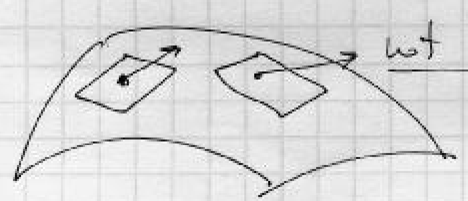
Lecture 9 — Connexions & parallel transport

- **Motivation:** $L_X Y$ is <u>not</u> a directional derivative:
$$L_{fX} \neq f L_X \quad \text{(on vector fields)}.$$

<u>Explanation</u>: depends on the <u>field</u> X rather than its value $X(p)$ at any given point.

Derivation requires „parallel transport", albeit for small distances.
Easy to do in the scalar case, problematic in general.

<u>Principal example</u>: Embedded submanifolds.



→ not in the tangent plane

Orthogonal projection — changes the lengths (bad!)

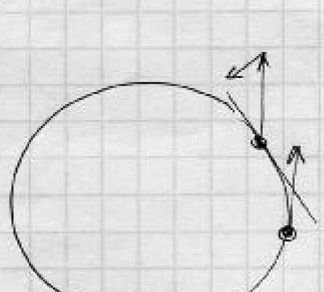
Starting point: $\mathbb{R}^n$

Absolute freedom

<u>Way out</u>: „infinitesimally small" steps: [shift by ε + orthoproj differs by ε^2 from an orthogonal transform

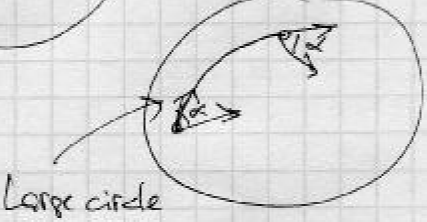
As $\varepsilon \to 0$, limit!

<u>Example 1.</u> <u>Circle</u>:



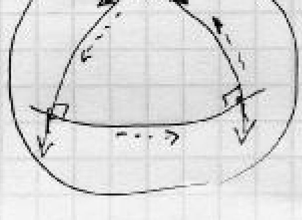
Parallel transport = v.f. of a constant velocity rotation.

<u>Example 2. Sphere.</u>

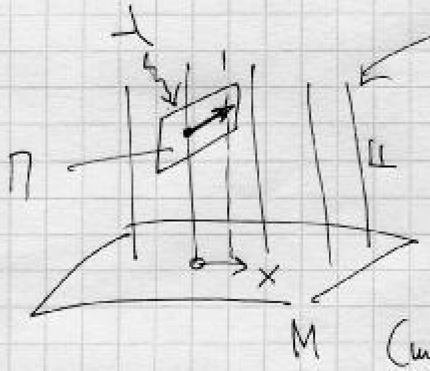


<u>Warning!</u> Parallel transport depends on the path!!!

Closed path ⟶ Rotation by the angle $= \pi/2$



How the parallel transport can be described?



Linear spaces attached to each point $a \in M$

M (manifold)

X = direction

Y = vector to translate "infinitesimal"

Ω = plane of deformations

$F \times M$ = Total space

$\dim \Pi = \dim M$; Π is a "graph"

$$\Pi_Y \in T_Y (F \times M) \Leftarrow \text{Distribution on } F \times M$$

Equations defining the distribution:

$$dy_i = \sum \theta_{ik}(x,y)\, dx_k \quad \text{- Equations defining } \Pi$$

If we want that the transport to an infinitely close "fiber" to be LINEAR: θ should be linear in y: $\theta_{ik} = \sum \theta_{ik\ell}(x)\, y_\ell\, dx_k$

$\Downarrow$

One can form a MATRIX 1-FORM $\Omega = \begin{pmatrix} \omega_{11} & \dots & \omega_{1m} \\ \omega_{m1} & \dots & \omega_{mm} \end{pmatrix}$

and write the equation in the vector form:

$$\boxed{dy = \Omega y}$$

(Ω = matrix form)

Giving such form is equivalent to defining a connexion

Problem: When the parallel transport is independent of the path? $\iff$ Distribution is integrable!

Digression: Frobenius theorem in the language of 1-forms:

$\Pi = \{\omega_1 = 0, \dots, \omega_m = 0\}$; It is called involutive, if $d\omega_i \in$ ideal in $\Lambda^2(\cdot)$, spanned by ω_i

Involutivity "old sense" = Involutivity "new sense"

$$\Leftarrow \quad d\omega_k(X_i, X_j) = X_i \underset{0}{\underbrace{\omega_k(X_j)}} - X_j \underset{0}{\underbrace{\omega_k(X_i)}} - \omega_k([X_i, X_j])$$

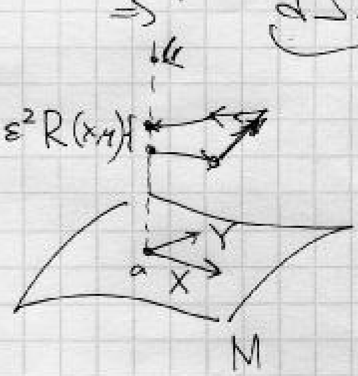
Applying this to $dy - \Omega y$ (vector - 1 - form, comp./wise)

$$d\Omega \cdot y - \Omega \wedge dy \underset{\text{mod } I}{=} (d\Omega - \Omega\wedge\Omega)y = 0$$

identically $\forall y$

$$\Rightarrow \underbrace{d\Omega - \Omega\wedge\Omega}_{\text{Matrix 2. form}} = 0$$

NB: Why $\Omega\wedge\Omega \neq 0$?



$R(X,Y)$ = automorphism of the fiber over $a \in M$

Curvature tensor: the result of parallel translation over the parallelogram $(\varepsilon X, \varepsilon Y)$

$$\Delta = \text{identity} + \varepsilon^2 R(X,Y)$$

Dual approach: instead of parallel translation, - differential operators

Covariant derivative $\nabla_X Y \leftarrow$ Leibnitz in Y

$\uparrow$ linear in X

Christoffel symbols: $\nabla_{\partial_i} \partial_j = \sum_k \Gamma^k_{ij} \partial_k$ n^3 coeffs.

$\partial_k = \frac{\partial}{\partial x_k}$ basic vector fields.

Then:

$$R(X,Y)\cdot Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

(A) Verify that R is tensorial w.r.t. X, Y:

$$R(fX,Y)Z = f\nabla_X\nabla_Y Z - \nabla_Y(f\nabla_X Z) - \nabla_{[fX,Y]}Z$$

$$= f R(X,Y) Z$$

$\cancel{L_Y f} \cdot \nabla_X Z - \cancel{L_Y f} \nabla_X Z$ + ..

terms involving df

(B) Once tensoriality is proven, choose any two vectors and extend them as coordinate vector fields:

i.e., WLOG $\quad \nabla = d - \Omega$

$$X = \frac{\partial}{\partial x}, \quad Y = \frac{\partial}{\partial y} \qquad \Omega = \underset{\uparrow}{A}\,dx + \underset{\uparrow}{B}\,dy$$

In these coordinates, $[X,Y]=0$ $\qquad$ matrix functions

$$R = \nabla_X \nabla_Y - \nabla_Y \nabla_X = \left[\frac{\partial}{\partial x} - A, \frac{\partial}{\partial y} - B \right]$$

$$R \cdot Z = \left(-\frac{\partial A}{\partial y} + \frac{\partial B}{\partial x} \right) Z + (BA - AB) Z$$

$$d\Omega = \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y} \right) dx \wedge dy \qquad \Omega \wedge \Omega = (AB - BA)\, dx \wedge dy$$

$$i_{X,Y}(d\Omega - \Omega \wedge \Omega) Z = R \cdot Z$$

∇ : Sections $\longrightarrow$ fiber-valued 1-forms

∇ a derivation:

$$\nabla(fX) = f\nabla X + \underbrace{df \otimes X}_{\text{fiber-valued 1-form}}$$

$\nabla_X = i_X \nabla$ - section

Iteration of ∇ (Similar to the de Rham complex)

$$\Lambda^k(B) \otimes \Gamma(\pi)$$

fiber-valued k-forms

$$\nabla_{(k)} : \Lambda^k \otimes \Gamma(\pi) \longrightarrow \Lambda^{k+1}(B) \otimes \Gamma(\pi)$$

$$\nabla_{(k+1)}(\alpha \wedge S^k) = d\alpha \wedge S^k + \alpha \otimes \nabla_{(k)} S^k$$

Language: Sections of the bundle

$$X : \underset{\text{base}}{B} \longrightarrow \underset{\text{base}}{B} \times \underset{\text{fiber}}{F}$$

such that $\pi \circ X = id_B$

(fiber-valued vector-functions)

Notation:

$\pi : B \times F \longrightarrow B$ bundle

$\Gamma(\pi)$ = sections

Curvature via the complex:

$$\Gamma(\pi) \xrightarrow{\nabla} \Gamma(\pi)\otimes\Lambda^1(B) \xrightarrow{\nabla} \Gamma(\pi)\otimes\Lambda^2(B) \longrightarrow \cdots$$

$$\nabla(\alpha\wedge \underset{\substack{\uparrow\\ \text{section}}}{s}) = d\alpha\wedge s + (-1)^{\deg\alpha}\,\alpha\wedge\nabla s$$

(Leibniz rule)

$$\nabla^2(fs) = \nabla(df\cdot s + f\cdot\nabla s) =$$
$$= \underbrace{d^2 f}_{\substack{\shortparallel\\ 0}}\cdot s + \cancel{df\wedge\nabla s} + f\nabla^2 s - \cancel{df\wedge\nabla s}$$
$$= f\,\nabla^2 s$$

Hence ∇^2 is a tensor:

$$i_X\, i_Y\, \nabla^2 s = \underbrace{R(X,Y)}_{\substack{\text{Linear automorphism}\\ \text{of the fiber into itself}\\ \text{antisymmetric w.r.t. } X,Y}}\cdot s$$

"Curvature tensor"

∇^2 = Curvature (matrix) 2. form

Connections on the tangent bundle. Lecture 10

$\nabla_X Y - \nabla_Y X$ can be compared with $[X,Y]$

Def: Connection is symmetric if the difference is zero.

Computation: if Γ_{ij}^k = k-th component of $\nabla_{\frac{\partial}{\partial x_i}} \frac{\partial}{\partial x_j}$, then (since coordinate vector fields commute)

Symmetric $\Longleftrightarrow$ $\Gamma_{ij}^k = \Gamma_{ji}^k$

Riemannian manifolds: Additional structure

$\langle \cdot , \cdot \rangle$ - positive definite bilinear form on the tangent bundle TM

Defined in local coordinates by n^2 smooth functions $g_{ij} = \langle \frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j} \rangle$

Scalar product $\Rightarrow$ lengths, angles, ...

Example: embedded submflds. E.g., smooth surface in $\mathbb{R}^3$ with std. structure

Explore: Connections between Riemannian structures and connexions

Axiom of compatibility (Leibnitz rule)

Levi-Civita connexion if symmetric

$$\nabla_X \langle Y, Z \rangle = \langle Y, \nabla_X Z \rangle + \langle \nabla_X Y, Z \rangle$$

Explanation: if two fields are parallel along a curve, then their scalar product is constant.

= all parallel transports are isometric operators.

Principal theorem of Riemannian geometry.

There exists a unique LC connexion for any RM. (Symm., metric preserving)

◁ Write a system of eq's for Γ_{ij}^k:

$$\underbrace{\frac{\partial}{\partial x_k} g_{ij}}_{\text{known functions}} = \langle \nabla_k e_i, e_j \rangle + \langle e_i, \nabla_k e_j \rangle = \sum_{l=1}^{n} \left(\Gamma_{ki}^l g_{lj} + \Gamma_{kj}^l g_{il} \right)$$

Why it is solvable? Pointwise!

- WLOG $g_{ij} = \delta_{ij}$ - system becomes simple ▷

Corollary: There exists a unique covariant derivative which preserves vectors tangent to a hypersurface in $\mathbb{R}^n$.

How to construct this LCC explicitly?

Let $\bar{\nabla}$ = "vector d" be the standard flat connection on $\mathbb{R}^n$

$M^{n-1} \subseteq \mathbb{R}^n$ a hypersurface, N = normal vector field on it

Problem: find another connection ∇ s.t. $\forall X, Y$ tangent to M, $\nabla_X Y$ again tangent.

– add a matrix valued 1-form: with the normal image

$$\nabla = \bar{\nabla} + a \cdot N$$

$$\nabla_X Y = \bar{\nabla}_X Y + \underbrace{a(X,Y)}_{\text{number, bilinear form}} \cdot N$$

$\bar{\nabla}$ symmetric
∇ symmetric
$\Downarrow$
$a(X,Y) = a(Y,X)$

To be tangent, again to M, we must postulate

$\forall X, Y \quad \langle \nabla_X Y, N \rangle = 0.$

This determines $a(X,Y)$ uniquely:

$$0 = \langle \bar{\nabla}_X Y, N \rangle + a(X,Y) = \underbrace{\bar{\nabla}_X \langle Y, N \rangle}_{= 0} - \langle Y, \bar{\nabla}_X N \rangle$$

$\Rightarrow \quad a(X,Y) = - \langle Y, \bar{\nabla}_X N \rangle$ unique possibility.

Why it is symmetric?
Why it is compatible with $\langle \cdot , \cdot \rangle$?

First check symmetry:

$$-\langle \bar\nabla_X N, Y\rangle \overset{?}{=} -\langle \bar\nabla_Y N, X\rangle$$

$$\parallel \qquad\qquad \parallel$$

$$-\underbrace{\nabla_X\langle N,Y\rangle}_{0} + \langle N, \bar\nabla_X Y\rangle \qquad -\underbrace{\nabla_X\langle N,X\rangle}_{0} + \langle N, \bar\nabla_Y X\rangle$$

But $\bar\nabla_X Y - \bar\nabla_Y X = [X,Y]$
(since $\bar\nabla$ is symmetric)

X, Y tanget to M $\Rightarrow$ $[X,Y]$ also tanget to M

$$\Downarrow$$

$$\langle N, [X,Y]\rangle = 0.$$

Second, check the compatibility; easier:

$$\nabla_X\langle Y, Z\rangle \underset{\uparrow}{=} \bar\nabla_X\langle Y, Z\rangle \underset{\uparrow}{=} \langle \bar\nabla_X Y, Z\rangle + \langle Y, \bar\nabla_X Z\rangle$$

both are simply the Lie derivatives on scalar functions

Since $\bar\nabla$ is $\langle\cdot,\cdot\rangle$-compatible

$$\underset{\uparrow}{=} \langle \nabla_X Y, Z\rangle + \langle Y, \nabla_X Z\rangle$$

Since $\nabla - \bar\nabla$ is a normal to M, hence to Y and Z.

Geodesic curves: γ : $\nabla_{\dot\gamma}\dot\gamma = 0$
(parallel to themselves)

$$\bar\nabla_{\dot\gamma}\dot\gamma = a(\dot\gamma,\dot\gamma)N$$

$$\parallel$$

$$\frac{d^2}{dt^2}\gamma$$

Acceleration normal to surface

LC connexion on a hypersurface