INTRODUCTION TO MANIFOLDS — III

Algebra of vector fields. Lie derivative(s).

1. Notations. The space of all C^∞-smooth vector fields on a manifold M is denoted by $X(M)$. If $v \in X(M)$ is a vector field, then $v(x) \in T_xM \simeq \mathbb{R}^n$ is its value at a point $x \in M$.

The `flow` of a vector field v is denoted by v^t:

$$\forall t \in \mathbb{R} \qquad v^t \colon M \to M$$

is a smooth map (automorphism) of M taking a point $x \in M$ into the point $v^t(x) \in M$ which is the t-endpoint of an integral trajectory for the field v, starting at the point x.

♣ **Problem 1.** Prove that the flow maps for a field v on a compact manifold M form a one-parameter group:

$$\forall t, s \in \mathbb{R} \qquad v^{t+s} = v^t \circ v^s = v^s \circ v^t,$$

and all v^t are diffeomorphisms of M.

♣ **Problem 2.** What means the formula

$$\left.\frac{d}{ds}\right|_{s=0} v^s = v$$

and is it true?

2. Star conventions. The space of all C^∞-smooth functions is denoted by $C^\infty(M)$. If $F \colon M \to M$ is a smooth map (not necessary a diffeomorphism), then there appears a contravariant map

$$F^* \colon C^\infty(M) \to C^\infty(M), \qquad F^* \colon f \mapsto F^* f, \quad F^*(x) = f(F(x)).$$

If $F \colon M \to N$ is a smooth map between two different manifolds, then

$$F^* \colon C^\infty(N) \to C^\infty(M).$$

Note that the direction of the arrows is reversed!

♣ **Problem 3.** Prove that $C^\infty(M)$ is a commutative associative algebra over $\mathbb{R}$ with respect to pointwise addition, subtraction and multiplication of functions. Prove that F^* is a homomorphism of this algebra (preserves all the operations). If $F \colon M \to N$, then $F^* \colon C^\infty(N) \to C^\infty(M)$ is a homomorphism also.

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Another star is associated with differentials: if $F\colon M_1 \to M_2$ is a `diffeomorphism`, then

$$F_*\colon X(M_1) \to X(M_2), \qquad v \mapsto F_* v, \quad (F_* v)(x) = \frac{\partial F}{\partial x}(x) \cdot v(x),$$

is a covariant (acts in the same direction) map which is:

(1) additive: $F_*(v + w) = F_* v + F_* w$;
(2) homogeneous: $\forall f \in C^\infty(M) \quad F_*(fv) = (F^*)^{-1} f \cdot F_* v$. (explain this formula!),

Why F_* is in general not defined, if F is just a smooth map and not a diffeomorphism?

3. Vector fields as differential operators.

♡ **Definition.** If $v \in X(M)$, then the `Lie derivative` L_v is

$$L_v\colon C^\infty(M) \to C^\infty(M), \qquad L_v f = \lim_{t \to 0} \frac{1}{t} \left((v^t)^* f - f \right).$$

In coordinates:

$$L_v f(a) = \lim_{t \to 0} \frac{f(a + tv + o(t)) - f(a)}{t} = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j}(a) v_j.$$

Properties of the Lie derivative:

(1) $L_v\colon C^\infty(M) \to C^\infty(M)$ is a `linear operator`:

$$L_v(f + g) = L_v f + L_v g, \quad L_v(\lambda f) = \lambda L_v f;$$

(2) the `Leibnitz identity` holds:

$$L_v(fg) = L_v f \cdot g + f \cdot L_v g.$$

(3) The Lie derivative linearly depends on v:

$$\forall f \in C^\infty(M),\ v, w \in X(M) \qquad L_{fv} = f L_v, \qquad L_{v+w} = L_v + L_w.$$

♣ **Problem 4.** Prove that the Lie derivative is local: for any function $f \in C^\infty(M)$ and any vector field v the value $L_v f(a)$ depends only on $v(a)$, so that for any other field w such that $w(a) = v(a)$, $L_v f(a) = L_w f(a)$.

Theorem. *Any differential operator, that is, a map $D\colon C^\infty(M) \to C^\infty(M)$ satisfying*

$$D(f + g) = Df + Dg, \quad D(\lambda f) = \lambda Df, \quad D(fg) = f\, Dg + Df \cdot g, \quad (\texttt{DiffOper})$$

is a Lie derivative along a certain vector field $v \in X$.

Idea of the proof. In local coordinates any function can be written as

$$f(x) = f(a) + \sum_{k=1}^{n} (x_k - a_k) f_k(x), \qquad f_k(a) = \frac{\partial f}{\partial x_k}(a).$$

Applying the Leibnitz identity, we conclude that $D = L_v$, where v is the vector field with components $v_k = D(x_k - a_k)$. □

Thus sometimes the notation

$$v = \sum_{k=1}^{n} v_k(x) \frac{\partial}{\partial x_k}$$

is used: such a notation understood as a differential operator, is a vector field from the geometric point of view.

4. Commutator. If $v, w \in X(M)$, then $D = L_v L_w - L_w L_v$ is a differential operator. Indeed, the Leibnitz formula is trivially satisfied, therefore $D = L_u$, where $u \in X(M)$.

♣ **Problem 5.** Check it!

♡ **Definition.** If $L_u = L_v L_w - L_w L_v$, then u is a `commutator` of v and w:

$$u = [v, w].$$

In coordinates:

$$L_u f = L_v \left(\sum_k \frac{\partial f}{\partial x_k} w_k \right) - L_w(\cdots) =$$

$$\sum_{k,j} \left(\frac{\partial^2 f}{\partial x_k \partial x_j} w_k v_j + \frac{\partial f}{\partial x_k} \frac{\partial w_k}{\partial x_j} v_j \right) - (\cdots) =$$

$$\sum_j \left(\sum_k \frac{\partial w_j}{\partial x_k} v_k - \sum_k \frac{\partial v_j}{\partial x_k} w_k \right) \frac{\partial f}{\partial x_j},$$

therefore

$$[v, w] = \sum_j \left(\sum_k \frac{\partial w_j}{\partial x_k} v_k - \sum_k \frac{\partial v_j}{\partial x_k} w_k \right) \frac{\partial}{\partial x_j}.$$

♣ **Problem 6.**

$$\left. \frac{\partial^2}{\partial s \partial t} \right|_{s=0, t=0} (f \circ v^t \circ w^s - f \circ w^s \circ v^t) = L_{[v,w]} f.$$

♣ **Problem 7.**

$$[v, w] = -[w, v].$$

♣ **Problem 8.** Prove the `Jacobi identity`

$$[[u, v], w] + [[v, w], u] + [[w, u], v] = 0.$$

5. Lie derivation of vector fields.

♡ **Definition.** The Lie derivative of a vector field w along another field v is

$$L_v w = \lim_{t\to 0} \frac{1}{t}(v^t_* w - w \circ v^t).$$

♣ **Problem 9.** Check that the above definition makes sense.

Properties of the Lie derivative: if $v, w \in X(M)$, $f \in C^\infty(M)$**, then:**

(1) $L_v v = 0$.
(2) L_v **is linear map from** $X(M)$ **to itself.**
(3) $L_v(fw) = (L_v f)w + f\, L_v w$ (**the Leibnitz property**).

Theorem.

$$L_v w = [v, w] \qquad \text{(or } [w, v]?\text{)}$$

Proof. Let

$$a = \left.\frac{\partial^2}{\partial s \partial t}\right|_{s=0,t=0} (f \circ v^t \circ w^s - f \circ w^s \circ v^t).$$

Then

$$a = \lim_{t\to 0} \frac{1}{t} \left(\left.\frac{\partial}{\partial s}\right|_{s=0} (\cdots) \right),$$

but

$$\left.\frac{\partial}{\partial s}\right|_{s=0} v^t \circ w^s = v^t_* w,$$

therefore

$$\left.\frac{\partial}{\partial s}\right|_{s=0} f \circ v^t \circ w^s = L_{v^t_* w} f,$$

while

$$\left.\frac{\partial}{\partial s}\right|_{s=0} f \circ w^s \circ v^t = L_{w \circ v^t} f,$$

and finally

$$a = L_{L_v w} f. \quad \square$$

♣ **Problem 10.** Is the Lie derivative of a vector field local in the following sense: if two fields $v_1, v_2 \in X(M)$ are coinciding on an open neighborhood of a certain point $a \in M$, then for any other field $v \in X(M)$

$$(L_{v_1} w)(a) = (L_{v_2} w)(a).$$

Is it true that the above value is determined by the (common) value $v_i(a)$?

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