Stable Moving Least-Squares

Yaron Lipman

School of Mathematical Sciences, Tel-Aviv University

Abstract

It is a common procedure for scattered data approximation to use local polynomial fitting in the least-squares sense. An important instance is the Moving Least-Squares where the corresponding weights of the data site vary smoothly, resulting in a smooth approximation. In this paper we build upon the techniques presented by Wendland and present a somewhat simpler error analysis of the MLS approximation. Then, we show by example that the $\sqrt{N}$ factor, which appears in the bound on the Lebesgue constant in [10], where N is the number of points used in the approximation, can be realized. Hence, we device a method for choosing the weights smoothly so that the corresponding Lebesgue constant can be bounded independently of N. This is done by employing Voronoi weights. We conclude with some numerical examples exhibiting the effectiveness of the suggested method for highly irregular data sites.

Key words:
Moving Least-Squares, scattered data approximation.

1 Introduction

The Moving Least-Squares (MLS) method is a method for scattered data approximation [1–3,5,7,4]. Given a scattered data set $(X,F) = \{(x_i, f_i)\}_{i=1}^N$ in some domain, $X \subset \Omega \subset \mathbb{R}^d$, the m-degree MLS method, fits for each point $x \in \Omega$, a polynomial $p \in \Pi_m(\Omega)$ and evaluates it at x. Here, $\Pi_m(\Omega)$ denotes the d-variate polynomial space of total degree m. The local polynomial is fitted in a weighted least-squares sense with weights decaying smoothly with the distance from x, resulting in a smooth overall approximant.

An error analysis for the m-degree MLS has been given by [8]. Levin proved (under some conditions on X) an $O(h^{m+1})$ approximation order of the m-degree MLS approximant, where h is the *fill distance* of the data X defined later on. Later, Wendland [10,11] succeeded in formulating a more concrete error analysis where the constants in the error bound are explicitly formulated in terms of the problem parameters. Wendland has exploited the recent *norming sets* idea by Jetter , Stockler and Ward [6] which allows one to find a norm on $\mathbb{R}^N$ equivalent to a norm on $\Pi_m(\Omega)$. Wendland established the following error bound to the MLS approximation:

$$\left|f(x) - MLS_f(x)\right| \leq \|f - p^*\|_{L_\infty(B(x,C_2h))} \left(1 + C_1 (\#I_\delta(x))^{1/2}\right), \tag{1.1}$$

where h is the fill distance of X, $B(x,r)$ denotes a closed ball of radius r centered at x, $\delta = C_3 h$, $\#I_\delta(x)$ is the number of points in $X \cap B(x,\delta)$, and p^* is the local best approximating polynomial to f. The constants C_1, C_2, C_3 are given explicitly in terms of the domain parameters, the degree polynomial m and the weights in the MLS approximation. Wendland's error analysis implies that the MLS error bound depends upon the local points' density. In particular, Wendland's error bound may become very large in the case of non-quasi-uniform data. In this context there are two interesting questions: Can this bound be realized, that is, is it sharp? If so, how can the MLS procedure be alleviated in situations of non-uniform data sites' distribution? In this paper we show that the answer to the first question is affirmative and suggests a way of choosing the MLS weights to produce an approximant which its corresponding error bound is independent of the points' distribution. We call the new version Stable Moving Least-Squares (Stable MLS).

The paper is organized as follows: In Section 2 we suggest a new formulation of the MLS operator, based on a certain extension of the inverse *sampling operator* [6,10], which generalizes the existing MLS formulation. Using the norming sets methodology by Jetter, Stockler, Ward and Wendland we are able to formulate a somewhat simpler and shorter error analysis to the MLS approximation. In this section we also prove a simple error formula for the MLS approximant. In Section 3 we describe how to choose the MLS weight function by using not only radial weight distributions but "spatially-aware" weight distribution to achieve stable approximation independently of the points' density. We conclude in Section 4 with several numerical experiments.

2 The MLS operator as an extension of the inverse sampling operator

First, let us lay out a definition of the MLS approximation operator in a more general context. We will show that the MLS operator can be defined by setting a family *semi-inner-products* in $\mathbb{R}^N$, where N is the number of data sites. Recall that a semi-inner-product does not possess the positivity property of an inner-product.

Let us consider domain $\Omega \subset \mathbb{R}^d$ which satisfies the *cone condition* (see E.g. [11]),

Definition 2.1 *A domain $\Omega \subset \mathbb{R}^d$ is said to satisfy the* cone condition *if there exist constants $r > 0, \theta \in (0,\pi)$ and a vector valued function $\xi : \Omega \to S^{d-1} \subset \mathbb{R}^d$, where S^{d-1} is the unit sphere, such that for every point $x \in \Omega$ the cone*

$$C(x,\xi(x),r,\theta) := \{x + \lambda y : y \in \mathbb{R}^d, \|y\|_2 = 1, \langle y, \xi(x)\rangle \geq \cos\theta, \lambda \in [0,r]\}$$

is contained in Ω. $\|\cdot\|_2$ *is the Euclidian norm in* $\mathbb{R}^d$.

Let $X = \{x_i\}_{i=1}^N \subset \Omega$ be a set of irregular data sites inside the domain, and $f_i = f(x_i)$ samples of some smooth function $f \in C^{m+1}(\Omega)$ at those data sites. Let us also define the *fill distance* h of X in Ω:

Definition 2.2 $h = h_{X,\Omega} := \max_{y\in\Omega} \min_{x_i\in X} \|y - x_i\|_2$.

h is the radius of the largest open ball with a center in Ω which does not contain a point from X.

Let $x \in \Omega$ be an arbitrary point, and denote $V = \Pi_m(\Omega)$. Define the sampling operator [11] $T : V \to T(V) \subset \mathbb{R}^N$ by

$$T_X(p) = T(p) = (p(x_1), ..., p(x_N)).$$

In the function space $C^{m+1}(\Omega)$ we use the maximum norm $\|f\|_{L_\infty(\Omega)}$, and in $\mathbb{R}^N$ we use a weighted semi-inner-product $\langle\cdot,\cdot\rangle_x$: $\xi, \nu \in \mathbb{R}^N$,

$$\langle \xi, \nu \rangle_x = \sum_{i=1}^N w_i^x \xi_i \nu_i,$$

$w_i^x \geq 0$, with the induced semi-norm $\|\cdot\|_x$. Here we differ from [11] where the l_∞ norm is used in $\mathbb{R}^N$.

If T is injective, then X is said to be a *norming set* [6,11]. Let us define:

Definition 2.3 *X is* norming set w.r.t. $\|\cdot\|_x$ *if* $\|T(v)\|_x = 0$ *implies that* $v = 0$, *for all* $v \in V$.

Correspondingly, the *norming constant* is defined to be

$$\|T^{-1}\| = \sup_{0\neq z\in T(V)} \frac{\|T^{-1}z\|_{L_\infty(\Omega)}}{\|z\|_x} = \sup_{0\neq p\in V} \frac{\|p\|_{L_\infty(\Omega)}}{\|T(p)\|_x}. \tag{2.1}$$

Define $L^x : \mathbb{R}^N \to T(V) \subset \mathbb{R}^N$ to be the best approximation operator from the subspace $T(V)$ in the semi-norm $\|\cdot\|_x$, namely, the least-squares projection. Let us prove that L^x is well-defined:

Theorem 2.4 *Let X be a norming set w.r.t.* $\|\cdot\|_x$ *, then for every* $z \in \mathbb{R}^N$ *there exist a unique projection* $L^x(z)$*, such that*

$$\|z - L^x(z)\|_x \leq \|z - z'\|_x,$$

for all $z' \in T(V)$*, and equality holds only for* $z' = L^x(z)$.

Proof. $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_x$ is an inner-product on $T(V)$ by assumption. Denote by $\{e_1,...,e_J\} \subset T(V)$, $J = \dim(V)$ an orthonormal basis to $T(V)$. Define $L^x(\cdot)$ by:

$$L^x(z) = \sum_{j=1}^{J} \langle z, e_j\rangle e_j.$$

It is easy to check that

$$z - L^x(z) \in T(V)^{\perp}. \tag{2.2}$$

Therefore, by Pythagoras Theorem for all $z' \in T(V)$,

$$\|z - z'\|_x^2 = \|z - L^x(z)\|_x^2 + \|L^x(z) - z'\|_x^2 \geq \|z - L^x(z)\|_x^2,$$

and the existence is proved. For uniqueness, let $s \in T(V)$ be such that

$$\|z - s\|_x \leq \|z - L^x(z)\|_x.$$

Then, from the first part of the proof we have that

$$\|z - s\|_x = \|z - L^x(z)\|_x. \tag{2.3}$$

From (2.2) it follows that $z - L^x(x) \perp L^x(z) - s$ and therefore from (2.3)

$$\|z - L^x(z)\|_x^2 + \|L^x(z) - s\|_x^2 = \|z - s\|_x^2 = \|z - L^x(z)\|_x^2,$$

subtracting $\|z - L^x(z)\|_x^2$ from right-most and left-most sides we get

$$L^x(z) - s = 0,$$

from the fact that $\|\cdot\|_x$ is a norm on $T(V)$. ■

Henceforth, let us assume X is a norming set w.r.t $\|\cdot\|_x$. Now, let us use the projection $L^x(\cdot)$ in order to define the MLS operator as the following extension of T^{-1} to $\mathbb{R}^N$:

$$\mathscr{M}_{f,X}(x) := \left\{T^{-1}L^x f\right\}[x],$$

where $f = (f_1,...,f_N)^t \in \mathbb{R}^N$ is the data. We also define the MLS local fitted polynomial at x by

$$\mathscr{P}_{f,X,x}(y) := \left\{T^{-1}L^x f\right\}[y],$$

where y is the argument of the polynomial. Furthermore,

$$\mathscr{M}_{f,X}(x) = \mathscr{P}_{f,X,x}(x).$$

The MLS operator reproduces polynomials, this can easily be seen from the above definition since L^x is the identity operator on $T(V)$: Let $p \in V$

$$\mathscr{M}_{p,X}(x) := \{T^{-1}L^x T(p)\}[x] = \{T^{-1}T(p)\}[x] = p(x). \tag{2.4}$$

An interesting point is that the MLS operator can be seen as a norm preserving extension of the inverse sampling operator. For that end let us define a norm on the (sub-) space S of linear operators $A : \mathbb{R}^N \to V$ such that $A(z) = 0$ for all z satisfying $\|z\|_x = 0$. We denote

$$\|A\| = \sup_{\substack{z \in \mathbb{R}^N \\ \|z\|_x > 0}} \frac{\|A(z)\|_{L_\infty(\Omega)}}{\|z\|_x}. \tag{2.5}$$

This defines a norm on the above described linear space of operators since if $\|A\| = 0$, then for z satisfying $\|z\|_x > 0$ we have from the definition (2.5) that $A(z) = 0$, and for z satisfying $\|z\|_x = 0$ we have $A(z) = 0$ from the definition of S. Note that our extension of the inverse sampling operator is an operator in that space, that is $\|T^{-1}L^x z\|_{L_\infty(\Omega)} = 0$ for z satisfying $\|z\|_x = 0$. This can be understood from Theorem 2.4 by taking z satisfying $\|z\|_x = 0$ and $z' = 0$ and getting

$$0 \leq \|z - L^x(z)\|_x \leq \|z\|_x = 0$$

and therefore $L^x(z) = 0$.

Theorem 2.5 *The MLS extension of the inverse sampling operator is norm-preserving, that is*

$$\|T^{-1}\| = \|T^{-1}L^x\|,$$

where $T^{-1} : T(V) \to V$, $T^{-1}L^x : \mathbb{R}^N \to V$ *with the norm defined in (2.5).*

Proof. First, for all $z \in \mathbb{R}^N$, $\|z\|_x > 0$

$$\frac{\|T^{-1}L^x(z)\|^2_{L_\infty(\Omega)}}{\|z\|_x^2} = \frac{\|T^{-1}L^x(z)\|^2_{L_\infty(\Omega)}}{\|z - L^x(z)\|_x^2 + \|L^x(z)\|_x^2} \leq \frac{\|T^{-1}L^x(z)\|^2_{L_\infty(\Omega)}}{\|L^x(z)\|_x^2} \leq \|T^{-1}\|^2.$$

The fact that $\|T^{-1}\| \leq \|T^{-1}L^x\|$ follows from the fact that $T^{-1} = T^{-1}L^x$ on $T(V)$. ■

Let us write the MLS operator in coordinates, that is, using the bases $B(y) = \{b_1(y), \ldots, b_J(y)\}$ for V and $\tilde{B} = \{\tilde{b}_1, \ldots, \tilde{b}_J\}$, where $\tilde{b}_j = \left(b_j(x_1), \ldots, b_j(x_N)\right)^t$ for $T(V)$. By the normal equations for the MLS projection,

$$\tilde{B}^t W \tilde{B} \Lambda = \tilde{B}^t W T(f), \tag{2.6}$$

where $\tilde{B} = \left(\tilde{b}_1, \ldots, \tilde{b}_J\right)$, $W = diag\left(w_1^x, \ldots, w_N^x\right)$, $\Lambda = (\lambda_1, \ldots, \lambda_J)^t$. Therefore the MLS polynomial $\mathscr{P}_{f,X,x}$ can be calculated by

$$\mathscr{P}_{f,X,x}(y) = B(y)\left(\tilde{B}^t W \tilde{B}\right)^{-1} \tilde{B}^t W T(f) = B(y)\Lambda = \sum_{j=1}^{J} \lambda_j b_j(y). \tag{2.7}$$

First, we prove a simple and useful error formula for the MLS polynomial. This formula can be seen as a generalization of the well-known error formula for univariate interpolation [9]

Theorem 2.6 *Let $f \in C^{m+1}(\Omega)$, $\alpha = (\alpha^1, ..., \alpha^d)$, $|\alpha| \le m$ and $\nu = (\nu^1, ..., \nu^d)$, $|\nu| = m+1$ multi-indices. There exist $\xi_{i,\nu} \in (0,1)$ such that*

$$D^\alpha f(x) - D^\alpha \mathscr{P}_{f,X,x}(x) = -\alpha! \sum_{i=1}^{N} \sum_{|\nu|=m+1} \frac{D^\nu f(x + \xi_{i,\nu}(x_i - x))}{\nu!} (x_i - x)^\nu [T^{-1}L^x]_{\alpha,i}, \tag{2.8}$$

where $[T^{-1}L^x]_{\alpha,i}$ is the (α, i) coordinate of the matrix $\left(\tilde{B}^t W \tilde{B}\right)^{-1} \tilde{B}^t W$ which corresponds to the transformation $T^{-1}L^x$ in the bases $E, \tilde{B}$ where E is the standard basis.

Proof. First, note that the MLS operator is invariant to translations, that is,

$$\mathscr{M}_{f(\cdot),X}(x) = \mathscr{M}_{f(\cdot - c),X+c}(x+c).$$

Therefore, we can assume w.l.o.g that $x = 0$.

Expanding f to its Taylor series around $x = 0$, evaluated at $x = x_i$,

$$f(x_i) = \sum_{|\nu| \le m} \frac{D^\nu f(0)}{\nu!} x_i^\nu + \sum_{|\nu|=m+1} \frac{D^\nu f(\xi_{i,\nu} x_i)}{\nu!} x_i^\nu,$$

where $\xi_{i,\nu} \in (0,1)$. Using the polynomial reproduction property (2.4) we have

$$f(0) - \mathscr{P}_{f,X,0}(0) = f(0) - T^{-1}L^0 f(X)[0] = -T^{-1}L^0 R,$$

where

$$R = \left(\sum_{|\nu|=m+1} \frac{D^\nu f(\xi_{1,\nu} x_1)}{\nu!} x_1^\nu, ..., \sum_{|\nu|=m+1} \frac{D^\nu f(\xi_{N,\nu} x_N)}{\nu!} x_N^\nu \right)^t.$$

We can choose the basis $b_i(y)$ freely, so let us take the standard basis $(y)^\beta$, $|\beta| \le m$. In that case

$$D^\alpha|_{y=0} T^{-1}L^0 R = (D^\alpha B(0)) \left(\tilde{B}^t W \tilde{B}\right)^{-1} \tilde{B}^t W R. \tag{2.9}$$

Note that

$$D^\alpha B(0) = \alpha! e_\alpha,$$

where $e_\alpha \in \mathbb{R}^{1\times J}$ is the zero vector except for a one at the α entry. Therefore we have

$$D^\alpha|_{y=0}\left(f(y)-\mathscr{P}_{f,X,0}(y)\right) = -D^\alpha B(0)\left(\tilde{B}^t W\tilde{B}\right)^{-1}\tilde{B}^t WR = -\alpha!\sum_{i=1}^{N}[T^{-1}L^x]_{\alpha,i}R_i,$$

where $[T^{-1}L^x]_{\alpha,i}$ is the (α,i) entry of the matrix $\left(\tilde{B}^t W\tilde{B}\right)^{-1}\tilde{B}^t W$. Note that for $\alpha=\bar{0}$, $[T^{-1}L^x]_{\alpha,i}$ are actually the basis (shape) functions of the MLS. ■

It should be noted that in the proof we use the extra assumption that the line between the data points x and x_i is contained in the domain Ω.

2.2 *Error analysis*

Let us now present a convergence analysis of the MLS operator, based on the new MLS formulation and the notion of norming sets.

A useful error bound can be achieved by the following argumentation: Let X be a norming set w.r.t $\|\cdot\|_x$, denote by δ_x the support of $\langle\cdot,\cdot\rangle_x$, that is,

$$\delta_x = \max_{i:w_i^x>0}\|x-x_i\|_2.$$

For any set $\mathscr{D}\subset B(x,\delta_x)\cap\Omega$ such that $x\in\mathscr{D}$,we have

$$\begin{aligned}|\mathscr{M}_{f,X}(x)-f(x)| &\le \|T^{-1}L^xT(f)-f\|_{L_\infty(\mathscr{D})}\\ &\le \|f-p^*\|_{L_\infty(\mathscr{D})}+\|T^{-1}L^xT(f)-T^{-1}L^xT(p^*)\|_{L_\infty(\mathscr{D})},\end{aligned}$$

and since $\|T(f)-T(p^*)\|_x \le \|f-p^*\|_{L_\infty(B(x,\delta_x)\cap\Omega)}\left(\sum_{i=1}^N w_i^x\right)^{1/2}$ we have,

$$|\mathscr{M}_{f,X}(x)-f(x)| \le \|f-p^*\|_{L_\infty(B(x,\delta_x)\cap\Omega)}\left\{1+\|T^{-1}L^x\|\|1\|_x\right\}, \tag{2.10}$$

where

$$\|T^{-1}L^x\| = \sup_{\substack{z\in\mathbb{R}^N\\ \|z\|_x>0}}\frac{\|T^{-1}L^x(z)\|_{L_\infty(\mathscr{D})}}{\|z\|_x}.$$

From Theorem 2.5, (2.10) becomes

$$|\mathscr{M}_{f,X}(x)-f(x)| \le \|f-p^*\|_{L_\infty(B(x,\delta_x)\cap\Omega)}\left\{1+\|T^{-1}\|\|1\|_x\right\}, \tag{2.11}$$

where $\|T^{-1}\| = \sup_{0\neq z\in T(V)} \frac{\|T^{-1}(z)\|_{L_\infty(\mathscr{D})}}{\|z\|_x}$ is the norming constant. The Lebesgue constant L_c is defined to be the minimal constant such that

$$|\mathscr{M}_{f,X}(x) - f(x)| \leq L_c \|f - p^*\|_{L_\infty(B(x,\delta_x)\cap\Omega)}.$$

From Eq. (2.11) we have

$$L_c \leq 1 + \|T^{-1}\|\|1\|_x. \tag{2.12}$$

According to (2.11), in order to prove the MLS approximation order, one should use a semi-inner-product $\langle\cdot,\cdot\rangle_x$ such that $\delta_x \leq \mathscr{C}h$, where $\mathscr{C}$ is some constant. In that case $\|f - p^*\|_{L_\infty(B(x,\delta_x)\cap\Omega)}$ will exhibit the desired (full) approximation order $O(h^{m+1})$. This can be seen by using for example the truncated Taylor series for f around x. For such δ_x we look for a sub-domain $\mathscr{D} \subset B(x,\delta_x)\cap\Omega$ such that the norming constant $\|T^{-1}\|$, and therefore the Lebesgue constant, can be bounded independently of the fill distance h.

We proceed by adopting some argumentations from Wendland [11]. In particular, we use the following three results, where the last one is a slight modification of Wendland's original result.

Lemma 2.7 *A cone $C = C(x,\xi,r,\theta)$ contains the ball $B(x + \frac{h}{\sin\theta}\xi, h)$ for $h \leq \frac{r\sin\theta}{1+\sin\theta}$.*

For a domain Ω which satisfies the cone condition with constants r,θ, let us set a constant $\kappa = \kappa(\theta) = \frac{3\sin^2\theta}{16(1+\sin\theta)^2}$.

Lemma 2.8 *A cone $C = C(x,\xi,r,\theta)$ with $r > 0$ and $0 < \theta \leq \pi/5$, satisfies the cone condition with constants $\tilde{\theta} = \theta$ and $\tilde{r} = \sqrt{3\kappa}r$.*

Lemma 2.9 *Let Ω be a domain which satisfies the cone condition with constants r,θ. Assume the fill distance h of the set $X = \{x_i\}_{i=1}^N$ satisfies $h \leq \frac{\kappa}{m^2}r$. Let $x \in \Omega$ be an arbitrary point, and set $\delta = \frac{m^2}{\kappa}h$, $C = C(x,\xi,\delta,\theta)$ its corresponding cone. Then, for every $p \in \Pi_m(C)$, $\|p\|_{L_\infty(C)} = 1$, there exists a ball $B_h = B(y,h) \subset C$ and for every point $x_i \in X \cap B_h$ $|p(x_i)| \geq \frac{1}{2}$.*

Proof. First note that the condition on h assures that $C \subset \Omega$. Next, consider $p \in \Pi_m(C)$, $\|p\|_{L_\infty(C)} = 1$ and let $x^* \in C$ be such that $|p(x^*)| = \|p\|_{L_\infty(C)} = 1$. By Lemma 2.8 we have that C satisfies the cone condition with constants $\tilde{r} = \sqrt{3\kappa}\delta$ and $\tilde{\theta} = \theta$. Hence there exists a cone $\tilde{C} = C(x^*,\tilde{\xi}(x^*),\tilde{r},\theta) \subset C$. Now, since

$$h = \frac{\kappa\delta}{m^2} = \frac{\kappa}{m^2}\frac{\tilde{r}}{\sqrt{3\kappa}} = \frac{1}{4m^2}\tilde{r}\frac{\sin\theta}{1+\sin\theta} \leq \tilde{r}\frac{\sin\theta}{1+\sin\theta}, \tag{2.13}$$

from Lemma 2.7 we have that $B_h := B(y,h) \subset \tilde{C}$ where $y = x^* + \frac{h}{\sin\theta}\tilde{\xi}(x^*)$. Notice that $B_h \cap X \neq \varnothing$. Next, for any $x_i \in B_h \cap X$ we apply Markov's inequality

$$|\tilde{p}'(t)| \leq \frac{2}{\tilde{r}} m^2 \|\tilde{p}\|_{L_\infty[0,\tilde{r}]}$$

to the univariate polynomial

$$\tilde{p}(t) := p\left(x^* + t\frac{x_i - x^*}{\|x_i - x^*\|}\right).$$

Using that

$$\|x^* - x_i\| \leq \|x^* - y\| + \|y - x_i\| \leq h\frac{1+\sin\theta}{\sin\theta},$$

and $4m^2 h = \tilde{r}\frac{\sin\theta}{1+\sin\theta}$ (see Eq. (2.13)) we get:

$$|p(x^*) - p(x_i)| \leq \int_0^{\|x_i - x^*\|} |\tilde{p}'(t)|dt \leq \|x^* - x_i\| \frac{2}{\tilde{r}} m^2 \|\tilde{p}\|_{L_\infty[0,\tilde{r}]} \leq \frac{1+\sin\theta}{\sin\theta}\frac{2h}{\tilde{r}} m^2 \leq \frac{1}{2}.$$

Recalling that $|p(x^*)| = 1$ the result follows. ■

Using the above lemmas we can prove the following theorem which is a modified result of Wendland's result [10].

Theorem 2.10 *Let Ω be a domain which satisfies the cone condition with constants r, θ. Fix $x \in \Omega$. Assuming that the fill distance h of the set $X = \{x_i\}_{i=1}^N$ satisfies $h \leq \frac{\kappa}{m^2} r$. Then, X is a norming set w.r.t $\|\cdot\|_x$, and for $\delta = \frac{m^2}{\kappa} h$, the Lebesgue constant for approximation at x is bounded as follows,*

$$L_c \leq 1 + 2\|1\|_x \left(\inf_{B(y,h)\subset C} \|T(\chi_{B(y,h)}(\cdot))\|_x\right)^{-1},$$

where $C = C(x, \xi, \delta, \theta)$, $1 = (1, 1, .., 1) \in \mathbb{R}^N$ and $\chi_{B(y,h)}(\cdot)$ is the characteristic function of the set $B(y, h)$.

Proof. Denote $V = \Pi_m(C)$, where $C = C(x, \xi(x), \delta, \theta)$ is a cone such that $C \subset \Omega$ (The assumption on h and δ assures that $\delta \leq r$). The Lebesgue constant L_c satisfies (2.12). Next, we wish to bound the norming constant

$$\|T^{-1}\| = \sup_{p\in V} \frac{\|p\|_{L_\infty(C)}}{\|T(p)\|_x} = \sup_{\|p\|_{L_\infty(C)}=1} \frac{1}{\|T(p)\|_x}. \tag{2.14}$$

By Lemma 2.9, for every $p \in V$, $\|p\|_{L_\infty(C)} = 1$ there exist a ball $B_h = B(y, h) \subset C$ such that for every point $x_i \in X \cap B_h$ $|p(x_i)| \geq \frac{1}{2}$. Thus it follows that $\|T(p)\|_x \geq \frac{1}{2}\|T(\chi_{B(y,h)})\|_x$. Now from (2.14) we have

$$\|T^{-1}\| \leq 2\left(\inf_{\{y:B(y,h)\subset C\}} \|T(\chi_{B(y,h)})\|_x\right)^{-1},$$

and using (2.12) we obtain the desired result. ■

For each choice of the weights $\bar{w}(x) = (w_1^x, ..., w_N^x)$, one gets a method of approximation. For example taking a compact support, smooth, decreasing function $\phi(r)$, such that

$$\phi(0) = 1 \ \ , \ \ \phi(1) = 1/2 \ \ , \ \ \phi(2) = 0,$$

and setting

$$w_i^x = \phi\left(\frac{\|x - x_i\|}{\delta}\right)$$

we get the standard MLS method. Set

$$\delta = \frac{m^2}{\kappa}h = \frac{16(1+\sin\theta)^2 m^2 h}{3\sin^2\theta}.$$

In this case,

$$\delta_x \leq 2\delta.$$

Theorem 2.10, can then be used to bound the Lebesgue constant L_c:

$$L_c \leq 1 + 2\left(\#I_{2\delta}\frac{\max_{i\in I_{2\delta}} w_i^x}{\min_{i\in I_\delta} w_i^x}\right)^{1/2} \leq 1 + 2\sqrt{2\#I_{2\delta}}, \tag{2.15}$$

where $I_\mu = I_\mu(x)$ denotes the indices set $I_\mu(x) := \{i : x_i \in B(x,\mu) \cap X\}$, and $\#I_\mu(x)$ is the size of this set. Now, taking $\mathscr{D} = C(x, \xi, \delta, \theta)$ in (2.11) we get the desired result:

$$|\mathscr{M}_{f,X}(x) - f(x)| \leq \|f - p^*\|_{L_\infty(B(x,2\delta)\cap\Omega)}(1 + 2\sqrt{2\#I_{2\delta}}).$$

This bound is similar to the one presented in [10], and we see that the square root of the number of points $\#I_{2\delta}$ used in the approximation appears in the bound. As the following example shows, this bound can indeed be realized (up to a constant factor).

Consider $\Omega = [0, 3/2] \subset \mathbb{R}$ and $X = \{0, 1, ..., 1, 1+\varepsilon, ..., 1+\varepsilon\}$, where the points 1 and $1+\varepsilon$ repeat n times each and $\varepsilon = \frac{1}{\sqrt{n}}$. Taking linear polynomials, i.e., $m = 1$ and $B(y) = \{1, y\}$, and constant unit weights, the *Lebesgue functions* (see [10]), which is another way of defining the Lebesgue constant, for point evaluation at $x = 0$ turns out to be

$$L_c = 1 + \frac{2n\sqrt{n}}{3n + 2\sqrt{n} + 1} \geq C_1\sqrt{n},$$

for some constant C_1. To see this note that from Equation (2.7) the approximation at $x = 0$ is

$$\mathscr{P}_{f,X,0}(0) = B(0)\left(\tilde{B}^t\tilde{B}\right)^{-1}\tilde{B}^t T(f).$$

Therefore the Lebesgue function at the point $x = 0$ is the sum of absolute values of the first row of the matrix $\left(\tilde{B}^t\tilde{B}\right)^{-1}\tilde{B}^t$. This matrix can be written explicitly for the specified X to yield the above term of the Lebesgue power function. It should be noted that in this work we defined the Lebesgue constant a little differently than the power function, however, the power function is the actual term which appears in the derived error bound in this paper and previous works.

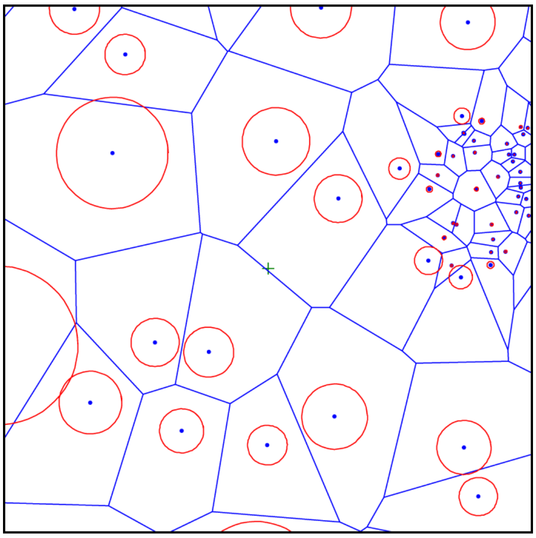


Fig. 1. In this figure we show the Voronoi diagram for irregular data samples, and the corresponding $|D_i|$ for each site is proportional to the radii of the red circles.

3 Stable Moving Least-Squares (Stable MLS)

The dependence of the error in the MLS approximation process on the number of points used in the local approximation (see Eq. (2.15)) is a drawback, especially in cases of highly non-uniform scattered data, that is, cases where the points' density changes drastically over the domain.

In order to overcome this drawback, we advocate a different choice of $\bar{w}(x)$ for which the Lebesgue constant is bounded independently of the number of points used in the local approximation. Here we shall make use of the result in Theorem 2.10 which can be used to express the bound in terms of the weights.
An important restriction is that we will not take $w_i^x = 0$ unless $\|x_i - x\|_2 > \delta_x$. The reason is that we do not want to enlarge the fill distance h of the set which will in turn lead to larger support δ used in the approximation and hence increase the bound on the best approximation part of inequality (2.11): $\|f - p^*\|_{L_\infty(\mathscr{D})}$. From this it is clear that simply throwing away points might deteriorate the approximation.

We subdivide Ω as $\Omega = \cup_{i \in I_\delta} D_i$, where D_i is the Voronoi cell for x_i inside the domain Ω with volume $|D_i|$, see for example Figure 1. If there are repetitions of points $x_{i_1} = x_{i_2} = \ldots = x_{i_k}$ we take a single representative x_{i_1} in the construction of the Voronoi cells. Note that since $|D_i \cap D_j| = 0$ for $i \neq j$ we have

$$|\cup_i D_i| = \sum_i |D_i|.$$

We define the weights to be

$$w_i^x = \phi\left(\frac{\|x-x_i\|_2}{\delta}\right)\frac{|D_i|}{\#\{x_j \in X : x_j = x_i\}}, \tag{3.1}$$

where $\#\{x_j \in X : x_j = x_i\}$ is the number of repetition of x_i in X.

Lemma 3.1 *Let $X = \{x_i\}_{i=1}^N \subset \Omega$ be data sites in some domain Ω with corresponding fill distance h. Let $\Omega = \cup_{i=1}^N D_i$ be a decomposition of the domain Ω into the Voronoi cells of the data sites X. Then the following holds:*

(1) For every $x \in \Omega$ s.t. $B(x,3h) \subset \Omega$, where h is the fill distance of X and $I_{3h}(x)$ is defined as above,

$$\sum_{i \in I_{3h}(x)} |D_i| \geq |B(x,h)|.$$

(2) $\sum_{i \in I_\delta(x)} w_i^x \leq |B(x,\delta+2h)|$

Proof. 1. Let $z \in B(x,h)$ be an arbitrary point. Since h is the fill distance there exist some $x_i \in X \cap B(x,h)$ with $\|z-x_i\|_2 < 2h \leq \|z-x_j\|_2$ for every $j \in \{1,2,...,N\}\setminus I_{3h}$. Therefore z will be in some Voronoi cell D_i, $i \in I_{3h}$. Hence $B(x,h) \subset \cup_{i \in I_{3h}} D_i$ and Claim 1 follows.

For 2, let $z \in \Omega \setminus B(x,\delta+2h)$ (note that $B(x,\delta+2h)$ is a closed set here). Assume, in negation, that $z \in \cup_{i \in I_\delta} D_i$ then there exists $x_i \in X \cap B(x,\delta)$ such that $\|x_i - z\|_2 \leq \|x_j - z\|_2$ for every $x_j \in X \cap (\Omega \setminus B(x,\delta))$. Consider the ball $B_* := B(z + h\frac{x_i - z}{\|x_i - z\|_2}, h)$. There exist some $x_* \in X \cap B_*$, and by triangle inequality we get $\|z - x_*\|_2 \leq 2h$ however $\|x_i - z\|_2 > 2h$, and this is a contradiction. Therefore $z \in D_j$ where x_j in not in $B(x,\delta)$, so $\sum_{i \in I_\delta} |D_i| = |\cup_{i \in I_\delta} D_i| \leq |B(x,\delta+2h)|$, using also $|\phi(r)| \leq 1$, Claim 2 follows. ■

Lemma 3.1 leads to the following result:

Theorem 3.2 *Let Ω be a domain which satisfies the cone condition with constants r,θ. Fix $x \in \Omega$. Assuming that the fill distance h of the set $X = \{x_i\}_{i=1}^N$ satisfy $3h \leq \frac{\kappa}{m^2} r$, set $\delta = 3\frac{m^2}{\kappa} h$, and define $\bar{w}(x) \in \mathbb{R}^N$ by (3.1). Then the Lebesgue constant is bounded as follows,*

$$L_c \leq 1 + 2\sqrt{2}\left(2 + 6\frac{m^2}{\kappa}\right)^{d/2},$$

and the error bound for the approximation is in turn,

$$|\mathscr{M}_{f,X}(x) - f(x)| \leq \|f - p^*\|_{L_\infty(B_\delta)}\left\{1 + 2\sqrt{2}\left(2 + 6\frac{m^2}{\kappa}\right)^{d/2}\right\}.$$

Proof. By Theorem 2.10 where h is replaced by $3h$ we have

$$L_c \leq \left\{ 1 + 2 \left(\frac{\sum_{i \in I_{2\delta}(x)} w_i^x}{\inf_{\{y : B(y,3h) \subset B(x,\delta)\}} \left\{ \sum_{i \in I_{3h}(y)} w_i^x \right\}} \right)^{1/2} \right\}.$$

Using Lemma 3.1 we have $\sum_{i \in I_{3h}} w_i^x \geq \frac{1}{2} \sum_{i \in I_{3h}} |D_i| \geq \frac{1}{2} |B(x,h)|$ and $\sum_{i \in I_{2\delta}} w_i^x \leq |B(x, 2\delta + 2h)|$. Hence

$$\inf_{\{y : B(y,3h) \subset B(x,\delta)\}} \left\{ \sum_{i \in I_{3h}(y)} w_i^x \right\} \geq \frac{1}{2} |B(x,h)|,$$

and therefore,

$$\frac{\sum_{i \in I_{2\delta}} w_i^x}{\inf_{\{y : B(y,3h) \subset B(x,\delta)\}} \left\{ \sum_{i \in I_{3h}(y)} w_i^x \right\}} \leq 2 \frac{|B(x, 2\delta + 2h)|}{|B(x,h)|}.$$

Hence the Lebesgue constant can be bounded independently of the number of points used:

$$L_c \leq 1 + 2\sqrt{2} \left(\frac{|B(x, 2\delta + 2h)|}{|B(x,h)|} \right)^{1/2} = 1 + 2\sqrt{2} \left(\frac{2\delta + 2h}{h} \right)^{d/2}.$$

In our case $\delta = 3 \frac{m^2}{\kappa} h$ and the theorem follows. ■

Remark 3.3 *The MLS approximation with Voronoi weights is smooth. Note that defining the weight $\bar{w}(x)$ as suggested in Theorem 3.2 obviously keeps the weights smooth and therefore results in a smooth (as smooth as ϕ) MLS approximant [8,11]. For example, $m = 0$ results in the Shepard's type approximant*

$$\mathscr{M}_{f,X}(x) = \frac{\sum_i f_i \, \phi\left(\frac{\|x - x_i\|_2}{\delta} \right) |D_i|}{\sum_i \phi\left(\frac{\|x - x_i\|_2}{\delta} \right) |D_i|},$$

where here we omit the repetition term $\#\{x_j \in X : x_j = x_i\}$ for brevity.

4 Numerical experiments

In this section we present some numerical experiments comparing the classical MLS operator to the Stable MLS presented above. The benefit in using the Stable MLS comes into play mostly in highly irregular data samples X. This is demonstrated by two examples: First, in Figure 2 we compare the *Lebesgue functions* (see [10]) for several scenarios of irregular data samples. Note that the Lebesgue functions remain almost unchanged (≈ 2) in all cases presented, and recall that the Lebesgue functions are greater or equal to 1.

Second, we show that highly irregular data sites may cause some artifacts in the MLS approximant, which are alleviated by the Stable MLS, see Figure 3.

As to the computational complexity of the newly proposed method, it has the same computational complexity of the standard MLS apart from a preprocessing step. The weights used in the Stable MLS operator (3.1) are simply the usual radial weights $\phi(\|x-x_i\|_2/\delta)$ scaled by the factors $\psi_i = \frac{|D_i|}{\#\{x_j \in X : x_j = x_i\}}$. These factors are computed in a preprocess step where the volume of the corresponding Voronoi cells of the data X are computed, divided by the multiplicity of each point.

5 Acknowledgements

This work has been done as part of the author's PhD studies under the supervision of Prof. David Levin. The Author further wants to thank the anonymous reviewers for their comments.

References

[1] G. Backus and F. Gilbert, *Numerical applications of a formalism for geophysical inverse problems*, Geophys. J.R. Astr. Soc. **13** (1967), 247–276.

[2] ———, *The resolving power of gross earth data*, Geophys. J.R. Astr. Soc. **16** (1968), 169–205.

[3] ———, *Uniqueness in the inversion of inaccurate gross earth data*, Trans. Roy. Soc. London, Ser. A **266** (1970), 123–192.

[4] L. Bos and K. Salkauskas, *Moving least-squares are backus-gilbert optimal*, J. Approx. Theory **59** (1989), 267–275.

[5] R. Franke and G. Nielson, *Smooth interpolation of large sets of scattered data*, International Journal for Numerical Methods in Engineering **15** (1980), no. 11, 1691–1704.

[6] Kurt Jetter, Joachim Stöckler, and Joseph D. Ward, *Error estimates for scattered data interpolation on spheres*, Mathematics of Computation **68** (1999), no. 226, 733–747.

[7] P. Lancaster and K. Salkauskas, *Surfaces generated by moving least squares methods*, Math. Comp. **37** (1981), no. 155, 141–158.

[8] David Levin, *The approximation power of moving least-squares*, Math. Comput. **67** (1998), no. 224, 1517–1531.

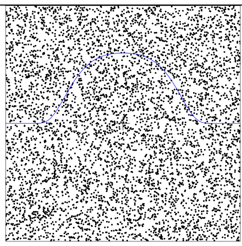
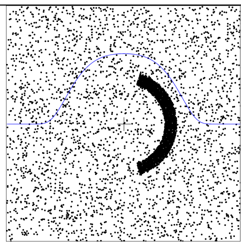
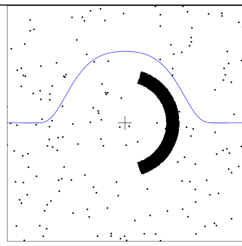
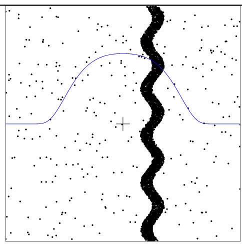
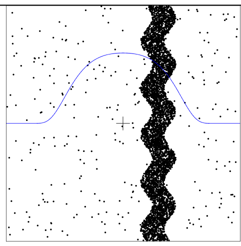
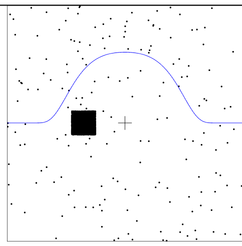
			
percentage	0.0	0.5	0.97
L_c MLS	1.92	2.49	10.59
L_c SMLS	1.89	1.98	2.04
ratio	0.98	0.8	0.19
			
percentage	0.95	0.95	0.97
L_c MLS	5.83	4.72	17.86
L_c SMLS	1.9	2.06	2.81
ratio	0.33	0.44	0.16

Fig. 2. In this example we compare the standard MLS with the Stable MLS (SMLS) in several scenarios. We have augmented a uniform random distribution of points by an extra set of points (circular arc, sinusoidal line and a square). The augmented set of points contains the prescribed *percentage* of the overall 6K points. We have calculated the Lebesgue function at the point marked by '+' (for definition of the Lebesgue function see [10]). The Lebesgue functions L_c are listed and also the ratio between the Lebesgue constant of the stable MLS and the standard MLS. It can be seen that the advantage of the Stable MLS is significant in the more irregular data settings. The weight function used is drawn in blue.

[9] Carl de Boor Samuel D. Conte, *Elementary numerical analysis*, McGraw-Hill Education, 1980.

[10] Holger Wendland, *Local polynomial reproduction and moving least squares approximation*, IMA Journal of Numerical Analysis **21** (2001), no. 1, 285–300.

[11] ______, *Scattered data approximation*, Cambridge University Press, 2005.

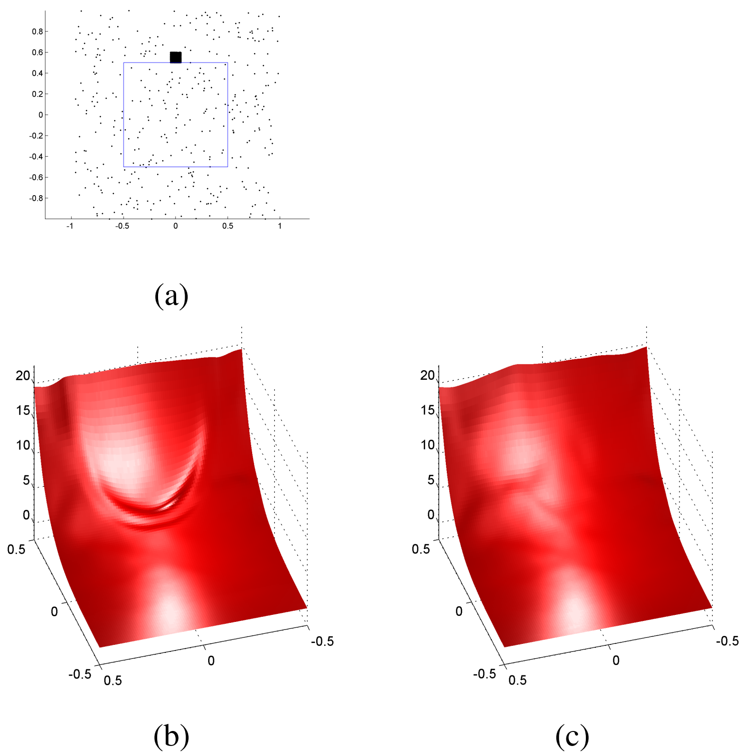


Fig. 3. A highly irregular data distribution (3K point) with a cluster of points is depicted in (a). The cluster contains about 90% of the points in the domain. The function $f(x,y) = e^{6x}$ is sampled over the data points and approximated using standard MLS in (b). The approximation is computed over the blue rectangle domain in (a). (c) shows the corresponding Stable MLS approximation. Note the artifact in the MLS approximant which is rectified in the Stable MLS approximant.