., torus embeddings).

What are they good for?

- ?
- They are not good for proving Riemann hypothesis.
- Stupid question!

Plan

- $\mathbf{F}_1$-algebras
- Banach $\mathbf{F}_1$-algebras
- K-affinoid algebras
- K-affinoid spaces
- K-analytic spaces
- Non-Archimedean analytic spaces
- Complex analytic spaces

Definition of $\mathbf{F}_1$-algebras

Definition

- An $\mathbf{F}_1$*-algebra* is a commutative multiplicative monoid A provided with elements $1 = 1_A$ and $0 = 0_A$ such that $1 \cdot f = f$ and $0 \cdot f = 0$ for all $f \in A$.
- A *homomorphism of* $\mathbf{F}_1$*-algebras* $\varphi : A \to B$ is a map compatible with the operations on A and B and takes 0_A and 1_A to 0_B and 1_B, respectively.
- A is *integral* if $fh = gh$ implies either $f = g$ or $h = 0$.
- A is an $\mathbf{F}_1$*-field* if every nonzero element of A is invertible (i.e., $\check{A} = A^*$, where $\check{A} = A \backslash \{0\}$).

If S is a sub-semigroup of A, one can define the *localization* $A \to S^{-1}A$ *of* A *with respect to* S. If A has no zero divisors, the localization of A with respect to $\check{A}$ is the *fraction* $\mathbf{F}_1$*-field* of A.

Examples of $\mathbf{F}_1$-algebras

Examples

- The *field of one element* $\mathbf{F}_1 = \{0, 1\}$.
- The *trivial* $\mathbf{F}_1$*-algebra* $\{0\}$ (with $0 = 1$).
- The multiplicative monoid $A^{\cdot}$ of any commutative ring A with unity is an $\mathbf{F}_1$-algebra. For example, $\mathbf{F}_1 = \mathbf{F}_2^{\cdot}$.
- $\mathbf{R}_+$ is an $\mathbf{F}_1$-field, and $\mathbf{Z}_+$ is an integral $\mathbf{F}_1$-algebra.
- Given an $\mathbf{F}_1$-algebra A, the set $A[T_1, \dots, T_n]$ consisting of 0 and $aT_1^{\mu_1} \dots T_n^{\mu_n}$ with $a \in \check{A}$ and $\mu_1, \dots, \mu_n \in \mathbf{Z}_+$ and provided with the evident multiplication is an $\mathbf{F}_1$-algebra.
- Any finite idempotent $\mathbf{F}_1$-algebra A, provided with the partial ordering ($e \leq f$ if $ef = f$) is a lattice. Conversely, any lattice structure on a finite set A gives rise to an $\mathbf{F}_1$-algebra structure on A: $ef = \sup(e, f)$, $0 = \sup(e)$ and $1 = \inf(e)$.

Ideals of $\mathbf{F}_1$-algebras

Definition

An *ideal* of an $\mathbf{F}_1$-algebra A is an equivalence relation which is compatible with the operation on A, i.e., a subset $E \subset A \times A$ which is an equivalence relation and an $\mathbf{F}_1$-subalgebra.

Given an ideal $E \subset A \times A$, the set of equivalence classes A/E is an $\mathbf{F}_1$-algebra.

Definition

A *Zariski ideal* is a subset $\mathbf{a} \subset A$ with the property that $fg \in \mathbf{a}$ whenever $f \in \mathbf{a}$ and $g \in A$.

Examples of ideals

Examples

- The diagonal $\Delta(A) \subset A \times A$ is the minimal ideal of A.
- If $\mathbf{a}$ is a Zariski ideal, the set $\Delta(A) \cup (\mathbf{a} \times \mathbf{a})$ is an ideal, and the corresponding quotient is denoted by $A/\mathbf{a}$.
- The union of any family of Zariski ideals is a Zariski ideal. The maximal (nontrivial) Zariski ideal of A is $\mathbf{m} = A \backslash A^*$.
- The *nilradical of A* is $\mathrm{nil}(A) = \{(f,g) \big|$ there is $n \geq 1$ with $f^i = g^i$ for all $i \geq n\}$. A is *reduced* if $\mathrm{nil}(A) = \Delta(A)$.
- If G is a subgroup of A^*, the set of pairs of the form (f, fg) with $f \in A$ and $g \in G$ is an ideal, and the corresponding quotient is the set A/G of orbits under the action of G on A. If A is an $\mathbf{F}_1$-field, each ideal of A is of this form.
- Given a homomorphism $\varphi : A \to B$, its *kernel* is the ideal $\mathrm{Ker}(\varphi) = \{(f,g) \in A \times A \big| \varphi(f) = \varphi(g)\}$, and its *Zariski kernel* is the Zariski ideal $\mathrm{Zker}(\varphi) = \{f \in A \big| \varphi(f) = 0\}$.

Prime ideals and spectra

Definition

- An ideal Π of A is *prime* if it is nontrivial and possesses the property that, if $(fh, gh) \in \Pi$, then either $(f, g) \in \Pi$ or $(h, 0) \in \Pi$, i.e., the quotient A/Π is a nontrivial integral $\mathbf{F}_1$-algebra.
- A Zariski ideal $\mathfrak{p} \subset A$ is *prime* if it is nontrivial and possesses the property that, if $fg \in \mathfrak{p}$, then either $f \in \mathfrak{p}$ or $g \in \mathfrak{p}$, i.e., the quotient $A/\mathfrak{p}$ is nontrivial and has no zero divisors.

The sets of prime ideals and of Zariski prime ideals of A are denoted by $\mathrm{Fspec}(A)$ and $\mathrm{Zspec}(A)$, the *spectrum* and the *Zariski spectrum*, respectively.

Prime ideals and spectra

For $x \in \mathrm{Fspec}(A)$ and $\mathfrak{p} \in \mathrm{Zspec}(A)$, the fraction $\mathbf{F}_1$-fields of A/Π_x and $A/\mathfrak{p}$ are denoted by $\kappa(x)$ and $\kappa(\mathfrak{p})$, respectively, and the image of $f \in A$ in $\kappa(x)$ is denoted by $f(x)$.

There is a surjective map

$$\mathrm{Fspec}(A) \to \mathrm{Zspec}(A) : x \mapsto \mathfrak{p}_x = \{f \in A \big| f(x) = 0\} \ .$$

Fact

There is a canonical bijection between the preimage of $\mathfrak{p} \in \mathrm{Zspec}(A)$ and the set of subgroups of $\kappa(\mathfrak{p})^*$.

The prime ideal of A that corresponds to the unit subgroup of $\kappa(\mathfrak{p})^*$ is denoted by $\Pi_\mathfrak{p}$. One has $\mathrm{nil}(A) = \bigcap_\mathfrak{p} \Pi_\mathfrak{p}$ and

$$\Pi_\mathfrak{p} = \{(f, g) \big| \text{either } f, g \in \mathfrak{p}, \text{ or } fh = gh \text{ for some } h \notin \mathfrak{p}\} \ .$$

The spectrum $\mathrm{Fspec}(A)$ of an $\mathbf{F}_1$-algebra A is provided with the topology whose base is formed by the sets $\bigcap_{i=1}^{n} D(f_i, g_i)$, where for $f, g \in A$ one sets $D(f, g) = \{x \in \mathrm{Fspec}(A) \big| f(x) \neq g(x)\}$.

Fact

$\mathrm{Fspec}(A)$ *is a quasicompact sober topological space.*

Corollary

The map $x \mapsto \overline{\{x\}}$ *gives rise to a bijection between* $\mathrm{Fspec}(A)$ *and the set of closed irreducible subsets of* $\mathrm{Fspec}(A)$*, and one has* $\overline{\{x\}} = \mathrm{Fspec}(A/\Pi_x)$.

Topology on the spectrum

Let $\mathcal{X} = \mathrm{Fspec}(A)$. For a Zariski prime ideal $\mathfrak{p} \subset A$, we set:

- $\mathcal{X}_\mathfrak{p} = \{x \in \mathcal{X} \big| f(x) = 0 \text{ for all } f \in \mathfrak{p}\}$;
- $\check{\mathcal{X}}_\mathfrak{p} = \{x \in \mathcal{X}_\mathfrak{p} \big| f(x) \neq 0 \text{ for all } f \notin \mathfrak{p}\}$;
- $\mathcal{X}^{(\mathfrak{p})} = \overline{\check{\mathcal{X}}_\mathfrak{p}}$.

Corollary

- *One has* $\mathcal{X}^{(\mathfrak{p})} = \overline{\{\Pi_\mathfrak{p}\}} = \mathrm{Fspec}(A/\Pi_\mathfrak{p})$*;*
- *the map* $\mathfrak{p} \mapsto \mathcal{X}^{(\mathfrak{p})}$ *gives rise to a bijection between the set of minimal prime ideals of A and the set of irreducible components of* $\mathcal{X}$*.*

The natural topology on the Zariski spectrum $\mathrm{Zspec}(A)$ is not interesting, and it is better to consider $\mathrm{Zspec}(A)$ as a partially ordered set with $\mathfrak{p} \leq \mathfrak{q}$ if $\mathfrak{q} \subset \mathfrak{p}$. This partial ordering admits the infimum operation.

Weakly decomposable $\mathbf{F}_1$-algebras

Definition

A $\mathbf{F}_1$-algebra A is said to be *weakly decomposable* if the set of irreducible components of $\mathcal{X} = \mathrm{Fspec}(A)$ is finite.

Fact

Suppose A is weakly decomposable, and let $\mathcal{U} \in \pi_0(\mathcal{X})$. Then

- *there is a unique maximal Zariski prime ideal $\mathfrak{p} = \mathfrak{p}^{(\mathcal{U})}$ with $\check{\mathcal{X}}_{\mathfrak{p}} \subset \mathcal{U}$; we set $\mathcal{U} \leq \mathcal{V}$ if $\mathfrak{p}^{(\mathcal{U})} \leq \mathfrak{p}^{(\mathcal{V})}$ (i.e., $\mathfrak{p}^{(\mathcal{U})} \supset \mathfrak{p}^{(\mathcal{V})}$);*
- *there is a unique maximal idempotent $e \in A$ with $e\big|_{\mathcal{U}} = 1$;*
- *there is an isomorphism of finite partially ordered sets $\check{I}_A \stackrel{\sim}{\to} \pi_0(\mathcal{X})$, where I_A is the idempotent $\mathbf{F}_1$-subalgebra of A.*

Definition of Banach $\mathbf{F}_1$-algebras

Definition

A *Banach* $\mathbf{F}_1$*-algebra* is an $\mathbf{F}_1$-algebra A provided with a *Banach norm*, i.e., a function $|| \ || : A \to \mathbf{R}_+$ possessing the following two properties:
(1) $||f|| = 0$ if and only if $f = 0$;
(2) $||fg|| \leq ||f|| \cdot ||g||$ for all $f, g \in A$.

Banach $\mathbf{F}_1$-algebras form a category with respect to *bounded homomorphisms*, i.e., homomorphisms of $\mathbf{F}_1$-algebras $\varphi : A \to B$ for which there exists a constant $C > 0$ with $||\varphi(f)|| \leq C||f||$ for all $f \in A$.

Examples of Banach $\mathbf{F}_1$-algebras

Examples

- A Banach $\mathbf{F}_1$-algebra K is a *real valuation* $\mathbf{F}_1$ *-field* if it is an $\mathbf{F}_1$-field and its norm is multiplicative. In this case $|K| = \{|\lambda| \big| \lambda \in K\}$ is an $\mathbf{F}_1$-subfield of $\mathbf{R}_+$, and $K/K^{**} \overset{\sim}{\to} |K|$, where $K^{**} = \{\lambda \in K^* \big| |\lambda| = 1\}$.
- The multiplicative monoid $A^{\cdot}$ of any commutative Banach ring A can be considered as a Banach $\mathbf{F}_1$-algebra.
- For a Banach $\mathbf{F}_1$-algebra A and numbers $r_1, \dots, r_n > 0$, the $\mathbf{F}_1$-algebra $A[T_1, \dots, T_n]$ provided with the norm $||aT_1^{\mu_1} \dots T_n^{\mu_n}|| = ||a|| r_1^{\mu_1} \dots r_n^{\mu_n}$ is a Banach $\mathbf{F}_1$-algebra denoted by $A\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\}$.
- Given an ideal E of A, the function $||\overline{f}|| = \inf\{||f|| \big| f \in \overline{f}\}$ is a Banach norm on A/E if and only if E is *closed*. The *closure of* E is $\overline{E} = E \cup (\mathbf{a} \times \mathbf{a})$, where $\mathbf{a} = \{f \in A \big| ||\overline{f}|| = 0\}$.

The spectrum of a Banach $\mathbf{F}_1$-algebra

Definition

The *spectrum* $\mathcal{M}(A)$ of a Banach $\mathbf{F}_1$-algebra A is the set of all bounded homomorphisms of Banach $\mathbf{F}_1$-algebras $|\ | : A \to \mathbf{R}_+$.

For a point $x \in \mathcal{M}(A)$, the seminorm $|\ |_x : A \to \mathbf{R}_+$ gives rise to a multiplicative norm on the $\mathbf{F}_1$-field $\mathcal{H}(x) = \kappa(\mathfrak{p}_x)$, where $\mathfrak{p}_x = \mathrm{Zker}(|\ |_x)$ and to a character $A \to \mathcal{H}(x) : f \mapsto f(x)$.

The spectrum $\mathcal{M}(A)$ is provided with the weakest topology with respect to which all real valued functions of the form $x \mapsto |f(x)| = |f|_x$ are continuous.

Fact

If A is nontrivial, $\mathcal{M}(A)$ is a nonempty compact space.

Corollary

$f \in A$ is invertible $\Longleftrightarrow f(x) \neq 0$ for all $x \in \mathcal{M}(A)$.

The spectral radius and Gelfand transform

The *spectral radius* of an element $f \in A$ is the number

$$\rho(f) = \lim_{n\to\infty} \sqrt[n]{||f^n||} = \inf_n \sqrt[n]{||f^n||} \,.$$

The function $f \mapsto \rho(f)$ is a bounded seminorm on A.

Fact

For any $f \in A$, one has $\rho(f) = \max_{x\in\mathcal{M}(A)} |f(x)|$.

Corollary

If $X = \mathcal{M}(A)$, the Gelfand transform $^\wedge : A \to \mathcal{C}(X)$ *is isometric with respect to the spectral norm.*

One has $\mathrm{Ker}(^\wedge) = \{(f,g) \big| |f(x)| = |g(x)| \text{ for all } x \in X\}$. If $|A| = A/\mathrm{Ker}(^\wedge)$ and $\widehat{A} = \mathrm{Im}(^\wedge)$, then $\mathcal{M}(\widehat{A}) \stackrel{\sim}{\to} \mathcal{M}(|A|) \stackrel{\sim}{\to} \mathcal{M}(A)$. The canonical bounded homomorphism $|A| \to \widehat{A}$ is a bijection, but is not an isomorphism in general.

K-affinoid algebras

Let K be a real valuation $\mathbf{F}_1$-field.

Definition

A K*-affinoid algebra* is a Banach K-algebra A for which there exists an admissible epimorphism $K\{r_1^{-1}T_1, \dots, r_n^{-1}T_n\} \to A$.

Example

Every real valuation K-field K' with finitely generated group $\mathrm{Coker}(K^* \to K'^*)$ is a K-affinoid algebra. For example, $\mathcal{H}(x)$ of every point $x \in \mathcal{M}(A)$ is a K-affinoid algebra.

Example: R-polytopal algebras

Let R be an $\mathbf{F}_1$-subfield of $\mathbf{R}_+$.

Definition

- An *R-affinoid polytope* in $\mathbf{R}_+^n$ is a subset defined by a finite number of equalities $f(t) = g(t)$ with $f, g \in R[T_1, \dots, T_n]$ and inequalities $t_i \leq r_i$ with $r_i > 0$ for all $1 \leq i \leq n$.
- For an R-affinoid polytope $V \subset \mathbf{R}_+^n$, A_V denotes the Banach R-algebra of continuous functions $V \to \mathbf{R}_+$ which are restrictions of functions from $R[T_1, \dots, T_n]$.
- An *R-polytopal algebra* is a Banach R-algebra isomorphic to the R-algebra A_V of some R-affinoid polytope $V \subset \mathbf{R}_+^n$.

Fact

Any R-polytopal algebra is R-affinoid.

The spectrum of a K-affinoid algebra

Let A be a K-affinoid algebra, and fix an epimorphism as above. Then $\mathcal{M}(A)$ is identified with a compact subset $X \subset \mathbf{R}_+^n$.

Fact

- *The kernel of the induced admissible epimorphism $|K|\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to A/K^{**}$ is finitely generated and, in particular, X is an $|K|$-affinoid polytope.*
- *The bijection $|A| \to \widehat{A}$ is an isomorphism of $|K|$-polytopal algebras.*
- *$Zspec(A)$ is finite, and A is weakly decomposable.*
- *The map $X \to \mathrm{Zspec}(A) : x \mapsto \mathfrak{p}_x$ is surjective.*

For a Zariski prime ideal $\mathfrak{p} \subset A$, we set:

- $X_\mathfrak{p} = \{x \in X \big| f(x) = 0 \text{ for all } f \in \mathfrak{p}\}$;
- $\check{X}_\mathfrak{p} = \{x \in X_\mathfrak{p} \big| f(x) \neq 0 \text{ for all } f \notin \mathfrak{p}\}$;
- $X^{(\mathfrak{p})} = \overline{\check{X}_\mathfrak{p}}$.

Properties of the spectrum

Fact

- $X_{\mathfrak{p}} = \mathcal{M}(A/\mathfrak{p})$.
- *The prime ideals* $\Pi_{\mathfrak{p}}$ *are closed.*
- $X^{(\mathfrak{p})} = \mathcal{M}(A^{(\mathfrak{p})})$, *where* $A^{(\mathfrak{p})} = A/\Pi_{\mathfrak{p}}$; *in particular, if* A *is integral, then* $X = X^{(0)}$.
- *If* A *is polytopal without zero divisors and* $X = X^{(0)}$, *then* A *is integral.*
- *For every* $U \in \pi_0(X)$, *there is a unique maximal Zariski prime ideal* $\mathfrak{p} = \mathfrak{p}^{(U)}$ *with* $\check{X}_{\mathfrak{p}} \subset U$; *we set* $U \leq V$ *if* $\mathfrak{p}^{(U)} \leq \mathfrak{p}^{(V)}$ *(i.e.,* $\mathfrak{p}^{(U)} \supset \mathfrak{p}^{(V)}$*);*
- *There is a unique maximal idempotent* $e \in A$ *with* $e\big|_U = 1$;
- *There is an isomorphism of finite partially ordered sets* $\check{I}_A \xrightarrow{\sim} \pi_0(X)$.

Properties of K-affinoid algebras

Let A be a K-affinoid algebra.

Fact

- *For every non-nilpotent element $f \in A$, there exists $C > 0$ such that $||f^n|| \le C\rho(f)^n$ for all $n \ge 1$.*
- *If A is reduced, then there exists $C > 0$ such that $||f|| \le C\rho(f)$ for all $f \in A$.*
- *If $\mathrm{nil}(A)$ is prime, then $\mathrm{Ker}(A \to \widehat{A}) = \{(f, g) \big| f^n = g^n h$ for some $n \ge 1$ and $h \in A$ with $|h(x)| = 1$ for all $x \in \mathcal{M}(A)\}$.*

Corollary

Let $\varphi : A \to B$ be a bounded homomorphism to a K-affinoid algebra B. Given $f_1, \ldots, f_n \in B$ and $r_1, \ldots, r_n > 0$ with $r_i \ge \rho(f_i)$, there exists a unique bounded homomorphism $A\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\} \to B$ extending φ and sending T_i to f_i.

K-affinoid spaces

The category K-$\mathcal{A}ff$ of *K-affinoid spaces* is, by definition, the category opposite to that of K-affinoid algebras.

Definition

A closed subset $V \subset X = \mathcal{M}(A)$ is an *affinoid domain* if there is a homomorphism of K-affinoid algebras $A \to A_V$ such that

- the image of $\mathcal{M}(A_V)$ in X lies in V;
- any homomorphism of K-affinoid algebras $A \to B$ such that the image of $\mathcal{M}(B)$ in X lies in V goes through a unique homomorphism of K-affinoid algebras $A_V \to B$.

Fact

- *The induced map $\mathcal{M}(A_V) \to V$ is bijective and, for every point $y \in \mathcal{M}(A_V)$ with the image $x \in X$, there is an isometric isomorphism $\mathcal{H}(x) \stackrel{\sim}{\to} \mathcal{H}(y)$;*
- *the induced map $\mathrm{Zspec}(A_V) \to \mathrm{Zspec}(A)$ is injective.*

Rational and Weierstrass domains

Fact

- *Given* $f_1, \dots, f_n, g \in A$ *and* $p_1, \dots, p_n, q > 0$*, the subset* $V = \{x \in X \big| |f_i(x)| \le p_i |g(x)|, |g(x)| \ge q\}$ *is an affinoid domain with respect to the homomorphism* $A \to A_V = A\{p_1^{-1}T_1, \dots, p_m^{-1}T_m, qS\}/E$ *, where* E *is the closed ideal generated by the pairs* (gT_i, f_i) *and* $(gS, 1)$*;*
- *the canonical homomorphism* $A_g \to A_V$ *is surjective, and its kernel coincides with the Zariski kernel.*

An affinoid domain of the above form is called *rational*. If $g = 1$ and $q = 1$, it is called *Weierstrass*.

Fact

Every point of X *has a fundamental system of neighborhoods consisting of rational domains.*

Description of affinoid domains

For Zariski prime ideals $\mathfrak{p} \le \mathfrak{q}$ (i.e., $\mathfrak{p} \supset \mathfrak{q}$), the canonical bounded homomorphism $A^{(\mathfrak{p})} = A/\Pi_{\mathfrak{p}} \to A^{(\mathfrak{q})}$ induces a continuous map $\tau_{\mathfrak{q}\mathfrak{p}} : X^{(\mathfrak{q})} = \mathcal{M}(A^{(\mathfrak{q})}) \to X^{(\mathfrak{p})}$.

For a subset $U \subset X$, we set $U_{\mathfrak{p}} = U \cap X_{\mathfrak{p}}$, $\check{U}_{\mathfrak{p}} = U \cap \check{X}_{\mathfrak{p}}$, $U^{(\mathfrak{p})} = U \cap X^{(\mathfrak{p})}$, and $\mathcal{I}(U) = \{\mathfrak{p} \in \mathrm{Zspec}(A) \big| \check{U}_{\mathfrak{p}} \neq \emptyset\}$.

Fact

A subset $U \subset X$ is an affinoid domain if and only if

- *for every $\mathfrak{p} \in \mathcal{I}(U)$, $U^{(\mathfrak{p})}$ is a connected rational domain in $X^{(\mathfrak{p})}$;*
- *if $\mathfrak{p} \le \mathfrak{q}$ in $\mathcal{I}(U)$, then $\tau_{\mathfrak{q}\mathfrak{p}}(U^{(\mathfrak{q})}) \subset U^{(\mathfrak{p})}$;*
- *the set $\mathcal{I}(V)$ is preserved under the infimum operation.*

Properties of affinoid domains

Let U be an affinoid domain in X.

Fact

- *If U is connected then it is rational, and it is Weierstrass if and only if $U \cap X_{\mathbf{m}} \neq \emptyset$;*
- *if X is connected and $U \cap X_{\mathbf{m}} \neq \emptyset$, then U is connected.*
- *there exists a decreasing sequence of affinoid domains $U_1 \supset U_2 \supset \cdots \supset U$ such that*
  - *U_{n+1} is a Weierstrass domain in U_n and lies in the topological interior of U_n in X;*
  - $\bigcap_{n=1}^{\infty} U_n = U$;
  - *the canonical homomorphisms $A_{U_n} \to A_U$ are bijections.*

Acyclic K-affinoid spaces

For a finite affinoid covering $\mathcal{U} = \{U_i\}_{i \in I}$ of a K-affinoid space $X = \mathcal{M}(A)$, we set

$$A_{\mathcal{U}} = \mathrm{Ker}(\prod_{i \in I} A_{U_i} \rightrightarrows \prod_{i,j \in I} A_{U_i \cap U_j})$$

and provide $A_{\mathcal{U}}$ with the supremum norm.

Fact

The canonical map $A \to A_{\mathcal{U}}$ is an admissible monomorphism.

Definition

X is said to be *acyclic* if the above map is a bijection for any $\mathcal{U}$.

Acyclic K-affinoid spaces

Fact

Each point $x \in X$ has an affinoid neighborhood U such that every affinoid domain $x \in V \subset U$ is acyclic.

Corollary

- *Given finite affinoid coverings $\mathcal{U}$ and $\mathcal{V}$ such that $\mathcal{V}$ is a refinement of $\mathcal{U}$, the canonical map $A_{\mathcal{U}} \to A_{\mathcal{V}}$ is an admissible monomorphism;*
- *there exists $\mathcal{U}$ such that, for any refinement $\mathcal{V}$, the above admissible monomorphism is a bijection.*

The full subcategory of K-$\mathcal{A}ff$ consisting of acyclic K-affinoid spaces is denoted by K-$\mathcal{C}aff$, and the category with the same family of objects and affinoid domain embeddings as morphisms is denoted by K-$\mathcal{C}aff^{ad}$.

Definition

- A family τ of subsets of a topological space X is a *quasinet* if, for every $x \in X$, there exist $V_1, \dots, V_n \in \tau$ such that $x \in V_1 \cap \dots \cap V_n$ and $V_1 \cup \dots \cup V_n$ is a neighborhood of x.
- A quasinet τ is a *net* if, for every pair $U, V \in \tau$, $\tau|_{U \cap V}$ is a quasinet on $U \cap V$.

We consider τ as a category and denote by $\mathcal{T}$ the canonical function $\tau \to \mathcal{T}op$ to the category of topological spaces $\mathcal{T}op$. We also denote by $\mathcal{T}^a$ the canonical functor K-$\mathcal{C}aff^{ad} \to \mathcal{T}op$.

Definition

A *K-analytic space* is a triple (X, A, τ), where X is a locally Hausdorff topological space, τ is a net of compact subsets of X, and A is an *acyclic affinoid atlas on X with the net* τ, i.e., a pair consisting of a functor $A : \tau \to K$-$\mathcal{C}aff^{ad}$ and an isomorphism of functors $\mathcal{T}^a \circ A \xrightarrow{\sim} \mathcal{T}$.

The category $K\text{-}\widetilde{\mathcal{A}n}$

Definition

A *strong morphism* $\varphi : (X, A, \tau) \to (X', A', \tau)$ is a pair:

- a continuous map $\varphi : X \to X'$, such that for every $U \in \tau$ there exists $U' \in \tau'$ with $\varphi(U) \subset U'$,
- a compatible system of morphisms of K-affinoid spaces $\varphi_{U/U'} : U \to U'$ with $\varphi_{U/U'} = \varphi\big|_U$ (as maps) for all pairs $U \in \tau$ and $U' \in \tau'$ with $\varphi(U) \subset U'$.

Strong morphisms define a category $K\text{-}\widetilde{\mathcal{A}n}$.

Definition

A strong morphism $\varphi : (X, A, \tau) \to (X', A', \tau')$ is a *quasi-isomorphism* if

- φ is a homeomorphism of topological spaces $X \xrightarrow{\sim} X'$,
- for every pair $U \in \tau$ and $U' \in \tau'$ with $\varphi(U) \subset U'$, $\varphi_{U/U'}$ is an affinoid domain embedding.

The category K-$\mathcal{A}n$

Definition

The *category of K-analytic spaces K-$\mathcal{A}n$* is the category of fractions of K-$\widetilde{\mathcal{A}n}$ with respect to the system of quasi-isomorphisms.

Morphisms in the category K-$\mathcal{A}n$ can be described as follows First of all

Fact

- *If W is an acyclic affinoid domain in some $U \in \tau$, it is an acyclic affinoid domain in any $V \in \tau$ that contains W;*
- *the family $\overline{\tau}$ consisting of all W as above is a net on X, and there exists an acyclic affinoid atlas $\overline{A}$ on X with the net $\overline{\tau}$ which extends A.*

The category K-$\mathcal{A}n$

If σ is a net on X, we write $\sigma \prec \tau$ if $\sigma \subset \overline{\tau}$. Then $\overline{A}$ defines an acyclic K-affinoid atlas A_σ with the net σ, and there is a canonical quasi-isomorphism $(X, A_\sigma, \sigma) \to (X, A, \tau)$. The system of nets σ with $\sigma \prec \tau$ is filtered and, for any K-analytic space $(\mathcal{X}', A', \tau')$, one has

$$\mathrm{Hom}((X, A, \tau), (X', A', \tau')) = \varinjlim_{\sigma \prec \tau} \mathrm{Hom}_{\widetilde{\mathcal{A}n}}((X, A_\sigma, \sigma), (X', A', \tau')).$$

Example

Let $X = \mathcal{M}(A)$ be a K-affinoid space. Then the family τ_c of acyclic affinoid domains in X is a net, and there is an acyclic affinoid atlas A_c with the net τ_c. In this way we get a functor $K\text{-}\mathcal{A}ff \to K\text{-}\mathcal{A}n : X \mapsto (X, A_c, \tau_c)$. For a K-affinoid space $Y = \mathcal{M}(B)$, $\mathrm{Hom}((Y, B_c, \sigma_c), (X, A_c, \tau_c)) = \mathrm{Hom}(A, B_\mathcal{V})$, where $\mathcal{V}$ is a finite covering of Y by acyclic affinoid domains.

Analytic domains

Let (X, A, τ) be a K-analytic space.

Definition

A subset $Y \subset X$ is an *analytic domain* if $\overline{\tau}\big|_Y$ is a quasinet.

In this case $\overline{\tau}\big|_Y$ is a net and the atlas $\overline{A}$ defines a K-analytic space $(Y, \overline{A}, \overline{\tau}\big|_Y)$. A *p-affinoid domain* is an analytic domain isomorphic to a K-affinoid space. A K-analytic space is said to be *good* if every point has an acyclic affinoid neighborhood.

Fact

The family $\widehat{\tau}$ of acyclic affinoid domains is a net on X, and there is a unique (up to a canonical isomorphism) acyclic K-affinoid atlas $\widehat{A}$ on X with the net $\widehat{\tau}$ that extends A (the maximal acyclic affinoid atlas *on X).*

G-topology on a K-analytic space

Definition

The *G-topology* X_{G} is the Grothendieck topology on the family of analytic domains in X defined by the following pretopology: the set of coverings of an analytic domain $W \subset X$ is formed by families of analytic domains which are quasinets on W.

Any representable presheaf is a sheaf on X_{G}.

Example

The presheaf representable by $\mathcal{M}(K\{r^{-1}T\})$ is a sheaf on X_{G} denoted by $\mathcal{O}^r_{X_{\mathrm{G}}}$. The inductive limit $\varinjlim \mathcal{O}^r_X$ is a sheaf of K-algebras on X_{G} denoted by $\mathcal{O}_{X_{\mathrm{G}}}$ and called the *structural sheaf* on X_{G}. Its restriction to X is denoted by $\mathcal{O}_X$.

If V is acyclic affinoid, one has $\mathcal{O}(V) = A_V$. If V is compact, $\mathcal{O}(V)$ is a Banach K-algebra and, for any finite covering $\{V_i\}$ of V, the map $\mathcal{O}(V) \to \prod_i \mathcal{O}(V_i)$ is an admissible monomorphism.

Closed immersions and separated morphisms

A morphism $\varphi : Y \to X$ is a *closed immersion* if there is a quasinet τ on X such that, for every $U \in \tau$, $\varphi^{-1}(U) \to U$ is a closed immersion of acyclic K-affinoid spaces.

If $\varphi : Y \to X$ is a closed immersion and X is good, then every point of X has an acyclic affinoid neighborhood U with the above property.

The category K-$\mathcal{A}n$ admits fiber products, and a morphism $\varphi : Y \to X$ is called *separated* if the diagonal morphism $Y \to Y \times_X Y$ is a closed immersion.

If $\varphi : Y \to X$ is and X is Hausdorff, then so is Y. If X is Hausdorff and good, it is separated.

Piecewise K-affinoid spaces

Definition

- A K-analytic space is *piecewise K-affinoid* if it admits a closed immersion of X in a K-affinoid space.
- An analytic domain is said to be *piecewise affinoid* if it is isomorphic to a piecewise K-affinoid space.

Piecewise K-affinoid spaces form a category $K\text{-}\mathcal{P}aff$.

Fact

- *Piecewise K-affinoid spaces are good;*
- *the category of piecewise K-affinoid spaces is preserved under finite disjoint unions and fiber products;*
- *for every piecewise K-affinoid space X, there exists a finite family of closed immersions $\varphi : Y_i \to X$ with integral K-affinoid spaces Y_i such that $X = \bigcup_i \varphi_i(Y_i)$.*

The relative interior of a morphism

Let $\varphi : Y = \mathcal{M}(B) \to X = \mathcal{M}(A)$ be a morphism of K-affinoid spaces.

Definition

The *relative interior* of φ is the set $\mathrm{Int}(Y/X)$ consisting of the points $y \in Y$ with the following property: for every non-nilpotent element $g \in B$ with $|g(y)| = \rho(g)$, one has $g(y)^n = f(y)$ for some $n \geq 1$ and $f \in A$ with $|f(y)| = \rho(f)$.

For example, if $B = A\{r_1^{-1}T_1, \ldots, r_n^{-1}T_n\}$, then $\mathrm{Int}(Y/X) = \{y \in Y \big| |T_i(y)| < r_i \text{ for all } 1 \leq i \leq n\}$.

The relative interior of a morphism

Let $\varphi : Y \to X$ be a morphism of K-analytic spaces.

Definition

- The *relative interior* of φ is the set $\mathrm{Int}(Y/X)$ consisting of the points $y \in Y$ with the following property: there exist acyclic affinoid domains $U_1, \ldots, U_n$ with $x = \varphi(y) \in U_1 \cap \cdots \cap U_n$ such that $U_1 \cup \cdots \cup U_n$ is a neighborhood of x and, for every $1 \leq i \leq n$, there exists an acyclic affinoid neighborhood V_i of y in $\varphi^{-1}(U_i)$ with $y \in \mathrm{Int}(V_i/U_i)$.
- The *relative boundary* of φ is the set $\delta(Y/X) = Y \backslash \mathrm{Int}(Y/X)$.

If X and Y are K-affinoid, this definition is consistent with the previous one.

Properties of the relative interior

Fact

- $\mathrm{Int}(Y/X)$ *is an open subset of* Y.
- *If* Y *is an analytic domain in* X*, then* $\mathrm{Int}(Y/X)$ *is the topological interior of* Y *in* X.
- *if* $y \in \mathrm{Int}(Y/X)$ *then, for every acyclic affinoid domain* $\varphi(y) \in U \subset X$*, the point* y *has an acyclic affinoid neighborhood* V *in* $\varphi^{-1}(U)$ *with* $y \in \mathrm{Int}(V/U)$.
- *given a second morphism* $\psi : Z \to Y$*, one has* $\mathrm{Int}(Z/Y) \cap \psi^{-1}(\mathrm{Int}(Y/X)) \subset \mathrm{Int}(Z/X)$ *and, if* $\mathcal{H}(\psi(z)) \xrightarrow{\sim} \mathcal{H}(z)$ *for all* $z \in Z$*, the inclusion is an equality;*
- *if* $\varphi : Y \to X$ *is separated and* X *is* K*-affinoid, then for every p-affinoid domain* $V \subset Y$ *with* $V \subset \mathrm{Int}(Y/X)$ *there exists a bigger p-affinoid domain* $W \subset Y$ *such that* $V \subset \mathrm{Int}(W/X)$ *and* V *is a Weierstrass domain in* W.

Proper morphisms

Definition

A morphism of K-analytic spaces $\varphi : Y \to X$ is *proper* if it is compact (i.e., proper as a map of topological spaces) and has no boundary (i.e., $\delta(Y/X) = \emptyset$).

Fact

The class of proper morphisms is preserved under any base change and any composition.

Fréchet K-algebras

Definition

- A *Fréchet K-algebra* is a K-algebra A provided with an increasing sequence of seminorms $|\ |_1 \leq |\ |_2 \leq \dots$ such that if $|f|_i = 0$ for all $i \geq 1$, then $f = 0$.
- A homomorphism of Fréchet K-algebras $\varphi : A \to A'$ is *bounded* if for every $i \geq 1$ there exist $j \geq 1$ and $C > 0$ such that $|\varphi(f)|_i \leq C|f|_j$ for all $f \in A$.

There is a fully faithfull functor from the category of Banach K-algebras to that of Fréchet K-algebras.

The category of Fréchet K-algebras admits countable direct products. Indeed, given a sequence $A_1, A_2, \dots$ with seminorms $|\ |_1^{(i)} \leq |\ |_2^{(i)} \leq \dots$, their direct product $A = \prod_{i=1}^{\infty} A_i$ is provided with the seminorms $|f|_n = \max\limits_{i+j\leq n}\{|f_i|_j^{(i)}\}$ for $f = (f_1, f_2, \dots) \in A$.

Fréchet K-algebras

Definition

The *spectrum* $\mathcal{M}(A)$ of a Fréchet $\mathbf{F}_1$-algebra A is the space of all bounded homomorphisms of Fréchet $\mathbf{F}_1$-algebras $|\ |: A \to \mathbf{R}_+$ provided with the evident topology.

Fact

The K-algebra $\mathcal{O}(X)$ of every K-analytic space X countable at infinity can be provided with a unique structure of a Fréchet K-algebra so that, for every sequence of similar analytic domains $U_1, U_2, \ldots$ that cover X, the canonical homomorphism $\mathcal{O}(X) \to \prod_{n=1}^{\infty} \mathcal{O}(U_n)$ is an admissible monomorphism.

For such X, there is a canonical continuous map $X \to \mathcal{M}(\mathcal{O}(X))$.

Stein K-analytic spaces

Definition

A K-analytic space X is *Stein* if it is a union of an increasing sequence of acyclic affinoid domains $U_1 \subset U_2 \subset \dots$ such that each U_i is a Weierstrass domain in U_{i+1} and $U_i \subset \mathrm{Int}(U_i)$.

If X is Stein, then $X \xrightarrow{\sim} \mathcal{M}(\mathcal{O}(X))$.

Fact

- *The contravariant functor $X \mapsto \mathcal{O}(X)$ from the category of Stein K-analytic spaces to that of Fréchet K-algebras is fully faithful.*
- *Every point of a K-analytic space without boundary has a fundamental system of open Stein neighborhoods.*

A *Stein K-algebra* is a Fréchet K-algebra isomorphic to $\mathcal{O}(X)$ of a Stein K-analytic space X.

Non-Archimedean analytic spaces

Let k be a non-Archimedean field, and suppose we are given an isometric homomorphism of $\mathbf{F}_1$-fields $\phi : K \to k^{\cdot}$.

Definition

A *ϕ-morphism* from a k-analytic space Y to a K-analytic space X is a pair consisting of

- a continuous map $\varphi : Y \to X$ such that, for every point $y \in Y$, there exist affinoid domains $V_1, \ldots, V_n \subset Y$ such that $y \in V_1 \cap \cdots \cap V_n$, $V_1 \cup \cdots \cup V_n$ is a neighborhoods of y, and all $\varphi(V_i)$ lie in acyclic affinoid subdomains of X;
- a system of compatible bounded ϕ-homomorphisms $A_U \to \mathcal{B}^{\cdot}_V$ for all pairs consisting of an affinoid domain $V = \mathcal{M}(\mathcal{B}_V) \subset Y$ and an acyclic affinoid domain $U = \mathcal{M}(A_U) \subset X$ with $\varphi(V) \subset U$ such that the induced map $V \to U$ coincides with $\varphi\big|_V$.

Non-Archimedean analytic spaces

Let Φ_X be the functor from the category of k-analytic spaces to the category of sets that takes a k-analytic space Y to the set of ϕ-morphisms $\mathrm{Hom}_\phi(Y, X)$.

Fact

- *The functor Φ_X is representable by a k-analytic space $X^{(\phi)}$ and a compact ϕ-morphism $\pi = \pi_X : X^{(\phi)} \to X$;*
- *π induces a morphism of sites $\pi_{\mathrm{G}} : X^{(\phi)}_{\mathrm{G}} \to X_{\mathrm{G}}$ and a homomorphism of sheaves $\mathcal{O}_{X_{\mathrm{G}}} \to \pi_{\mathrm{G}*}\mathcal{O}_{X^{(\phi)}_{\mathrm{G}}}$;*
- *the functor $X \mapsto X^{(\phi)}$ commutes with fiber products and takes open and closed immersions, and proper morphisms to morphisms of the same type;*
- *the functor $X \mapsto X^{(\phi)}$ gives rise to a functor $K\text{-}\mathcal{P}aff \to k\text{-}\mathcal{A}ff$.*

Classes of analytic spaces

The following definitions makes sense also for the field of complex numbers $k = \mathbf{C}$. Let B be a K-algebra.

Definition

- B is *ϕ-nontrivial* if the stabilizer of any non-nilpotent element of B in K^* lies $\mathrm{Ker}(K^* \to k^*)$.
- B is *ϕ-special* if it is ϕ-nontrivial and, for every Zariski prime ideal $\mathfrak{p} \subset B$, the group $\mathrm{Ker}(K^* \to \kappa(\mathfrak{p})^*)$ has no torsion.
- B is *ϕ-superspecial* if it is ϕ-special and, for every pair of Zariski prime ideals $\mathfrak{p} \le \mathfrak{q}$, $\mathcal{Y}^{(\mathfrak{p})} \cap \mathcal{Y}^{(\mathfrak{q})}$ is a Zariski closed subset of $\mathcal{Y}^{(\mathfrak{p})}$, where $\mathcal{Y} = \mathrm{Spec}(B)$.
- A K-analytic space X is *ϕ-nontrivial* (resp. *ϕ-special*, resp. *ϕ-superspecial*) if the K-affinoid algebras of all of its affinoid domains possess the corresponding property.

For example, any irreducible ϕ-special X is ϕ-superspecial.

Properties of the map $\pi : X^{(\phi)} \to X$

Let X be a K-analytic space.

Fact

- *If X is ϕ-nontrivial, the map π is surjective and takes analytic domains of $X^{(\phi)}$ to analytic domains of X.*
- *If X is ϕ-superspecial, then*
  - *there is a canonical continuous section $\sigma : X \to X^{(\phi)}$ of the map π;*
  - *there is a strong deformation retraction $\Phi : X^{(\phi)} \times [0,1] \to X^{(\phi)}$ of $X^{(\phi)}$ to $\sigma(X)$.*

The full subcategory of K-$\mathcal{A}n$ consisting of ϕ-special K-analytic spaces is denoted by K-$\mathcal{A}n^{(\phi)}$.

Sheaves of Banach K-algebras on k-analytic spaces

Definition

- A *sheaf of Banach K-algebras* on a k-analytic space Y is a sheaf of K-algebras $\mathcal{A}$ on Y_{G} with the following structure:
  - $\mathcal{A}$ restricted to the family of compact analytic domains is induced by a functor to the category of Banach K-algebras;
  - if $\{V_i\}_{i\in I}$ is a finite covering of a compact analytic domain V by compact analytic domains, then $\mathcal{A}(V) \to \prod_{i\in I} \mathcal{A}(V_i)$ is an admissible monomorphism.
- a homomorphism of sheaves of Banach K-algebras $\mathcal{A} \to \mathcal{A}'$ is *bounded* if, for every compact analytic domain V, the homomorphism $\mathcal{A}(V) \to \mathcal{A}'(V)$ is bounded.

For example, $\mathcal{O}_{Y_{\mathrm{G}}}$ is a sheaf of Banach K-algebras.

Fact

For a K-analytic space X, $\pi_{\mathrm{G}}^{*}\mathcal{O}_{X_{\mathrm{G}}}$ is a sheaf of Banach K-algebras on $X^{(\phi)}$, and $\alpha_X : \pi_{\mathrm{G}}^{*}\mathcal{O}_{X_{\mathrm{G}}} \to \mathcal{O}_{X_{\mathrm{G}}^{(\phi)}}$ is bounded.

k-analytic spaces with a prelogarithmic K-structure

Definition

- A k-analytic space with a *prelogarithmic K-structure* is a triple $(Y, \mathcal{A}, \alpha)$ consisting of
  - a k-analytic space Y;
  - a sheaf of Banach K-algebras $\mathcal{A}$ on Y_{G};
  - a bounded ϕ-homomorphism of sheaves of Banach K-algebras $\mathcal{A} \to \mathcal{O}_{Y_{\mathrm{G}}}$.
- A morphism $(Y, \mathcal{A}, \alpha) \to (Y', \mathcal{A}', \alpha')$ is a pair consisting of
  - a morphism of k-analytic spaces $\varphi : Y \to Y'$;
  - a bounded homomorphism of sheaves of Banach K-algebras $\mathcal{A}' \to \varphi_* \mathcal{A}$, which is compatible with the homomorphism $\mathcal{O}_{Y'_{\mathrm{G}}} \to \varphi_* \mathcal{O}_{Y_{\mathrm{G}}}$.
- k-$\mathcal{A}n^{(\phi)}$ is the category of k-analytic spaces with a prelogarithmic K-structure.

Description of X in terms of $X^{(\phi)}$

Fact

- *The correspondence* $X \mapsto (X^{(\phi)}, \pi_{\mathrm{G}}^* \mathcal{O}_{X_{\mathrm{G}}}, \alpha_X)$ *gives rise to a fully faithful functor* $K\text{-}\mathcal{A}n^{(\phi)} \mapsto k\text{-}\mathcal{A}n^{(\phi)}$.
- *An object* $(Y, \mathcal{A}, \alpha)$ *of* $k\text{-}\mathcal{A}n^{(\phi)}$ *lies in the essential image of the above functor if and only if*
  - *the family of* k*-affinoid domains* $V \subset Y$ *such that* $\mathcal{A}(V)$ *is an acyclic* K*-affinoid algebra and* $V \overset{\sim}{\to} \mathcal{M}(\mathcal{A}(V))^{(\phi)}$ *is a net;*
  - *for every pair of affinoid domains* $V \subset W$ *from the above net,* $\mathcal{M}(\mathcal{A}(V)) \to \mathcal{M}(\mathcal{A}(W))$ *is an affinoid domain embedding.*

Suppose we are given an isometric homomorphism of $\mathbf{F}_1$-fields $K \to \mathbf{C}^{\cdot}$.

Definition

A *ϕ-morphism* from a complex analytic space Y to a K-analytic space without boundary X is a pair consisting of

- a continuous map $\varphi : Y \to X$;
- a system of compatible bounded ϕ-homomorphisms $\mathcal{O}(\mathcal{U}) \to \mathcal{O}(\mathcal{V})^{\cdot}$ for all pairs consisting of an open Stein subspaces $\mathcal{U} \subset X$ and $\mathcal{V} \subset Y$ with $\varphi(\mathcal{V}) \subset \mathcal{U}$ such that the induced map $\mathcal{V} \to \mathcal{U}$ coincides with $\varphi\big|_{\mathcal{V}}$.

If Y is the complex analytic point, then the set of ϕ-morphisms $\mathrm{Hom}_{\phi}(Y, X)$ is identified with the set of pairs consisting of a point $x \in X$ and an isometric ϕ-homomorphism $\mathcal{H}(x) \to \mathbf{C}^{\cdot}$. The latter set is defined for an arbitrary K-analytic space X, and is denoted by $X_{\phi}(\mathbf{C})$.

Complex analytic spaces

For a K-analytic space without boundary X, let Φ_X be the functor from the category of complex analytic spaces to that of sets that takes a complex analytic space Y to $\mathrm{Hom}_\phi(Y, X)$.

Fact

- *The functor Φ_X is representable by a complex analytic space $X^{(\phi)}$ and a compact ϕ-morphism $\pi = \pi_X : X^{(\phi)} \to X$;*
- *there is a canonical bijection $X_\phi(\mathbf{C}) \xrightarrow{\sim} X^{(\phi)}$ and, for any analytic domain $X' \subset X$, $\pi^{-1}(X') = X'_\phi(\mathbf{C})$;*
- *for any open Stein subspace $\mathcal{U} \subset X$, $\pi^{-1}(\mathcal{U})$ is an open Stein subspace of $X^{(\phi)}$;*
- *if X is separated then, for any p-affinoid domain $U \subset X$, $\pi^{-1}(U)$ is a Stein compact in $X^{(\phi)}$.*
- *the functor $X \mapsto X^{(\phi)}$ takes open and closed immersions and proper morphisms to morphisms of the same type.*

Properties of the map $\pi : X^{(\phi)} \to X$

Fact

- *if X is ϕ-nontrivial, π is an open surjective map, and its fibers are direct products of a real torus and a finite set;*
- *if X is ϕ-special, the fibers of π are real tori;*
- *if X is ϕ-special and irreducible, there is an action of a real torus S on $X^{(\phi)}$ such that $X^{(\phi)}/S \stackrel{\sim}{\to} X$.*

The full subcategory of K-$\mathcal{A}n$ consisting of ϕ-special K-analytic spaces without boundary is denoted by K-$\mathcal{C}an^{(\phi)}$.

Sheaves of Fréchet K-algebras on complex analytic spaces

Definition

- A *sheaf of Fréchet K-algebras* on a complex analytic space Y is a sheaf of K-algebras $\mathcal{A}$ on Y with the following structure:
  - $\mathcal{A}$ restricted to the family of open subspaces countable at infinity is induced by a functor to the category of Fréchet K-algebras;
  - if $\{\mathcal{V}_i\}_{i\in I}$ is a countable covering of a such an open subspace $\mathcal{V}$ by similar open subspaces, then $\mathcal{A}(V) \to \prod_{i\in I} \mathcal{A}(V_i)$ is an admissible monomorphism.
- a homomorphism of sheaves of Fréchet K-algebras $\mathcal{A} \to \mathcal{A}'$ is *bounded* if, for every $\mathcal{V}$ as above, the homomorphism $\mathcal{A}(\mathcal{V}) \to \mathcal{A}'(\mathcal{V})$ is bounded.

For example, $\mathcal{O}_Y$ is a sheaf of Banach K-algebras.

Complex analytic spaces with a prelogarithmic K-structure

Definition

- A complex analytic space with a *prelogarithmic K-structure* is a triple $(Y, \mathcal{A}, \alpha)$ consisting of
  - a complex analytic space Y;
  - a sheaf of Fréchet K-algebras $\mathcal{A}$ on Y;
  - a bounded ϕ-homomorphism of sheaves of Fréchet K-algebras $\mathcal{A} \to \mathcal{O}_Y$.
- A morphism $(Y, \mathcal{A}, \alpha) \to (Y', \mathcal{A}', \alpha')$ is a pair consisting of
  - a morphism of complex analytic spaces $\varphi : Y \to Y'$;
  - a bounded homomorphism of sheaves of Fréchet K-algebras $\mathcal{A}' \to \varphi_* \mathcal{A}$, which is compatible with the homomorphism $\mathcal{O}_{Y'} \to \varphi_* \mathcal{O}_Y$.
- $\mathbf{C}$-$\mathcal{A}n^{(\phi)}$ is the category of complex analytic spaces with a prelogarithmic K-structure.

Description of X in terms of $X^{(\phi)}$

Fact

- *The correspondence* $X \mapsto (X^{(\phi)}, \pi^*\mathcal{O}_X, \alpha_X)$ *gives rise to a fully faithful functor* $K\text{-}\mathcal{C}an^{(\phi)} \mapsto \mathbf{C}\text{-}\mathcal{A}n^{(\phi)}$.
- *An object* $(Y, \mathcal{A}, \alpha)$ *of* $\mathbf{C}\text{-}\mathcal{A}n^{(\phi)}$ *lies in the essential image of the above functor if and only if*
  - *the family of open Stein subspaces* $\mathcal{V} \subset Y$ *such that* $\mathcal{A}(\mathcal{V})$ *is a Stein* K*-algebra and* $\mathcal{V} \stackrel{\sim}{\to} \mathcal{M}(\mathcal{A}(\mathcal{V}))^{(\phi)}$ *is a basis of a topology;*
  - *for every pair of open Stein subspaces* $\mathcal{V} \subset \mathcal{W}$ *from the above basis,* $\mathcal{M}(\mathcal{A}(\mathcal{V})) \to \mathcal{M}(\mathcal{A}(\mathcal{W}))$ *is an open immersion.*