FORMS AND INTEGRATION — I

Differential forms: definitions

Part I: Linear Theory

Let $V \simeq \mathbb{R}^n$ be a linear space: we avoid the symbol $\mathbb{R}^n$ since the latter implicitly implies some coordinates.

♡ **Definition.** An `exterior` k`-form` on V is a map

$$\omega\colon \underbrace{V \times \cdot \times V}_{k \text{ times}} \to \mathbb{R}, \qquad (v_1, \dots, v_k) \mapsto \omega(v_1, \dots, v_k),$$

which is:

- linear in each argument, and
- antisymmetric: if $\sigma \in S_k$ is a permutation on k symbols, and $|\sigma| = \pm 1$ its parity, then

$$\omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) = (-1)^{|\sigma|}\omega(v_1, \dots, v_k).$$

The space of all k-forms on V is denoted by $\wedge^k(V^*)$: it is a linear space over $\mathbb{R}$. □

♢ *Example.* Linear forms are 1-forms: $\wedge^1(V^*) = V^*$.

♢ *Example.* If $\dim V = k$ and a coordinate system in V is chosen, and $v_j = (v_{j1}, \dots, v_{jk})$, then

$$\omega(v_1, \dots, v_k) = \det \begin{vmatrix} v_{11} & \dots & v_{k1} \\ \vdots & \ddots & \vdots \\ v_{1k} & \dots & v_{kk} \end{vmatrix}$$

is a k-form. We denote it by $\det_x$, x explicitly indicating the coordinate system.

♣ `Problem 1.` Prove that for any $u, v \in \mathbb{R}^3$ the two formulas,

$$\omega_2 = \det_x(u, \cdot, \cdot), \quad \omega_1 = \det_x(u, v, \cdot)$$

define 2- and 1-forms respectively.

In any coordinate system $(x_1, \dots, x_n)$ on $V \simeq \mathbb{R}^n$ a k-form can be associated with a tuple of reals: if $\alpha\colon \{1, \dots, k\} \to \{1, \dots, n\}$ is an `index map`, and $(\mathbf{e}_1, \dots, \mathbf{e}_n)$ a basis in V, then we define

$$a_\alpha = \omega(\mathbf{e}_{\alpha(1)}, \dots, \mathbf{e}_{\alpha(k)})$$

and consider the collection $\{a_\alpha\}$ with α ranging over all possible index maps.

Typeset by $\mathcal{AMS}$-TEX

♣ `Problem 2.` Prove that the form is uniquely determined by its coefficients a_α. □

♣ `Problem 3.` How many independent coefficients there are among a_α? □

♣ `Problem 4.` Compute $\dim \wedge^k(\mathbb{R}^n)$. □

♣ `Problem 5.` Prove that there are no nonzero k-forms on V if $k > \dim V$. □

♠ **1- and 2-forms.** Among all k-forms on an n-space, the cases of $k = 1, 2, n-1$ and n are of special importance.

♡ **Definition.** A 1-form is nonzero, if it is nonzero. A 2-form ω is `nondegenerate`, if

$$\forall v \in V \ \exists u \in V : \omega(u, v) \neq 0. \quad \square$$

♣ `Problem 6.` Prove that a 2-form is nondegenerate, if and only if the matrix composed of its coefficients, is nondegenerate.

$$\det \begin{vmatrix} a_{11} & \dots & a_{k1} \\ \vdots & \ddots & \vdots \\ a_{1k} & \dots & a_{kk} \end{vmatrix} \neq 0. \quad \square$$

♣ `Problem 7.` Prove that the above property is independent of the choice of a coordinate system. □

♣ `Problem 8.` Prove that there are no nondegenerate 2-forms on an `odd`-dimensional space. □

♣ `Problem 9.` Prove that $\dim V = n \implies \dim \wedge^n(V^*) = 1$.

♣ `Problem 10.` Prove that for a generic 2-form on an odd-dimensional space V, there exists exactly one vector (or, more precisely, the direction defined by this vector) such that

$$\forall u \in V \quad \omega(v, u) = 0.$$

There is an operation which takes k-forms into $(k-1)$-forms: if $v \in V$ is any vector, then the operation

$$i_v : \omega \mapsto i_v\omega, \qquad i_v\omega(v_1, \dots, v_{k-1}) = \omega(v, v_1, \dots, v_{k-1}),$$

is a linear operator on k-forms. For some reasons this operator is called ***intrinsic antidifferentiation*****.**

♠ **Functorial properties of forms.** Linear transformations of the space V induce linear transformations on the spaces $\wedge^k(V^*)$: if V, W are two (different, in general) linear spaces, and $A \colon V \to W$ is a linear map, then

$$\times^k A \colon V \times \dots \times V \to W \times \dots \times W$$

is a natural extension, and the operator $A^*\colon \wedge^k(W^*) \to \wedge^k(V^*)$ is a linear map defined via the diagram

$$\begin{array}{ccc} V\times\cdots\times V & \xrightarrow{\times^k A} & W\times\cdots\times W \\ {\scriptstyle A^*\omega}\big\downarrow & & \big\downarrow{\scriptstyle\omega} \\ \mathbb{R} & = & \mathbb{R} \end{array}$$

which is to be commutative. The form $A^*\omega$ is the `pullback` of the form ω by the map A.

♣ `Problem 11.` Write a normal formula defining the form $A^*\omega$.

♣ `Problem 12.` If

$$V \xrightarrow{A} W \xrightarrow{B} Z$$

is a chain of maps, then $(AB)^* = B^* \circ A^*$. □

♠ **Exterior multiplication = wedge product.** If $\omega_1, \dots, \omega_k$ are k 1-forms on V, then the `tensor product` $s = \omega_k \otimes \cdots \otimes \omega_k$ can be defined on $V \times \cdots \times V$:

$$s(v_1, \dots, v_k) = \omega_1(v_1) \cdots \omega_k(v_k).$$

♣ `Problem 13.` Is s a k-form? *Answer*: no.

♡ **Definition.** If $\omega \in \wedge^k(V^*)$, $\theta \in \wedge^r(V^*)$, then the `exterior product`

$$\Omega = \omega \wedge \theta \in \wedge^{k+r}(V^*)$$

is defined by the formula

$$\Omega(v_1, \dots, v_{k+r}) = \sum_{\sigma \in S_{k+r}} (-1)^{|\sigma|}\, \omega(v_{\sigma(1)}, \dots, v_{\sigma(k)}) \cdot \theta(v_{\sigma(k+1)}, \dots, v_{\sigma(k+r)}).$$

In other words: *to compute the wedge product $\omega\wedge\theta$ on $k+r$ vectors $v_1 \dots, v_{k+r}$, split them in all possible ways into a group of k and the rest of r elements, compute the product of the values taken by ω on the first set and by θ on the second one, multiply by the parity of the permutation, and average over all permutations.*

Notation:

$$\boxed{\wedge(V^*) = \prod_{k=1}^{\dim V} \wedge^k(V^*).}$$

The wedge product is a nice algebraic operation on $\wedge(V^*)$:

$$\omega \wedge \theta = (-1)^{\deg\omega\cdot\deg\theta}\theta \wedge \omega,$$
$$\omega \wedge (\theta_1 + \theta_2) = \omega \wedge \theta_1 + \omega \wedge \theta_2$$
$$\omega \wedge (\theta \wedge \psi) = (\omega \wedge \theta) \wedge \psi,$$
$$A^*(\omega \wedge \theta) = A^*\omega \wedge A^*\theta$$

♣ `Problem 14.` Let $\mathbf{e}^j \in \mathbb{R}^{n*}$ be basis covectors on $\mathbb{R}^n$. Then

$$\det_x(\cdot) = \mathbf{e}^1 \wedge \cdots \wedge \mathbf{e}^n.$$

The coordinate representation of any form[1]:

$$\boxed{\omega = \sum_{\alpha:\ \text{ordered}} a_\alpha \, \mathbf{e}^{\alpha(1)} \wedge \cdots \wedge \mathbf{e}^{\alpha(k)}.}$$

♠ **Appendix: exterior algebra and vector algebra in $\mathbb{R}^3$.** In this subsection we consider $\mathbb{R}^3 = \mathbb{E}^3$ being the `Euclidean space`, that is, with the scalar product $(u, v) \mapsto \langle u, v\rangle$.

♡ **Definition.** With each vector $v \in \mathbb{E}^3$ the following 1- and 2-form can be asociated:

$$\wedge^k(\mathbb{E}^3) \ni \theta_v(u) = \langle v, u\rangle\,, \qquad \wedge^2(\mathbb{E}^3) \ni \omega_v(u, w) = \text{Mixed product of } (v, u, w).$$

The volume form $\det \in \wedge^3(\mathbb{E}^3)$ is also well defined in this case (how?)

♣ `Problem 15.` Write these two forms in coordinates. □

♣ `Problem 16.`

$$\forall u, v \in \mathbb{E}^3 \quad \theta_u \wedge \theta_v = \omega_{u \times v},$$

where $u \times v$ is the vector product (the cross product) in $\mathbb{E}^3$. □

♣ `Problem 17.`

$$\theta_u \wedge \omega_v = \langle u, v\rangle \cdot \det\,. \quad \square$$

References

[A] [redacted]rnold V. I., *Mathematical methods of Classical mechanics, 2nd ed.* (*Graduate Texts in Mathematics, vol. 60*), Springer–Verlag, New-York, 1989, `wislib code 531.01515 ARN`. (in English)

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[1] An ordered α is a map $\alpha\colon \{1, \dots, k\} \to \{1, \dots, n\}$ with $\alpha(1) < \cdots < \alpha(k)$.