Approximating Piecewise-Smooth Functions

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Abstract

We consider the possibility of using locally supported quasi-interpolation operators for the approximation of univariate non-smooth functions. In such a case one usually expects the rate of approximation to be lower than that of smooth functions. It is shown in this paper that prior knowledge of the type of 'singularity' of the function can be used to regain the full approximation power of the quasi-interpolation method. The singularity types may include jumps in the derivatives at unknown locations, or even singularities of the form $(x-s)^{\alpha}$, with unknown s and α. The new approximation strategy includes singularity detection and high-order evaluation of the singularity parameters, such as the above s and α. Using the acquired singularity structure, a correction of the primary quasi-interpolation approximation is computed, yielding the final high-order approximation. The procedure is local, and the method is also applicable to a non-uniform data-point distribution. The paper includes some examples illustrating the high performance of the suggested method, supported by an analysis proving the approximation rates in some of the interesting cases.

1 Introduction

High-quality approximations of piecewise-smooth functions from a discrete set of function values is a challenging problem with many applications in fields such as numerical solutions of PDEs, image analysis and geometric modeling. A prominent approach to the problem is the so-called essentially non-oscillatory (ENO) and subcell resolution (SR) schemes introduced by Harten [9]. The ENO scheme constructs a piecewise-polynomial interpolant on a uniform grid which, loosely speaking, uses the smoothest consecutive data points in the vicinity of each data cell. The SR technique approximate the singularity location by intersecting two polynomials each from another side of the suspected singularity cell. In the spirit of ENO-SR many interesting works have been written using this simple but powerful idea. Recently, Arandifa *et al.*[1] gave a rigorous treatment to a variation of the technique, proving the expected approximation power on piecewise-smooth data. Archibald *et al.*[2, 3] have introduced the a polynomial annihilation technique for locating the cell which contains the singularity. A closely related problem is the removal of Gibbs phenomenon in the approximation of piecewise smooth functions. This problem is investigated in a series of papers by Gottlieb and Shu [8]. The methods suggested by Gottlieb and Shu are global in nature, using Fourier data information. These methods are also applicable for uniformly sampled function data, and recently even extended to non-uniform data by Gelb [7]. For a comprehensive review of these methods and recent developments see [11]. The method presented here relies upon local approximation tools, and thus the singularity detection algorithm is essentially local, similar to the ENO-SR scheme.

Let g be a real function on $[a,b]$, and let $X=\{x_i\}_1^N \subset [a,b]$ be a set of data points, $x_i < x_{i+1}$. We assume that $g \in PC^{m+1}[a,b]$,where $PC^{m+1}[a,b]$ denotes the function space of $C^m[a,b]$ functions with a piecewise continuous $m+1$ derivative. Consider approximations to g by quasi-interpolation operators of the type

$$Qg(x) = \sum_{i=1}^{N} q_i(x) g(x_i), \tag{1}$$

where $\{q_i\}_{i=1}^N$ are compactly supported functions, $q_i \in C^\mu[a,b]$, satisfying a polynomial reproduction property:

$$Qp = p,\ p \in \Pi_m, \tag{2}$$

where Π_m is the polynomial space of degree equal to or smaller than m. Such operators are very useful in achieving efficient approximations to smooth functions. Namely, let h, the separation distance, be the size of the largest open interval $J \subset [a,b]$ such that $J \cap X = \emptyset$, and let us assume that $supp\{q_i\} = O(h), i = 1,\dots,N$. Then, if the quasi-interpolation operator Q has a bounded Lebesque constant, $\sum_{i=1}^N |q_i(x)| \le C$, it follows that

$$g(x) - Qg(x) = O(h^{m+1}),\ x \in [a,b]. \tag{3}$$

In this paper we give as examples quasi-interpolation by splines [6] and by moving least-squares approximations [12].

The aim of this paper is to present a simple framework for enhancing such operators to well approximate piecewise-smooth functions. In particular we wish to obtain the same approximation order as in approximating smooth data, and furthermore, we would like the singular points in the approximant to be good approximations to the singular points of the approximated function.

2 Quasi-interpolation of piecewise-smooth functions

We consider piecewise-smooth functions $f \in PC^{m+1}[a,b] \setminus \{s\}$, $s \in (a,b)$, which are PC^{m+1} smooth except for a single point $x = s$. We assume that the singularity type is known, but its parameters, such as the singularity location and magnitude, are unknown. Throughout the paper we deal with jump singularities of the derivatives of f, but the discussion can easily be adapted to different singularity types. For instance, we also present an example with singularities of the type $(x-s)_+^{\alpha}$, $\alpha \in \mathbb{R}^+$ where $(x)_+ = x$ when $x \geq 0$ and $(x)_+ = 0$ otherwise. Note that in this example the $\lfloor \alpha \rfloor + 1$ derivative limit from the right of s is not finite.

We thus assume that f can be written as follows:

$$f(x) = g(x) + r(x), \tag{4}$$

where $g(x) \in PC^{m+1}[a,b]$, and

$$r(x) = \sum_{j=0}^{m} \frac{\Delta_j}{j!}(x-s)_+^j \ . \tag{5}$$

We note that

$$\Delta_j = f^{(j)}(s^+) - f^{(j)}(s^-).$$

The error in the quasi-interpolation approximation to f, using data values at the points X, can be written as

$$Ef(x) = f(x) - Qf(x) = g(x) - \sum_{i=1}^{N} q_i(x)g(x_i) + r(x) - \sum_{i=1}^{N} q_i(x)r(x_i) = Eg(x) + Er(x),$$

where $Eg(x), Er(x)$ denote the error functions of approximating $g(x), r(x)$ using Q, respectively.

We assume the quasi-interpolation operator Q has a bounded Lebesque constant and hence the expected $O(h^{m+1})$ approximation order is realized for smooth functions. Therefore, $|Eg(x)| = O(h^{m+1})$, where h is the fill-distance of the data X as defined above, and the deterioration of the approximation is latent within the term $Er(x)$.

If the singularity model (type) is known, then $Er(x)$ is known analytically. In our case, assuming r is of the form (5), we have

$$Er(x) = \sum_{j=0}^{m} \frac{\Delta_j}{j!} H_j(x;s), \tag{6}$$

where

$$H_j(x;s) = (x-s)_+^j - \sum_{i=1}^{N} q_i(x)(x_i - s)_+^j. \tag{7}$$

Hence, $Er(x) = Er(x;s,\bar{\Delta})$ is a simple known function of the singularity parameters s and $\bar{\Delta} = \{\Delta_j\}_{j=0}^m$. Also in our hands, are the values $\{Ef(x_i)\}$, and according to the above observations

$$Er(x_i) = Ef(x_i) + O(h^{m+1})\ . \tag{8}$$

Therefore, the general idea for enhancing the quasi-interpolating approximation operator is by fitting a function of the form (6) to the actual errors $\{Ef(x_i)\}$ in the quasi-interpolation approximation. As we argue below, the function $Er(x)$ is of finite support of size $O(h)$. Furthermore, the operator E annihilates certain polynomials, and this leads to natural orthogonality relations, with respect to the standard inner product. In order to take advantage of these properties we shall use the standard least-squares fitting. We expect that using other norms will give similar results. The overall procedure is thus as follows: First, we find approximations $s*$ and $\bar{\Delta}^* = \{\Delta_j^*\}_{j=0}^m$ to the singularity parameters s and $\bar{\Delta}$ by a least-squares fitting:

$$(s^*, \bar{\Delta}^*) := \underset{s',\bar{\Delta}'}{\operatorname{argmin}} \sum_{j=1}^{N} \left\{ Ef(x_j) - Er(x_j; s', \bar{\Delta}') \right\}^2 \tag{9}$$

Next, we correct the original quasi-interpolant, and define the new approximation to f as

$$\bar{Q}f(x) = Qf(x) + Er(x; s^*, \bar{\Delta}^*). \tag{10}$$

Remark 2.1 *The least-squares fitting problem (9) leads to a system of equations which is linear in the unknowns $\bar{\Delta}^*$ and algebraic (polynomial) in s^*.*

Remark 2.2 *The approximation $\bar{Q}f$ is piecewise C^μ, with possible jump discontinuities in the derivatives at s^*. It is easy to verify that if $g \in \Pi_m$ and r is of the form (5), then*

$$\bar{Q}(g+r) = g+r. \tag{11}$$

Note that the above reproduction property does not automatically provide an $O(h^{m+1})$ approximation order, since the overall process is non-linear.

Remark 2.3 *The functions $H_j(x;s)$ are of finite support of size $O(h)$. This follows from the definition of $H_j(x;s)$ and the fact that Q reproduce polynomials of degree less or equal to m. This consequently implies that the correction term $Er(x;s^*,\bar{\Delta}^*)$ is of finite support of size $O(h)$.*

In the following theorem we summarize the main properties of the new approximant $\bar{Q}f(x)$. For the sake of briefness and clarity we assume that the points x_i are equidistant, that is, $x_i = a+ih$, $i = 0,...,N$, $h = \frac{b-a}{N}$ and that the quasi-interpolation basis functions are all shifts of one function:

$$q_i(x) = q\left(\frac{x}{h} - i\right) ,$$

where $supp\{q(\mathbb{Z})\} = [-e,e] \cap \mathbb{Z},\ e = 2,3,....$ In fact, to achieve polynomial reproduction over the whole interval [a,b], one should use some special basis functions near the end points of the interval. However, since we assume that the singularity of f is at a fixed point $s \in (a,b)$, and since we consider asymptotics as $h \to 0$, and $q(\cdot)$ is of a finite support, it is enough to consider a shift invariant basis. In order to retain the polynomial reproduction over $[a,b]$, and maintain the simplicity of a shift invariant basis, we augment the point set with e extra points on each side, that is, $x_i = a+ih$, $i = -e,...,N+e$.

Theorem 2.1 *Assume Q is a quasi-interpolation operator reproducing polynomials in Π_m. Let f be a continuous function in $[a,b]$ of the form (4) with $\Delta_1 \neq 0$. The approximant $\bar{Q}f(x)$ defined above satisfies the following properties:*

1. *$\bar{Q}f$ has the same smoothness as Qf except for the point s^* where it has the singularity type of the singularity model.*
2. *$|s-s^*| = O(h^{m+1})$ and $|\Delta_j - \Delta_j^*| = O(h^{m+1-j})$, $j = 1,2,...,m$.*
3. *$\bar{Q}f$ has full approximation order, that is,*

$$|f(x) - \bar{Q}f(x)| = O(h^{m+1}).$$

Proof. Claim 1 follows directly from the definition of the correction term $E(x;s^*,\bar{\Delta}^*)$. In order to prove claims 2 and 3, we first note that the parameters s^* and Δ^* minimizing (9) certainly satisfy

$$\sum_{j=0}^{N}\left\{Ef(x_j)-Er(x_j;s^*,\bar{\Delta}^*)\right\}^2 \le \sum_{j=0}^{N}\left\{Ef(x_j)-Er(x_j;s,\bar{\Delta})\right\}^2 .$$

Existence of a minimizer s^* is proved in Appendix A. Next, it follows from Remark 2.3 that there is a fixed number (independent of h) of indices j such that $Er(x_j;s^*,\bar{\Delta}^*)\neq 0$ or $Er(x_j;s,\bar{\Delta})\neq 0$. Combining this with the fact that

$$Ef(x_j)-Er(x_j;s,\bar{\Delta})=O(h^{m+1}) ,$$

we get

$$Er(x_j;s,\bar{\Delta})-Er(x_j;s^*,\bar{\Delta}^*)=O(h^{m+1}). \tag{12}$$

Therefore, we can conclude that the corrected approximation gives the right approximation order at the data points:

$$|f(x_j)-\bar{Q}f(x_j)|=O(h^{m+1}) .$$

For the proof of claim 3 we shall use (12) in order to show that

$$Er(x;s,\bar{\Delta})-Er(x;s^*,\bar{\Delta}^*)=O(h^{m+1}) ,\ \forall x\in[a,b] .$$

This part of the proof is less obvious, and is rather lengthy:

From the definitions we have that

$$Er(x;s,\bar{\Delta})-Er(x;s^*,\bar{\Delta}^*)=$$

$$\textstyle\sum_{j=1}^{m}\frac{\Delta_j}{j!}\left[(x-s)_+^j-\sum_i q_i(x)(x_i-s)_+^j\right]-\sum_{j=1}^{m}\frac{\Delta_j^*}{j!}\left[(x-s^*)_+^j-\sum_i q_i(x)(x_i-s^*)_+^j\right] \tag{13}$$

Next, note that

$$(x-s)_+^j-\sum_i q_i(x)(x_i-s)_+^j=-(x-s)_-^j+\sum_i q_i(x)(x_i-s)_-^j,\ \ 0\le j\le m, \tag{14}$$

and similarly for s^*. These identities can be understood via the polynomial reproduction property of Q. Let us define the following polynomials of degree m:

$$p(x)=\sum_{j=1}^{m}\frac{\Delta_j}{j!}(x-s)^j ,\quad p^*(x)=\sum_{j=1}^{m}\frac{\Delta_j^*}{j!}(x-s^*)^j. \tag{15}$$

W.l.o.g. we assume that $s \le s^*$, otherwise the following can be adapted accordingly. Using (13) and (14) we get:

$$Er(x;s,\bar{\Delta}) - Er(x;s^*,\bar{\Delta}^*) = \sum_i q_i(x) \begin{cases} \Lambda_i^1 & x < s \\ \Lambda_i^2 & s \le x < s^* \\ \Lambda_i^3 & s^* \le x \end{cases}, \tag{16}$$

where

$$\Lambda_i^1 = \begin{cases} 0 & x_i < s \\ -p(x_i) & s \le x_i < s^* \\ p^*(x_i) - p(x_i) & s^* \le x_i \end{cases} \quad \Lambda_i^2 = \begin{cases} p(x_i) & x_i < s \\ 0 & s \le x_i < s^* \\ p^*(x_i) & s^* \le x_i \end{cases} \tag{17}$$

$$\Lambda_i^3 = \begin{cases} p(x_i) - p^*(x_i) & x_i < s \\ -p^*(x_i) & s \le x_i < s^* \\ 0 & s^* \le x_i \end{cases}, \quad i = -M-e, \ldots, N+e$$

Next, denote by ν the maximal index such that $x_\nu < s$. Then using (12) with

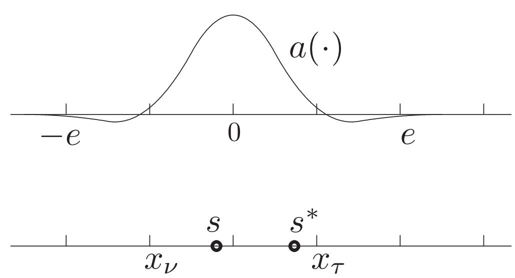


Figure 1: Illustration for the proof of Theorem 2.1.

$j = \nu - e + 1 \equiv \eta$, and using the compact support of q, we are left with

$$O(h^{m+1}) = Er(x_\eta;s,\bar{\Delta}) - Er(x_\eta;s^*,\bar{\Delta}^*) = q_{\nu+1}(x_\eta)\Lambda_{\nu+1}^1.$$

Since $q_{\nu+1}(x_\eta) = q(-e) \neq 0$, we get that

$$\Lambda_{\nu+1}^1 = O(h^{m+1}).$$

Next, by considering the point $x_{\eta+1}$ we get

$$O(h^{m+1}) = Er(x_{\eta+1};s,\bar{\Delta}) - Er(x_{\eta+1};s^*,\bar{\Delta}^*) =$$

$$q_{\nu+1}(x_{\eta+1})\Lambda_{\nu+1}^1 + q_{\nu+2}(x_{\eta+1})\Lambda_{\nu+2}^1 = O(h^{m+1}) + q_{\nu+2}(x_{\eta+1})\Lambda_{\nu+2}^1,$$

and since $q_{\nu+2}(x_{\eta+1}) = q(-e) \neq 0$ we have

$$\Lambda^1_{\nu+2} = O(h^{m+1}).$$

We can continue in the same manner showing

$$\Lambda^1_i = O(h^{m+1}) \ , \quad \nu+1 \leq i \leq \nu+e. \tag{18}$$

In a similar manner we also get

$$\Lambda^3_i = O(h^{m+1}) \ , \quad \tau-1 \geq i \geq \tau-e, \tag{19}$$

where τ is the minimal index such that $s^* \leq x_\tau$. Next, let us show that for a small enough h there is at most one mesh point between x_ν to x_τ, that is, either $\tau = \nu+1$ or $\tau = \nu+2$. Otherwise, if $x_{\nu+2} < s^*$, since $e \geq 2$, then $p(x_{\nu+2}) = -\Lambda^1_{\nu+2} = O(h^{m+1})$. But since $\Delta_1 \neq 0$, and $x_{\nu+2} - s > h$, we have $p(x_{\nu+2}) = \theta(h)$ which yields a contradiction for small enough h. So in the last part of the proof, we deal with each of the two possible cases: $\tau = \nu+1$ and $\tau = \nu+2$.

Case 1: If $\tau = \nu+1$, in this case we have

$$\begin{aligned} \Lambda^3_i &= p(x_i) - p^*(x_i) = O(h^{m+1}) \ , \quad i = \tau-e, \ldots, \nu, \\ \Lambda^1_i &= -p(x_i) + p^*(x_i) = O(h^{m+1}) \ , \quad i = \tau, \ldots, \nu+e, \end{aligned} \tag{20}$$

that is, we have $2e$ points where the polynomial $p - p^*$ has an $O(h^{m+1})$ value. Since $p - p^*$ is a polynomial of degree less than or equal to m and necessarily $2e \geq m+1$, we get that for $x \in [x_{\tau-e-2}, x_{\nu+e+2}]$

$$p(x) - p^*(x) = O(h^{m+1}). \tag{21}$$

This has several consequences: First, setting $x = s^*$ we get

$$p(s^*) = O(h^{m+1}),$$

hence, since $\Delta_1 \neq 0$, we get that $|s - s^*| = O(h^{m+1})$. Secondly, for $x < s$ ($s^* \leq x$), we see that all relevant Λ^1_i (Λ^3_i) are $O(h^{m+1})$ and therefore we have that

$$Er(x; s, \bar{\Delta}) - Er(x; s^*, \bar{\Delta}^*) = O(h^{m+1}) \ , \quad x < s, s^* \leq x.$$

Finally, for $s \leq x < s^*$, $Er(x; s, \bar{\Delta}) - Er(x; s^*, \bar{\Delta}^*) =$

$$\sum_{i=\tau-e-2}^{\nu+e+2} q_i(x)\Lambda^2_i = \sum_{i=\tau-e-1}^{\nu+e+2} q_i(x)p(x_i) + O(h^{m+1}) = p(x) + O(h^{m+1}) = O(h^{m+1}),$$

where the first equality uses (17) and (21) and the second equality is due to polynomial reproduction.

Case 2: If $\tau = \nu + 2$. In this case we only have $2e-1$ points ($i = \tau - e, ..., \nu + e$) where $p - p^*$ is $O(h^{m+1})$. In order to obtain such a relation at yet another point, we use the extra information we have, namely,

$$p(x_{\nu+1}) = -\Lambda^1_{\nu+1} = O(h^{m+1}). \tag{22}$$

Using (17), (18), (19) we have

$$O(h^{m+1}) = Er(x_{\nu+1}; s, \bar{\Delta}) - Er(x_{\nu+1}; s^*, \bar{\Delta}^*) = \sum_{i=\tau-e-1}^{\nu+e+1} q_i(x_{\nu+1})\Lambda^2_i =$$

$$\sum_{i=\tau-e-1}^{\nu+e+1} q_i(x_{\nu+1})p(x_i) + q_{\nu+e+1}(x_{\nu+1})\left(p^*(x_{\nu+e+1}) - p(x_{\nu+e+1})\right) + O(h^{m+1}) =$$

$$p(x_{\nu+1}) + q_{\nu+e+1}(x_{\nu+1})\left(p^*(x_{\nu+e+1}) - p(x_{\nu+e+1})\right) + O(h^{m+1}).$$

Observing that $q_{\nu+e+1}(x_{\nu+1}) = q(-e) \neq 0$ and using (22), we get

$$p^*(x_{\nu+e+1}) - p(x_{\nu+e+1}) = O(h^{m+1}),$$

and we can continue as in the first case.
Finally, let us prove $|\Delta_j - \Delta^*_j| = O(h^{m+1-j})$. First note that we have that there exist $2e$ consecutive points $x_{i_1}, ..., x_{i_{2e}} \in X$ such that $p(x_{i_j}) - p^*(x_{i_j}) = O(h^{m+1})$. Next let us write $p - p^*$ in the Lagrange basis

$$p(x) - p^*(x) = \sum_{j=1}^{2e} \left(p(x_{i_j}) - p^*(x_{i_j})\right) L_j(x),$$

where the functions $L_j(x) = \frac{\prod_{\chi \neq j}(x - x_{i_\chi})}{\prod_{\chi \neq j}(x_{i_j} - x_{i_\chi})}$, $j = 1, .., 2e$ form the Lagrange basis. Since

$$\left.\frac{d^J}{dx^J}\right|_{x \in [min\{x_{i_j}\}, max\{x_{i_j}\}]} L_j(x) = O(h^{-J}),$$

we get that

$$\left.\frac{d^J}{dx^J}\right|_{x=s^*} (p(x) - p^*(x)) = O(h^{m+1-J}).$$

On the other hand, for any $J = 1, ..., m$, differentiating $p(x) - p^*(x)$ J times and substitute $x = s^*$ we have

$$\left.\frac{d^J}{dx^J}\right|_{x=s^*} (p(x) - p^*(x)) = \Delta_J - \Delta^*_J + \sum_{j=J+1}^{m} \frac{\Delta_j}{(j-J)!}(s^* - s)^{j-J}.$$

Since $|s - s^*| = O(h^{m+1})$ the previous two equations imply that

$$|\Delta_J^* - \Delta_J| = O(h^{m+1-J}).$$

■

2.1 Algorithm

From a practical point of view, we suggest two algorithms for alleviating quasi-interpolants Q to accommodate piecewise-smooth data with known singularity family type $\bar{Q}$: First, in the case of jump in the first derivative we provide a closed-form formula for locating the singularity and approximating the singularity parameters. The algorithm and its analysis are provided in Section 3. The computational complexity of the algorithm in this case is $O(N)$ (where N is the number of data points).

Second, in the general case, the algorithm is as follows. First, an initial (rough) guess s^0 to the singularity location s is constructed. This can be done in several ways. We have used for example

$$s^0 = \underset{s'}{\operatorname{argmin}} \sum_j |Ef(x_j)||x_j - s'|^2.$$

This boils down to the centroid of the error Ef. Another option could be to use the ENO scheme for locating a cell or few cells that might contain the singularity. The initial guess provides us with $|s^0 - s| = O(h)$ approximation of the singularity location and its computational complexity is $O(N)$, where again N is the number of data points. It Should be noted that in the case that h is not small enough, the algorithm may provide an erroneous approximation, especially in cases of detecting high order singularities. In the second step a minimization of the functional (9) is done by minimizing the functional as a (rational) function of s' in a constant number of intervals adjacent to the initial guess s^0. That is, denote by $\bar{\Delta}' = \bar{\Delta}'(s')$ the minimizer of (9) with respect to a fixed s', then the functional

$$\sum_{j=1}^{N} \left\{ Ef(x_j) - Er(x_j; s', \bar{\Delta}'(s')) \right\}^2,$$

is minimized with respect to s' in the above mentioned intervals. The computational complexity of the second step is constant, that is, independent of N. This follows from the fact that the support of Er contains only a fixed number of data points (assuming a quasi-uniform data set).

3 A closed form solution for a special case

As explained in Remark 2.1, the least-squares fitting problem (9) leads to a system of equations which is linear in the unknowns $\bar{\Delta}^*$ and algebraic (polynomial) in s^*. However, in the special case where f has jump discontinuities at s only in its value and its first derivative we can present a closed form solution of (9). In this case we have also observed some superconvergence of the approximation s^* to s, which we prove below. This case is a particular instance of functions which are linear combinations of smooth functions and spline functions.

In that case (6) reduces to

$$Er(x;s',\bar{\Delta}') = \Delta_0' H_0(x;s') + \Delta_1' H_1(x;s'), \tag{23}$$

Assume $s \in (x_k, x_{k+1}]$, in such a case the sum to be minimized in (9) becomes

$$\begin{aligned}\sum_{j=1}^N \Big\{Ef(x_j) - \Delta_0' \left[(x_j - s')_+^0 - \textstyle\sum_{i=1}^N q_i(x_j)(x_i - s')_+^0\right] \\ -\Delta_1' \left[(x_j - s')_+ - \textstyle\sum_{i=1}^N q_i(x_j)(x_i - s')_+\right] \Big\}^2 =\end{aligned}$$

$$\begin{aligned}\sum_{j=1}^N \Big\{Ef(x_j) - \Delta_0' \left[(x_j - x_{k+1})_+^0 - \textstyle\sum_{i\geq k+1} q_i(x_j)\right] \\ -\Delta_1' \left[(x_j - x_{k+1})_+^0 (x_j - s') - \textstyle\sum_{i\geq k+1} q_i(x_j)(x_i - s')\right] \Big\}^2\end{aligned} \tag{24}$$

After rearranging we get

$$\begin{aligned}\sum_{j=1}^N \Big\{Ef(x_j) - (\Delta_1' s' - \Delta_0') \left[-(x_j - x_{k+1})_+^0 + \textstyle\sum_{i\geq k+1} q_i(x_j)\right] \\ -\Delta_1' \left[(x_j - x_{k+1})_+^0 x_j - \textstyle\sum_{i\geq k+1} q_i(x_j) x_i\right] \Big\}^2.\end{aligned} \tag{25}$$

And the functional is quadratic in the variables $(\Delta_1' s' - \Delta_0')$ and Δ_1'.

Obviously, the minimizer in this case is not unique. Therefore, let us consider the case where $\Delta_0 = 0$, that is, f is continuous. In that case (25) reduces to a 2×2 linear system in the variables

$$\tilde{\Delta}_1' := \Delta_1' \ , \ \tilde{s}' := \Delta_1' s'.$$

Denote the functions

$$\begin{aligned}\phi_k(x) &= x(x - x_{k+1})_+^0 - \textstyle\sum_{i\geq k+1} q_i(x) x_i, \\ \psi_k(x) &= -(x - x_{k+1})_+^0 + \textstyle\sum_{i\geq k+1} q_i(x).\end{aligned} \tag{26}$$

By the polynomial reproduction property of the quasi-interpolation operator it follows that both ϕ_k and ψ_k are of compact support of size $O(h)$. Then equation (25) turns into

$$\sum_{j=1}^{N} \left\{ Ef(x_j) - \tilde{s}' \psi_k(x_j) - \tilde{\Delta}_1' \phi_k(x_j) \right\}^2 . \quad (27)$$

Let us define the matrix $A = (\psi \ , \ \phi)$ where ϕ, ψ are column vectors defined by

$$\phi_j = \phi_k(x_j) \ , \quad \psi_j = \psi_k(x_j). \quad (28)$$

Then, the normal equations for system (27) is:

$$A^t A \begin{pmatrix} \tilde{s}' \\ \tilde{\Delta}_1' \end{pmatrix} = A^t \left(Ef(x_1), ..., Ef(x_N) \right)^t . \quad (29)$$

Hence, the algorithm for calculating the approximations s^*, Δ_1^* to the true singularity parameters s, Δ_1 can be described concisely as follows: for each interval $[x_i, x_{i+1}]$, minimize the functional in (9) constrained to the interval $[x_i, x_{i+1}]$, that is, solve (29): if $s' = \tilde{s}'/\tilde{\Delta}_1' \in [x_i, x_{i+1}]$ it is the minimum of (9) in that interval. Otherwise, evaluate the functional at the ends $s' = x_i, \Delta_1' = \Delta_1'(s')$ and the same for x_{i+1}. The minimum of the functional on this interval will be attained at one of the ends. Finally denote by s^*, Δ_1^* the values which yielded the global minimum of the functional (9).

Note that we have used the symbol $\Delta'(s')$ to denote, as remarked in (2.1), that fixing s' in functional (9) results in a linear system for Δ. In Appendix A this system is proved to be non-singular.

3.1 Approximation order analysis

An important virtue of the above method is that, by using $m-$th degree quasi-approximation operator, we get an $O(h^{2(m+B)-1})$ approximation order to the location of the singular point, where $m+B$ is the maximal degree of the polynomials reproduced by the quasi-interpolation operator at the *nodes*. In this section we provide the analysis of this approximation order. Regarding different types of singularities (not only jump in the first derivative) we have also observed some superconvergence properties; in Section 5 we present some numerical experiments demonstrating this phenomenon and compare it to the subcell resolution method of similar degree.

We return to the settings of equidistant point $x_i = a + ih$, $-e \le i \le N + e$, $h = \frac{b-a}{N}$, and to basis functions $q_i(\cdot)$ which are shifted versions of a "mother basis function" $q(x)$, that is,

$$q_i(x) = q\left(\frac{x}{h} - i\right) .$$

Let us set k such that $s \in [x_k, x_{k+1}]$. The normal equations (29) which are solved for the $\tilde{s}^*, \tilde{\Delta}_1^*$ are

$$A^t A \begin{pmatrix} \tilde{s}^* \\ \tilde{\Delta}_1^* \end{pmatrix} = A^t E f = A^t \left(A \begin{pmatrix} \tilde{s} \\ \tilde{\Delta}_1 \end{pmatrix} + \varepsilon \right) = A^t A \begin{pmatrix} \tilde{s} \\ \tilde{\Delta}_1 \end{pmatrix} + A^t \varepsilon, \qquad (30)$$

where ε stands for the errors in approximating the smooth part of f, that is $\varepsilon_j = Eg(x_j)$, and $\tilde{s} = s \cdot \Delta_1$, $\tilde{\Delta}_1 = \Delta_1$ where s, Δ_1 are the true singularity position and the jump in the first derivative, respectively.

In order to prove the desired approximation result $|s^* - s| = O(h^{2(m+B)-1})$, it is enough to prove

$$|\tilde{s}^* - \tilde{s}| = O(h^{2(m+B)-1}) \ , \ |\tilde{\Delta}_1^* - \tilde{\Delta}_1| = O(h^{2(m+B-1)}).$$

By (30) we have

$$\begin{pmatrix} \tilde{s}^* - \tilde{s} \\ \tilde{\Delta}_1^* - \tilde{\Delta}_1 \end{pmatrix} = (A^t A)^{-1} A^t \varepsilon. \qquad (31)$$

It is quite easy to get a bound of the form

$$\|(A^t A)^{-1} A^t\| = O(h^{-r}),$$

with some $r > 0$ where $\|\cdot\| = \sup_{v \neq 0} \frac{\|\cdot v\|_\infty}{\|v\|_\infty}$, $\|v\|_\infty = \max_i\{|v_i|\}$. And since $\varepsilon = O(h^{m+1})$ we will generally only get

$$\|(A^t A)^{-1} A^t \varepsilon\| = O(h^{-r+m+1}).$$

The key property which yields the higher approximation order is that the vectors ϕ and ψ are orthogonal to some polynomial vectors $X^\ell = \{x_i^\ell\}_{i=0}^N$, $\ell = 0, 1, ..., \ell'$. Then, if ε is smooth, it can be well approximated by polynomials, and therefore it will follow that $\|A^t \varepsilon\|$ decreases faster than $\|A^t\| \|\varepsilon\|$ as $h \to 0$.

In the following we prove the main ingredients of the superconvergence result.

Lemma 3.1 *Let $B \in \mathbb{N}^+$ such that Q reproduce polynomials of maximal degree $m+B$ on the nodes X; then we have the following orthogonality relations:*

$$\langle \phi, X^\ell \rangle = 0 \quad \ell = 0,1,\ldots,m+B-2,$$

$$\langle \psi, X^\ell \rangle = 0 \quad \ell = 0,1,\ldots,m+B-1.$$

Proof. First denote the operator T by

$$T(\xi)_j = \xi_j - \sum_i q_i(x_j)\xi_i, \tag{32}$$

then the vectors ϕ, ψ (28) can be written as follows:

$$\phi = T(r^1)\ , \quad \psi = T(r^0),$$

where $r^0_j = (x_j - x_{k+1})^0_+$ and $r^1_j = x_j(x_j - x_{k+1})^0_+$. Since T has finite support on $\mathbb{R}^\infty$ and $T(\{p(j)\}) = 0$ for all $p \in \Pi_{m+B}(\mathbb{R})$ it can be written as

$$T = R\Delta^{m+B+1} = \Delta^{m+B+1}R,$$

where Δ is the forward difference operator, that is, $(\Delta\xi)_j = \xi_{j+1} - \xi_j$ and R has compact support. Next we make use of the summation by parts (S.B.P.) formula:

$$\sum_{i=-M}^{N} f_i \Delta g_i = [(f).(g).]^{N+1}_{-M} - \sum_{i=-M}^{N} g_{i+1}\Delta f_i. \tag{33}$$

Using (33) $m+B-b$ times, $b=0,1$, and since $(\Delta^{m+B+1-\nu}r^b)_j = 0,\ j = 0, N+\nu$ for all $\nu \le m+B-b$ plus the fact that Δ^{m+B-b} annihilates polynomial sequences $\{p(j)\}$ for $p \in \Pi_{m+B-1-b}(\mathbb{R})$ the lemma is proved. ■

Lemma 3.2 *There exists a matrix $\mathcal{Q} \in \mathbb{R}^{N+1\times 2}$, such that*

1. *A can be written as*
$$A = \mathcal{Q}\begin{pmatrix} 1 & x_{k+1} \\ 0 & h \end{pmatrix},$$
where $\mathcal{Q}_{j,i} \neq 0, i = 1,2$ only for some fixed number of indices j around k.

2. *$\mathcal{Q}^t\mathcal{Q}$ and $\|\mathcal{Q}^t\|$ are independent of h.*

Proof. Using the notation of previous lemma,

$$A\begin{pmatrix} 1 & -x_{k+1} \\ 0 & 1 \end{pmatrix} = \left(T(x_\cdot - x_{k+1})^0_+, T(x_\cdot - x_{k+1})_+\right) =$$

$$\left(T(\cdot-(k+1))_+^0, T(\cdot-(k+1))_+\right)\begin{pmatrix} 1 & 0 \\ 0 & h \end{pmatrix}.$$

Therefore

$$A = \left(T(\cdot-(k+1))_+^0, T(\cdot-(k+1))_+\right)\begin{pmatrix} 1 & x_{k+1} \\ 0 & h \end{pmatrix} \equiv \mathcal{Q}\begin{pmatrix} 1 & x_{k+1} \\ 0 & h \end{pmatrix}.$$

This shows (1). Next, note that T is translation invariant. Indeed, denote by E the translation operator, that is $(EX)_j = x_{j-1}$, then

$$TE(\cdot) = ET(\cdot).$$

Hence, we have that

$$T(\cdot-(k+1))_+^0 = TE^{k+1}(\cdot)_+^0 = E^{k+1}T(\cdot)_+^0.$$

And similarly

$$T(\cdot-(k+1))_+ = E^{k+1}T(\cdot)_+.$$

We therefore see that the column vectors of the matrix $\mathcal{Q}$ consist of shifted versions of the constant (independent of h) vectors

$$T(\cdot)_+^0 \ , \ T(\cdot)_+.$$

Therefore, the second claim of the lemma is evident. ■

Lemma 3.3 *The normal matrix $A^tA \in \mathbb{R}^{2\times 2}$ is invertible and we have*

$$\|(A^tA)^{-1}A^t\| = C_1 h^{-1}. \tag{34}$$

Proof. For the first part it is enough to show that the vectors ϕ, ψ are linearly independent. Using (33) $\nu = m+B$ times, we get, as shown in Lemma 3.1,

$$\langle \psi, X^{m+B-1}\rangle = 0. \tag{35}$$

However, we also get

$$\begin{aligned} \langle \phi, X^{m+B-1}\rangle &= (-1)^{m+B-1}\left[(R\Delta r^1)_{\cdot+m+B-1}(\Delta^{m+B-1}X^{m+B-1})_{\cdot}\right]_0^{N+1} \\ &= (-1)^{m+B-1}h^{m+B-1}(m+B-1)!(R\Delta r^1)_{N+m+B} \\ &= (-1)^{m+B-1}h^{m+B}(m+B-1)!(R(\mathbf{1}))_{N+m+B} \\ &\neq 0, \end{aligned} \tag{36}$$

where the second equality follows from the fact that Δr^1 equals zero near the left boundary, and in the last inequality $\mathbf{1}_j = 1$, $R(\mathbf{1})_{N+m+B} = C_1 \neq 0$ since $m+B$ is the maximal reproduction degree of T on the nodes X. Finally, from (36) and (35) it follows that ϕ, ψ are linearly independent. Furthermore, (34) is evident from Lemma 3.2. ■

Finally, we can prove the stated approximation order result:

Theorem 3.1 *Let f be a function of the form*

$$f(x) = g(x) + \Delta_1 (x - s)_+,$$

where $g \in PC^{2(m+B)}$. Let Q be a quasi-interpolation operator of order m as defined in (1). Further assume that $|s - x_j| \geq ch^m, x_j \in X$ for some constant c. Then, the algorithm described in Section 3 results in the approximation

$$|s^* - s| = O(h^{2(m+B)-1}) \ ; \ |\Delta_1^* - \Delta_1| = O(h^{2(m+B)-1}).$$

Proof. It is enough to prove

$$|\tilde{s}^* - \tilde{s}| = O(h^{2(m+B)-1}) \ , \ |\tilde{\Delta}_1^* - \tilde{\Delta}_1| = O(h^{2(m+B-1)}).$$

By Theorem 2.1 we have that $|s^* - s| = O(h^{m+1})$. Combining this with the assumption that $|s - x_j| \geq ch^m$ we get that, for small enough h, s^* is in the same interval as s, that is $s^* \in [x_k, x_{k+1}]$. In this case we have from (31) that

$$|\tilde{\Delta}_1 - \tilde{\Delta}_1^*|, |\tilde{s} - \tilde{s}^*| \leq \|(A^t A)^{-1} A^t \varepsilon\|,$$

where $\varepsilon = R\Delta^{m+B+1} g(X)$. By Lemma 3.1 we have that both ϕ, ψ annihilates polynomials of degree $m + B - 2$, that is

$$\langle \phi, X^\ell \rangle = \langle \psi, X^\ell \rangle = 0 \ , \quad \ell = 0, 1, ..., m + B - 2.$$

Therefore by expanding g to its Taylor series through order $2(m+B) - 1$, that is $g = g_T + O(h^{2(m+B)})$, we have that

$$\|(A^t A)^{-1} A^t \varepsilon\| = \|(A^t A)^{-1} A^t O(h^{2(m+B)})\| = O(h^{2(m+B)-1}),$$

where in the last equality we used Lemma 3.3.

■

4 Noise at isolates data points

The classical approach to detecting and fixing noise at isolated data points [5, 4, 10] is based on using the divided difference operator. One assumes the data is corrupted with noise at some data point x_k, that is,

$$f(x_i) = g(x_i) + \varepsilon\delta(x_i - x_k),$$

where $\delta(z)$ equals one if $z = 0$, and zero otherwise. The method for detecting x_k and approximating the value ε is based on observing distinct pattern of the error

$$\Delta^m\{\varepsilon\delta(x_i - x_k)\}_i = \varepsilon \left(\begin{array}{c} m \\ i \end{array} \right) (-1)^{m-i},$$

This error increases in absolute value as m increases in contrast to the divided difference of the smooth part g which generally decreases as m increases. The above pattern is searched for in the data to locate the corrupted data point x_k, and then a suitable $\varepsilon^* \approx \varepsilon$ is calculated. Finally the input data points are corrected by subtracting this ε^* from the detected data point value. We are not aware of an explicit method for computing ε or any result addressing the approximation order of this method.

This method is closely related to our approach and can be easily understood as a particular instance where the singularity model used is a jump at a single data-point

$$r(x) = \varepsilon\delta(x - x_k).$$

Since we assume the noise is at one of the data points, we can explicitly check every data point x_j, and calculate ε_j^* to minimize (9). Using the normal equations the minimizer is

$$\varepsilon_j^* = \frac{\langle Ef(X), H(X;x_j)\rangle}{\langle H(X;x_j), H(X;x_j)\rangle},$$

where $\left\{H(X;x_j)\right\}_i = \delta(x_i - x_j) - q_k(x_i)$. Then one uses the j and corresponding ε_j^* which minimize (9) among all possible data points to rectify the input data at point x_j.

Using similar analysis to Section 3.1 it is easy to prove the following,

Theorem 4.1 *Let f be a function of the form*

$$f(x) = g(x) + \varepsilon\delta(x - x_k),$$

where $\varepsilon \geq 0$ and $g \in PC^{2(m+B)+2}$. Let Q be a quasi-interpolation operator of order m as defined in (1) which reconstructs polynomials of order $m+B$ at the data points. *Then, for small enough h, the above procedure finds x_k, and approximates the noise level ε to the order $O(h^{2(m+B)+2})$, that is*

$$|\varepsilon^* - \varepsilon| = O(h^{2(m+B)+2}).$$

Proof. For every $s' = x_j$ the functional in (9) becomes

$$\|Ef - \varepsilon_j^* H_j\|^2 = \left\langle Ef - \frac{\langle Ef, H_j\rangle}{\langle H_j, H_j\rangle} H_j, Ef - \frac{\langle Ef, H_j\rangle}{\langle H_j, H_j\rangle} H_j \right\rangle,$$

where we substituted the minimizer $\varepsilon_j^* = \frac{\langle Ef, H_j\rangle}{\langle H_j, H_j\rangle}$, and $H_j = H(X; x_j)$. Using the fact that $Ef = Eg + Er = O(h^{m+1}) + \varepsilon H_k$ we then get

$$\|Ef - \varepsilon_j^* H_j\|^2 = \|\varepsilon H_k - \frac{\langle \varepsilon H_k, H_j\rangle}{\langle H_j, H_j\rangle} H_j\|^2 + O(h^{m+1}),$$

which for small enough h attains its minimum only for $j = k$.

After finding x_k, the normal equations for (9) are

$$H^t H \varepsilon^* = H^t Ef(X) = H^t Eg(X) + H^t H \varepsilon,$$

where $H = H(X; x_k)$. Expanding g to its Taylor series through order $2(m+B)+1$, we get, similarly to Theorem 3.1,

$$H^t H(\varepsilon^* - \varepsilon) = H^t H Eg(X) = O(h^{2(m+B)+2}),$$

and since $H^t H \geq c > 0$ for some c independent of h we get the desired approximation order. ■

5 Numerical experiments and concluding remarks

In this section we present a few numerical experiments with the method presented above. First, in Figure 2 we present an example of approximating a function with the singularity model of the form $c|x-s|_+^\alpha$, where c, s, α are unknown. Here, we used cubic Moving Least Squares on irregular nodes as a quasi-interpolation operator Q. Experimentally, for the case of equidistant points an approximation order $O(h^8)$ to s has been observed.

Figure 3 demonstrates an approximation of a smooth function with a jump in the second derivative using only a few data points (8 or 17). Here we used the Moving Least Squares on irregular nodes. (c) and (e) demonstrate the robustness and accuracy of the optimization process. Following Remark 2.1, the minimization of the functional in (9) reduces to a non-linear minimization in a single variable s'. By Appendix A, in each interval the functional boils down to a rational function of s' with no poles.

Figure 4 demonstrates an approximation of a function with a jump in the first derivative. In this example we used quadratic quasi-interpolating splines on regular nodes.

In Figures 5-7 we compare the approximation order of the singularity location, that is $|s^* - s|$, with the subcell resolution method [1]. In these comparisons we have used the same number of nodes in both methods to locate the singularity within a given cell. In these examples we consider singularity models different from the one we treated in Section 3.1. As shown in Figure 7, the polynomial intersection method is not very efficient in the case of a smooth function with a jump in the second derivative.

In Figure 8 we present a two-dimensional example, where a 2D-function's singularity curve also possesses a point singularity. In this example we have used the univariate detection algorithm twice: First we approximated the singularity along the x-axis for each row of data points, and then used the resulting approximations as univariate functional data along the y-axis to locate its point singularity. The figure depicts the resulting approximated curve singularity along with the true singularity curve.

We experimented with the method applying it to a smooth function. The results are presented in Figure 9. Among other things, in this example the magnitude Δ_1^* tend to zero at the rate of $O(h^5)$. This experiment shows that "ghost singularities" may be identified by checking the convergence of Δ_1^* to zero. On the other hand, it shows that such singularities do not affect the approximation to the function.

Finally, we have experimented with data samples contaminated with noise (Figure 10). Generally, the tradeoff between the noise size ε and the mesh size h can be explained as follows: For a function with a jump singularity at its k-th derivative we expect, using a quasi-interpolation method Q of degree m, an $O(h^{m+1})$ approximation order away from the singularity and the error becomes $O(h^k)$ in a vicinity of the singularity. Hence, as long as the errors near the singularity are larger than the noise level ε, the detection method still works. An example is presented in Figure 10.

In this paper we have presented a new method for alleviating quasi-interpolation techniques to approximate piecewise-smooth data, retaining their approximation power. This is done by a general method of fitting the error in approximating the singularity to the error in the approximation. The method is particularly efficient for locally supported quasi-interpolation operators, and is shown to be very effective for uniform and non-uniform distributions of data points. We have demonstrated the generality of the method by applying it to several singularity models. In the case where not all the function's derivatives have jump discontinuity, we observed a higher order of approximation of the location and size of the jumps. In particular we have proved an $O(h^{2(m+b)-1})$ approximation order to the singularity location and magnitude for functions with a jump in their first derivative. In that case we have also provided a closed-form solution to the minimization problem.
We believe that the general methodology of locally fitting a singularity model to the errors in a non-interpolatory approximation procedure bears further possible applications in higher dimensions.

Appendix A

Theorem A.1 *Fixing s in (9) results in a non-singular linear system for* $\bar{\Delta} = (\Delta_1, \ldots, \Delta_m)$. *In particular the functional in (9) is a rational function of s in each interval* $[x_k, x_{k+1}]$, *with no poles in the interval.*

Proof. It is enough to show that the vectors $H_1(X;s), H_2(X;s), \ldots, H_m(X;s)$, where $(H_j(x;s))_i = H_j(x_i;s)$ and H is defined in (7), are linearly independent.
Note that

$$H_j(X;s) = T(X-s)_+^j = R\Delta^{m+1}(X-s)_+^j,$$

where T is defined in (32). Let $B \in \mathbb{N}^+$ such that Q reproduce polynomials of maximal degree $m+B$ on the nodes X. Using the summation by parts formula (33), $m+B-j$ and $m+B-j+1$ times we get

$$\begin{aligned} \langle H_j, X^\ell \rangle &= 0, \quad \ell \leq m+B-j-1 \\ \langle H_j, X^{m+B-j} \rangle &\neq 0, \end{aligned} \tag{37}$$

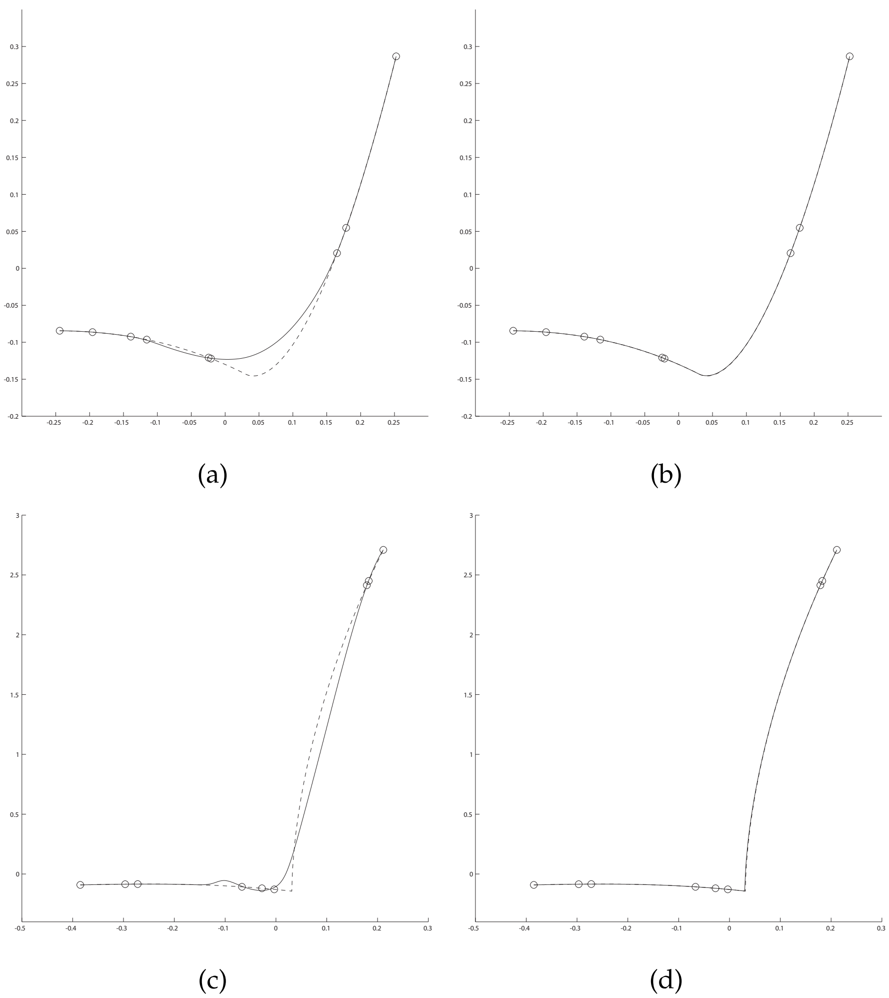


Figure 2: Approximation of a function with an unknown singularity of the type $c|x-s|_{+}^{\alpha}$, where the magnitude c, location s and the exponent α are unknown. (a,c) show the approximations using cubic MLS. A dashed line represents the unknown function f and the solid line denotes the MLS approximant. (b,d) show the MLS augmented with the fitted singular function. In (a-b) the data were taken from a smooth function with singularity at $s = \pi/100$ and exponent $\alpha = \sqrt{3}$ and in (c-d) the unknown exponent is $\alpha = 1/\sqrt{3}$.

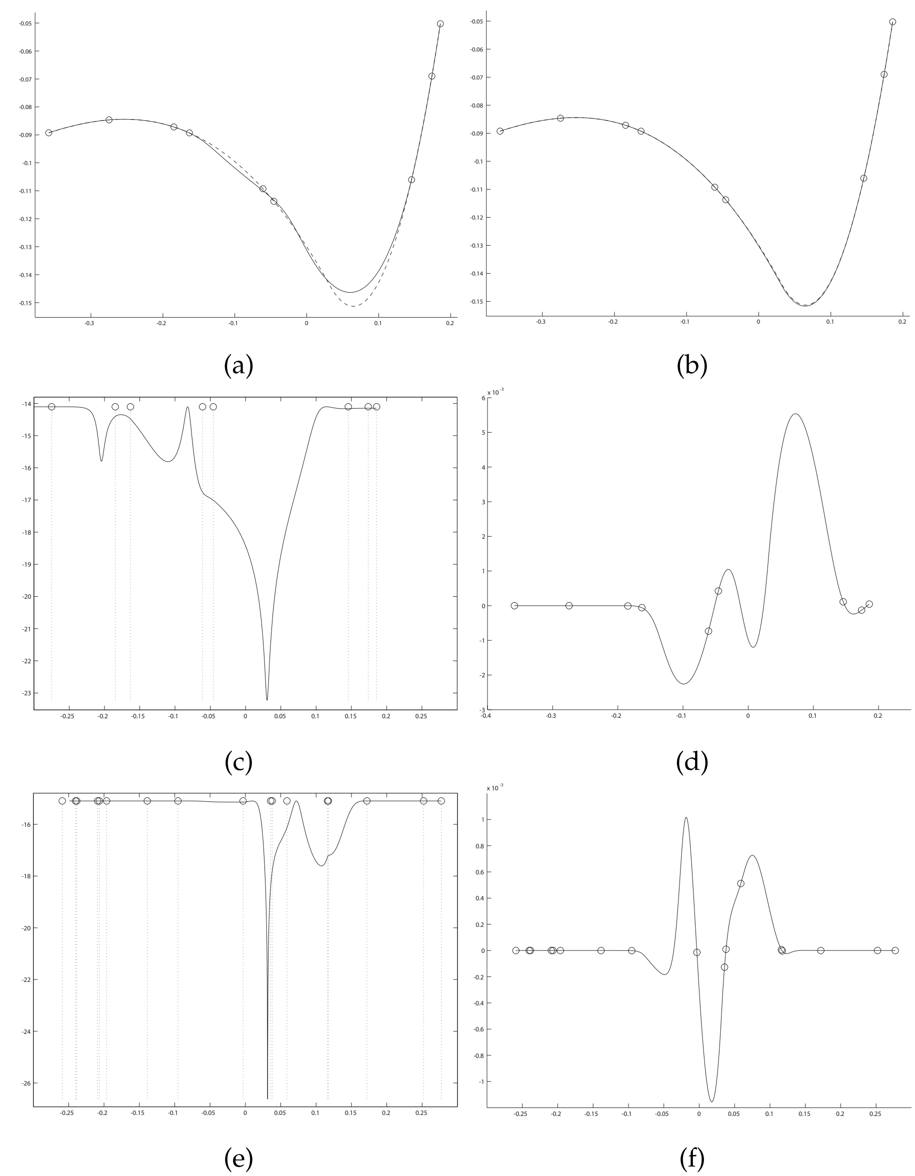
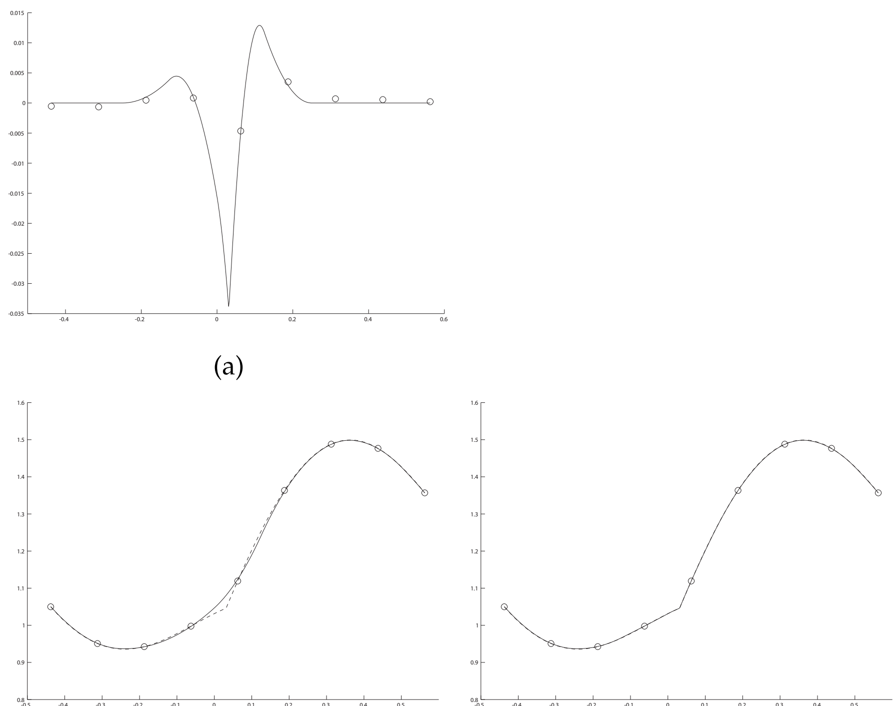
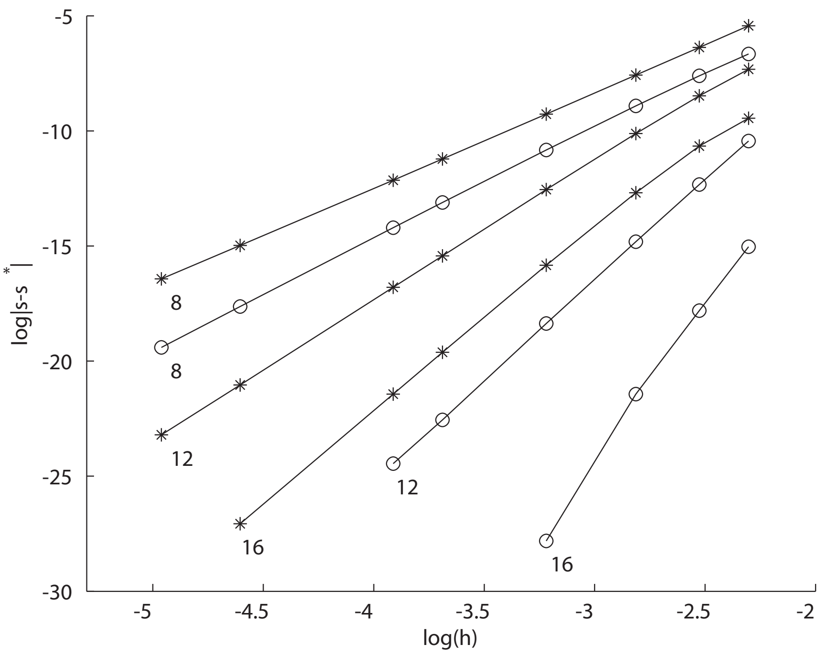


Figure 3: Approximation of a C^1 function with a jump discontinuity in the second derivative, that is, $f(x) = 8(x - \frac{\pi}{100})_+^2 + e^{-x^2} + 0.3\sin(2x) - x - 1$. The data sites are irregular in this case. (c) depicts the logarithm of the error functional value (9) as a function of the single parameter s'. (d) shows the fitted function $Er(x; s', \bar{\Delta}')$ to the error data $\{Ef(x_j)\}$, depicted with circles. In (a) the unknown function f is shown by a dashed line and the MLS approximation by a solid line. (b) shows the MLS augmented with the singular function (d). (e) and (f) show the logarithm of the error functional and the fitted function $Er(x; s', \bar{\Delta}')$ for twice the number of data points.



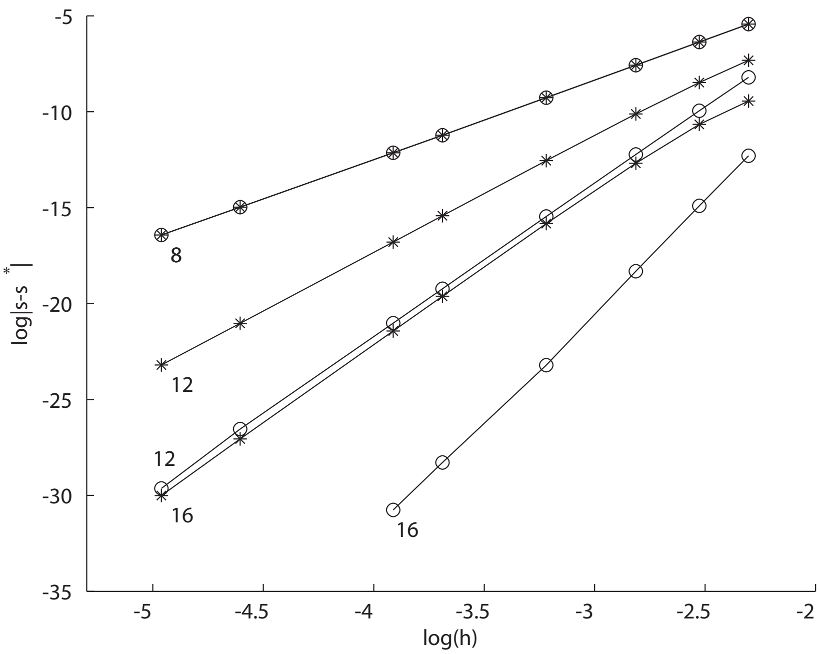
(a) (b) (c)

Figure 4: Approximation of a function with a jump discontinuity in the first derivative, that is, $f(x) = |x - \frac{\pi}{100}| + e^{-x^2} + 0.3\sin(5x)$. (a) depicts the fitted function $Er(x; s', \bar{\Delta}')$ to the error data $\{Ef(x_i)\}$ (circles). (b) shows the approximation using quadratic quasi-interpolating B-splines. In dashed line is the unknown function f. (c) shows the approximant (b) augmented with the singular function (a).



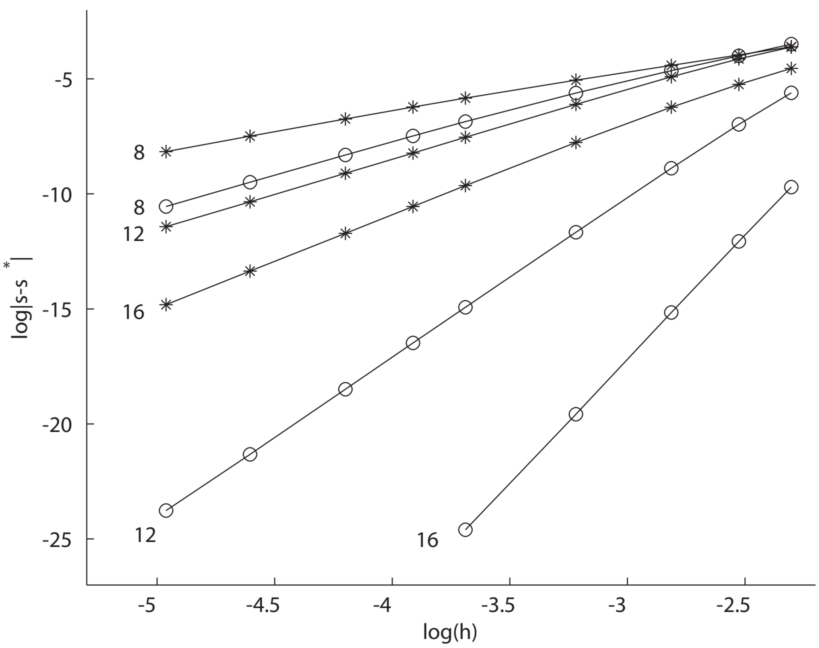
Total points number	8	12	16
Poly-intersect	$O(h^{4.1})$	$O(h^{6})$	$O(h^{7.7})$
Quasi-interp method	$O(h^{4.8})$	$O(h^{8.7})$	$O(h^{14})$

Figure 5: Comparison with the subcell resolution method w.r.t the total number of points used. The singularity model here takes into account a jump in the first and the second derivatives. The function used $f(x) = |x-\pi/3| + \sqrt{2}(x-\pi/3)_+^2 + e^{-x^2} + 0.3\sin(5x)$. A graph of $\log|s-s^*|$ versus $\log h$ is depicted. The total number of points used is written by the relevant curve. Circle markers denote the method presented in the paper. Asterisks markers show the polynomial intersection (subcell resolution) method.



Total points number	8	12	16
Poly-intersect	$O(h^{4.1})$	$O(h^{6})$	$O(h^{7.8})$
Quasi-interp method	$O(h^{4.1})$	$O(h^{8})$	$O(h^{11.5})$

Figure 6: Comparison with subcell resolution method w.r.t the number of points used. The singularity model here takes into account jumps upto (and including) the third derivative. The function used $f(x) = |x - \pi/3| + \sqrt{2}(x - \pi/3)_+^3 + e^{-x^2} + 0.3\sin(5x)$. A graph of $\log|s - s^*|$ versus $\log h$ is depicted. The total number of points used is written by the relevant curve. Circle markers denote the method presented in the paper. Asterisks markers show the polynomial intersection (subcell resolution) method.



Total points number	8	12	16
Poly-intersect	$O(h^{1.7})$	$O(h^{3})$	$O(h^{3.9})$
Quasi-interp method	$O(h^{2.7})$	$O(h^{6.9})$	$O(h^{10.8})$

Figure 7: Comparison with subcell resolution method w.r.t the number of points used. The singularity model here takes into account a continuous first derivative and a jump in the second and third derivative. The function used is $f(x) = \sqrt{2}(x-\pi/3)_+^2 + e^{-x^2} + 0.3\sin(5x)$. A graph of $\log|s-s^*|$ versus $\log h$ is depicted. The total number of points used is written by the relevant curve. Circle markers denote the method presented in the paper. Asterisks markers show the polynomial intersection (subcell resolution) method.

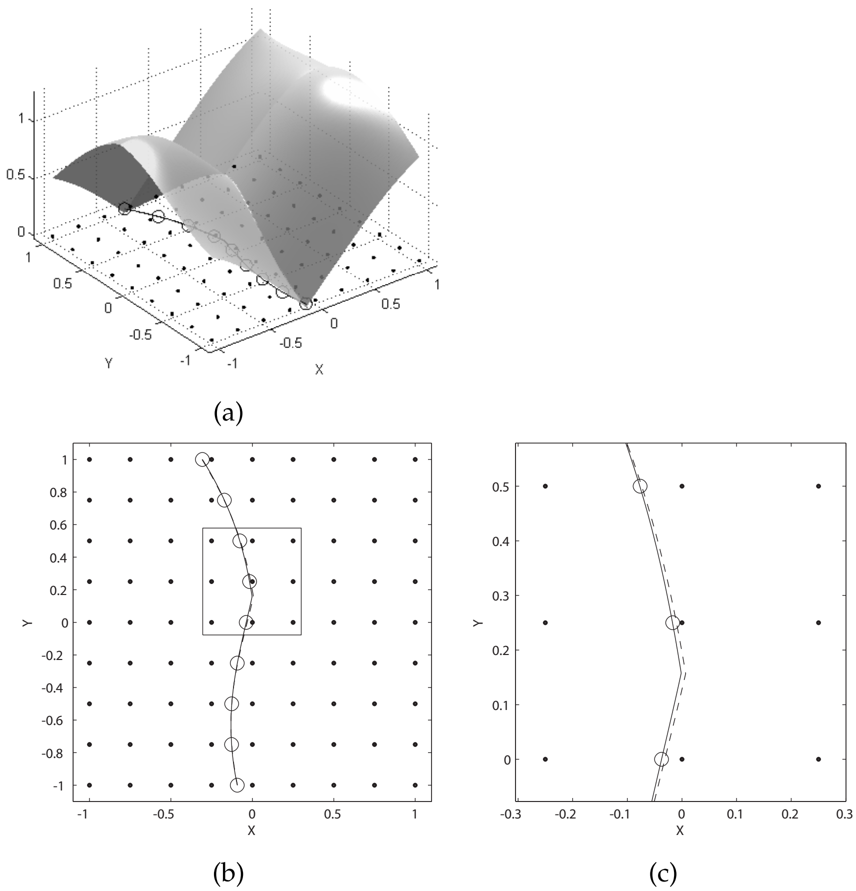
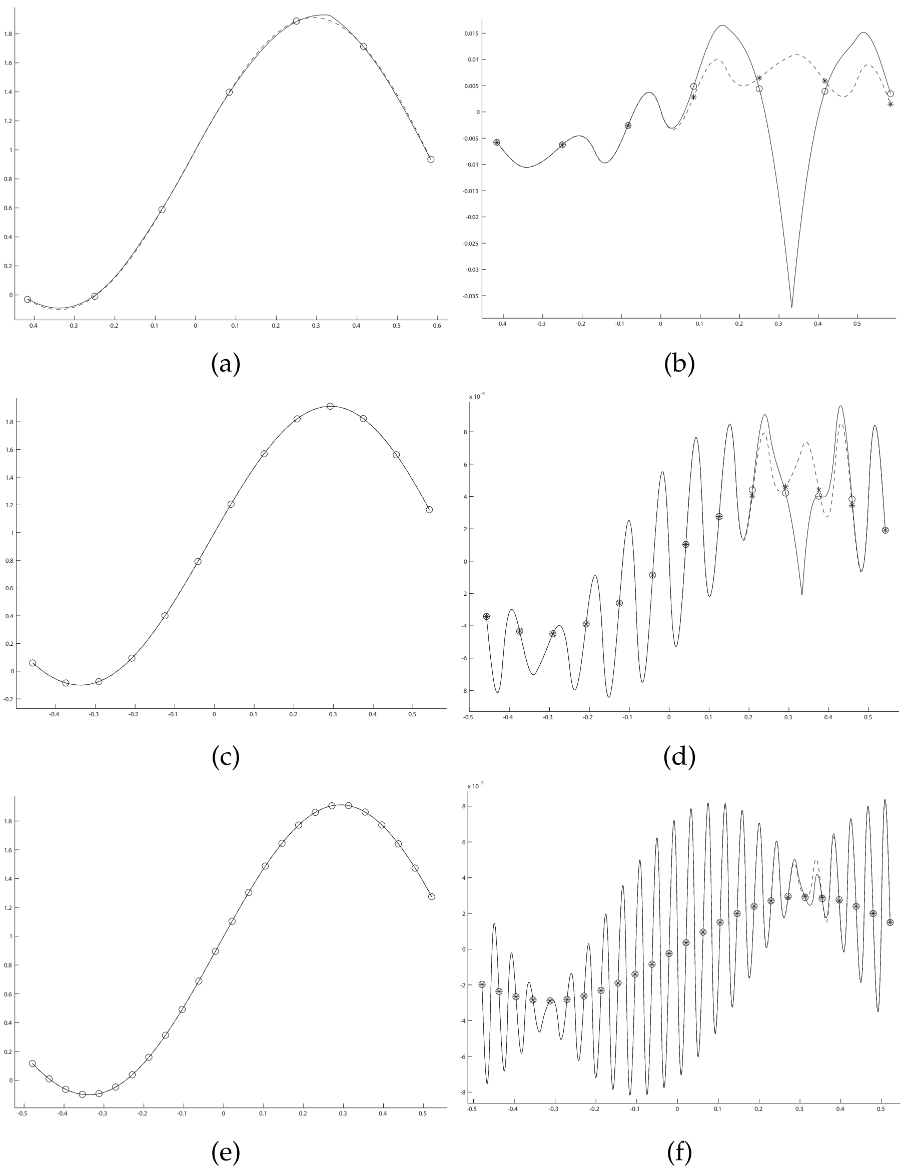


Figure 8: 2D example. The dashed curves in (b) and (c) denote the true singularity curve, and the continuous curves are the approximated singularity curve using a two-step procedure. The circles denote the $x-$singularity detection phase results. The function used in this case is $f(x,y) = e^{-y^2-0.5x^4} - 0.4 + \left|x + \frac{\pi}{100} - 0.1\sin(2.5y) + 0.4(y - \frac{\pi}{20})_+\right|$. Numerical evaluation of the singularity location along the curve, $s = \frac{\pi}{20}$, yields the approximation order $|s^* - \frac{\pi}{20}| = O(h^5)$.



h	0.16	0.08	0.04	0.02
Q max error *on nodes*	$6.46 \cdot 10^{-3}$	$4.60 \cdot 10^{-4}$	$2.95 \cdot 10^{-5}$	$1.85 \cdot 10^{-6}$
$\bar{Q}$ max error *on nodes*	$6.27 \cdot 10^{-3}$	$4.49 \cdot 10^{-4}$	$2.95 \cdot 10^{-5}$	$1.84 \cdot 10^{-6}$
Q max error	$1.09 \cdot 10^{-2}$	$8.55 \cdot 10^{-4}$	$8.37 \cdot 10^{-5}$	$9.36 \cdot 10^{-6}$
$\bar{Q}$ max error	$3.72 \cdot 10^{-2}$	$9.61 \cdot 10^{-4}$	$8.37 \cdot 10^{-5}$	$9.26 \cdot 10^{-6}$
$\lvert\Delta_1^*\rvert$	1.54	0.059	0.002	$6.13 \cdot 10^{-5}$

Figure 9: The results of applying the method (using quadratic quasi-interpolating B-splines) to data sampled from smooth function $f(x) = e^{-x^2} + \sin(5x)$. The left column depicts the $\bar{Q}$ approximant in a solid line and the function f in a dashed line. In the right column the error of $\bar{Q}$ is a solid line and the error of Q is a dashed line. In (a-b) $h = 0.16$, (c-d) $h = 0.08$, (e-f) $h = 0.04$. The table at the bottom summarizes the results. Note that the decreasing rate of Δ_1^* is $O(h^5)$ in this case.

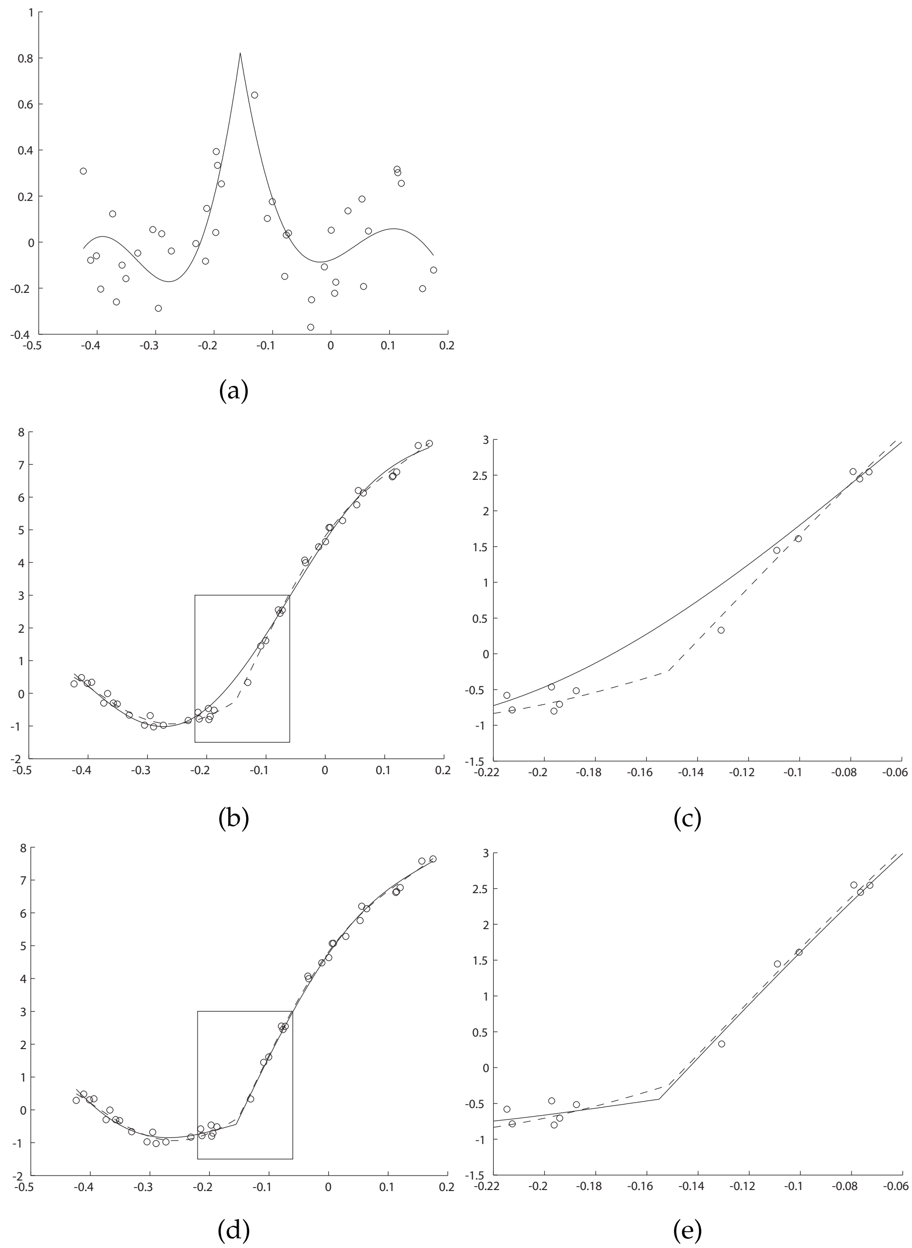


Figure 10: Approximation using a noisy data set sampled from a function with a jump discontinuity in the first derivative. (a) depicts the fitted function $Er(x;s',\bar{\Delta}')$ to the error data $\{Ef(x_i)\}$ (circles). (b) shows the MLS cubic quasi-interpolant. In dashed line is the unknown function f (zoom-in view in (c)). (d) shows the rectified quasi-interpolant $\bar{Q}$ (zoom-in view in (e)).

respectively. And since the vectors $X^B, ..., X^{m+B-1}$ are linearly independent the lemma follows. ■

As a simple consequence of this theorem we prove a global minimizer s^* of the functional in (9) exists:

Corollary A.1 *There exists a global minimizer $s^* \in [a, b]$ to the functional in (9).*

Proof. We show a minimizer to the functional where $\bar{\Delta}'$ is taken as a function of s', that is $\bar{\Delta}' = \bar{\Delta}'(s')$. By Theorem A.1 we have that the functional in (9) is a continuous function of s' defined on a closed interval and therefore from standard argumentation has a global minima inside this interval.■

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