Lecture 3

1 Recall:

M smooth manifold $\longleftrightarrow$ $\mathcal{O} = C^\infty(M)$ algebra of smooth functions on M

Vector field on M $\iff$ derivation of $\mathcal{O}$

$$X : \mathcal{O} \to \mathcal{O}, \quad \mathbb{R}\text{-linear}$$

$$X(fg) = f \cdot Xg + g \cdot Xf \quad \text{(Leibnitz rule)}$$

Example: $M \simeq U \subseteq \mathbb{R}^n$

$V : U \to \mathbb{R}^n$ „usual vector field" $(v_1, \dots, v_n)$

$X = \sum v_i \frac{\partial}{\partial x_i}$ diff. operator.

Conversely, every derivation of $\mathcal{O} = C^\infty(U)$ is like that.

Action by diffeos:

$$f : U \to U' \quad \Longleftrightarrow \quad f^* : \mathcal{O}' \to \mathcal{O}$$

diffeo $\quad U' \subseteq \mathbb{R}^n$?! $\qquad \mathcal{O}' = C^\infty(U'), \quad \mathcal{O} = C^\infty(U)$

$$X \in \mathcal{D}(U) \xrightarrow{f_*} X' = (f^*)^{-1} X f^* \in \mathcal{D}(U')$$

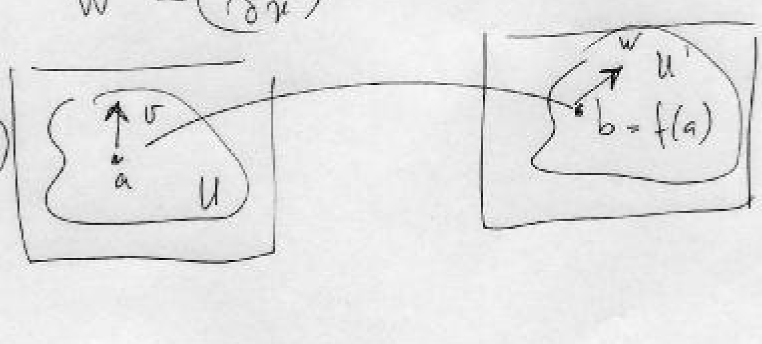
Exercise: $X \sim \sum_1^n v_j \frac{\partial}{\partial x_j}$ $\qquad X' = \sum_1^n w_j \frac{\partial}{\partial y_j}$

$$y_j = f_j(x)$$

$$w_j = \sum_{k=1}^n \frac{\partial f_j}{\partial x_k} v_k \quad \text{abbreviated to}$$

$$w = \left(\frac{\partial f}{\partial x}\right) v$$

Warning: w attached to $b = f(a)$



<u>Local character:</u> $f, g \in \mathfrak{U} = C^\infty(M)$

$f \equiv g$ in some nbhd of $a \in M \Rightarrow \forall X \in \mathfrak{D}(M)$
$Xf(a) = Xg(a)$

<u>Warning:</u> $f(a) = g(a) \not\Rightarrow Xf(a) = Xg(a)$!

◁ Let h be a cutoff function: $\begin{cases} h \equiv 1 \text{ near } a \\ h \equiv 0 \text{ away from } a \end{cases}$ $h(f-g) \equiv 0$ everywhere

$$0 \equiv X(h(f-g)) = Xh \cdot (f-g) + \underset{\substack{\text{|||} \\ 1 \text{ in a nbhd of } a}}{h} \cdot (Xf - Xg)$$ ▷

<u>Useful trick:</u>

$X \in \mathfrak{D}(M)$, we are interested only in $\mathcal{U} \subseteq M$

$\Rightarrow$ consider uX, where $u|_{\mathcal{U}} \equiv 1$, $u \equiv 0$ far away.

<u>One-parameter groups:</u> $\{H^t\}_{t \in \mathbb{R}}$ $H^t : \mathfrak{U} \to \mathfrak{U}$ automorphism of algebras.

<u>Definitions:</u> $H^{t+s} = H^t H^s$

$\Updownarrow$

$\{f^t\}$: $f^t : M \xrightarrow[\text{diffeo}]{} M$, $f^{t+s} = f^t \circ f^s$

<u>Property:</u> $\{H^t\}_{t \in \mathbb{R}}$ OPG$(\mathfrak{U}) \Rightarrow X = \lim_{t \to 0} \frac{H^t - id}{t}$

derivation! (<u>if</u> it exists!)

<u>Conversely:</u> Given $X = \begin{cases} \text{vector field} \\ \text{derivation} \end{cases}$

Construct an OPG $\begin{cases} \text{of diffeomorphisms of } M \\ \text{automorphisms of } \mathfrak{U} \end{cases}$ | Such that it is "differentiable" and

$$X = \frac{d}{dt}\Big|_{t=0} H^t$$

Thm If M compact, then there exists an OPG $H^t = \exp tX$.

... Proof: Local $\exists!$ of solutions

$\Rightarrow$ Globalization: from charts to the whole of M, from small to arbitrary t.

This allows to differentiate various objects along X

Ex 1. $$Xf = \frac{d}{dt}\Big|_{t=0} H^t f \quad \text{(tautology)}$$

Ex 2: $$X \bullet Y = \lim_{t\to 0} \frac{H^{-t} Y H^t - Y}{t}$$ — a new vector field.

Computation:

$$(X \bullet Y) f = \lim_{t\to 0} \frac{1}{t}\left(Y H^t f - H^t Y f \right)$$

$$= \lim_{t\to 0}\left[Y \cdot \frac{H^t f}{t} - \frac{1}{t} Y f + \frac{1}{t} Y f - H^t Y f \right]$$

$$= Y \cdot \lim_{t\to 0} \frac{H^t f - f}{t} - \lim_{t\to 0} \frac{H^t \cdot Yf - Yf}{t}$$

$$= YXf - XYf = [Y, X] f$$

Commutator.

Example: $[X, Y] = 0$ $H^{-t} Y H^t = Y$