Einstein relation for biased random walk on Galton–Watson trees

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Abstract

We prove the Einstein relation, relating the velocity under a small perturbation to the diffusivity in equilibrium, for certain biased random walks on Galton–Watson trees. This provides the first example where the Einstein relation is proved for motion in random media with arbitrary deep traps.

1 Introduction

Let ω be a rooted Galton–Watson tree with offspring distribution $\{p_k\}$, where $p_0 = 0$, $m = \sum kp_k > 1$ and $\sum b^k p_k < \infty$ for some $b > 1$. For a vertex $v \in \omega$, let $|v|$ denote the distance of v from the root of ω. Consider a (continuous–time) nearest-neighbor random walk $\{Y_t^\alpha\}_{t\geq 0}$ on $\mathcal{T}$, which when at a vertex v, jumps with rate 1 toward each child of v and at rate $\lambda = \lambda_\alpha = me^{-\alpha}$, $\alpha \in \mathbb{R}$, toward the parent of v.

It follows from [14] that if $\alpha = 0$, the random walk $\{Y_t^\alpha\}_{t\geq 0}$ is, for almost every tree ω, null recurrent (positive recurrent for $\alpha < 0$, transient for $\alpha > 0$). Further, an easy adaptation of [17] shows that $|Y^0_{[nt]}|/\sqrt{n}$ satisfies a (quenched, and hence also annealed) invariance principle (i.e., converges weakly to a multiple of the absolute value of a Brownian motion), with diffusivity

$$\mathcal{D}^0 = \frac{2m^2(m-1)}{\sum k^2 p_k - m}\,. \tag{1.1}$$

(Compare with [17, Corollary 1], and note that the factor 2 is due to the speed up of the continuous–time walk relative to the discrete–time walk considered there. See (2.10) below and also the derivation in [4].) On the other hand, see [16], when $\alpha > 0$, $|Y_t^\alpha|/t \to_{t\to\infty} \bar{v}_\alpha > 0$, almost surely, with $\bar{v}_\alpha$ deterministic. A consequence of our main result, Theorem 1.2 below, is the following.

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Theorem 1.1 (Einstein relation) *With notation and assumptions as above,*

$$\lim_{\alpha \searrow 0} \frac{\bar{v}_\alpha}{\alpha} = \frac{\mathcal{D}^0}{2} \,. \tag{1.2}$$

The relation (1.2) is known as an *Einstein relation.* It is straight forward to verify that for homogeneous random walks on $\mathbb{Z}_+$ (corresponding to deterministic Galton–Watson trees, that is, those with $p_k = 1$ for some $k \geq 1$), the Einstein relation holds.

In a weak limit (velocity rescaled with time) the Einstein relation is proved in a very general setup by Lebowitz and Rost (cf. [11]). See also [3] for general fluctuation-dissipation relations.

For the tagged particle in the symmetric exclusion process, the Einstein relation has been proved by Loulakis in $d \geq 3$ [12]. The approach of [12], based on perturbation theory and transient estimates, was adapted for bond diffusion in $\mathbb{Z}^d$ for special environment distributions (cf. [9]). For mixing dynamical random environments with spectral gap, a full perturbation expansion can also be proved (cf. [10]).

For a diffusion in random potential, the recent [8] proves the Einstein relation by following the strategy of [11], adding to it a good control (uniform in the environment) of suitably defined regeneration times in the transient regime. A major difference in our setup is the possibility of having "traps" of arbitrary strength in the environment; in particular, the presence of such traps does not allow one to obtain estimates on regeneration times that are uniform in the environment, and we have been unable to obtain sharp enough estimates on regeneration times that would allow us to mimic the strategy in [8]. On the other hand, the tree structure allows us to develop some estimates directly for hitting times via recursions, see Section 3. We emphasize that our work is (to the best of our knowledge) the first in which an Einstein relation is rigorously proved for motion in random environments with arbitrary strong traps.

In order to explore the full range of parameters α, we will work in a more general context than that described above, following [17]. This is described next.

Consider infinite trees $\mathcal{T}$ with no leaves, equipped with one (semi)-infinite directed path, denoted `Ray`, starting from a distinguished vertex called the **root** and denoted o. Using `Ray`, we define in a natural way the offsprings of a vertex $v \in \mathcal{T}$, and denote by $D_n(v)$ the collection of vertices that are descendants of v at distance n from v, with $Z_n(v) = |D_n(v)|$. See [17, Section 4] for precise definitions. For any vertex $v \in \mathcal{T}$, we let d_v denote the number of offspring of v, and write $\overleftarrow{v}$ for the parent of v. Finally, we write $\rho(v)$ for the *horocycle* distance of v from the root o. Note that $\rho(v)$ is positive if v is a a descendant of o and negative if it is an ancestor of ρ.

Let Ω_T denote the space of marked trees. As in [17] and motivated by [16], given an offspring distribution $\{p_k\}_{k\geq 0}$ satisfying our general assumptions, we introduce a reference probability `IGW` on Ω_T, as follows. Fix the root o and a semi-infinite ray, denoted `Ray`, emanating from it. Each vertex $v \in$ `Ray` with $v \neq o$ is assigned independently a size-biased number of offspring, that is $P_{\texttt{IGW}}(d_v = k) = kp_k/m$, one of which is identified with the descendant of v on `Ray`. To each offspring of $v \neq o$ not on `Ray`, and to o, one attaches an independent Galton-Watson tree of offspring distribution $\{p_k\}_{k\geq 0}$. Note that `IGW` makes the collection $\{d_v\}_{v\in\mathcal{T}}$ independent. We denote expectations with respect to `IGW` by $\langle\cdot\rangle_0$ (the reason for the notation will become apparent in Section 2.1 below).

As mentioned above and in contrast with [16] and [17], it will be convenient to work in continuous time, because it slightly simplifies the formulas (the adaptation needed to transfer the results to the discrete time setup of [16] are straight-forward). For background, we refer to [4], where the results in [16] and [17] are transferred to continuous time, in the more general setup of multi-type Galton–Watson trees. Given a marked tree $\mathcal{T}$ and $\alpha \in \mathbb{R}$, we define an α-biased random walk $\{X_t^\alpha\}_{t\geq 0}$ on $\mathcal{T}$ as the continuous time Markov process with state space the vertices of $\mathcal{T}$, $X_0^\alpha = o$, and so that when at v, the jump rate is 1 toward the each of the descendants of v, and the jump rate is $e^{-\alpha}m$ toward the parent $\overleftarrow{v}$. More explicitly, the generator of the random walk $\{X_t^\alpha\}_{t\geq 0}$ can be written as

$$\mathcal{L}_{\alpha,\mathcal{T}}F(v) = \sum_{x\in D_1(v)} (F(x) - F(v)) + e^{-\alpha}m\left(F(\overleftarrow{v}) - F(v)\right) . \tag{1.3}$$

Alternatively,

$$\mathcal{L}_{\alpha,\mathcal{T}}F(v) = -m\partial_-^*\partial_- F(v) + (e^{-\alpha} - 1)m\partial_- F(v) \tag{1.4}$$

where $\partial_- F(v) = F(\overleftarrow{v}) - F(v)$ and

$$\partial_-^* F(v) = \frac{1}{m}\sum_{x\in D_1(v)} F(x) - F(v) \tag{1.5}$$

Note that if $\alpha < 0$ the (average) drift is towards the ancestors, whereas if $\alpha > 0$ the (average) drift is towards the children. As in [16] and [17], we have that

$$\lim_{t\to\infty} \frac{\rho(X_t^\alpha)}{t} \to_{t\to\infty} v_\alpha\,, \quad \texttt{IGW} - \text{a.s.} \tag{1.6}$$

It is easy to verify that when $\alpha > 0$, then $v_\alpha = \bar{v}_\alpha$, and that $\text{sign}(v_\alpha) = \text{sign}(\alpha)$. Further, we have, again from [17], that $\rho(X^0_{[nt]})/\sqrt{n}$ satisfies the invariance principle (that is, converges weakly to a Brownian motion), with diffusivity constant $\mathcal{D}^0$ as in (1.1).

Our main result concerning walks on IGW-trees is the following.

Theorem 1.2 *With assumptions as above,*

$$\lim_{\alpha\to 0} \frac{v_\alpha}{\alpha} = \frac{\mathcal{D}^0}{2}\,. \tag{1.7}$$

Remark 1.3 *It is natural to expect that the Einstein relation holds in many related models, including Galton–Watson trees with only moment bounds on the offspring distribution, multi-type Galton–Watson trees as in [4], and walks in random environments on Galton–Watson trees, at least in the regime where a CLT with non-zero variance holds, see [5]. We do not explore these extensions here.*

The structure of the paper is as follows. In the next section, we consider the case of $\alpha < 0$, exhibit an invariant measure for the environment viewed from the point of view of the particle, and use it to prove the Einstein relation when $\alpha \nearrow 0$. Section 3 deals with the harder case of $\alpha \searrow 0$. We first prove an easier Einstein relation (or linear response) concerning escape probabilities of the walk, exploiting the tree structure to introduce certain recursions. Using that, we relate the Einstein relation for velocities to estimates on hitting times. A crucial role in obtaining these estimates, and an alternative formula for the velocity (Theorem 3.8) is obtained by the introduction, after [6], of a spine random walk, see Lemma 3.4.

2 The environment process, and proof of Theorem 1.2 for $\alpha \nearrow 0$.

As is often the case when motion in random media is concerned, it is advantageous to consider the evolution, in Ω_T, of the environment from the point of view of the particle. One of the reasons for our opting to work in continuous time is that when $\alpha = 0$, the invariant measure for that (Markov) process is simply `IGW`, in contrast with the more complicated measure `IGWR` of [17]. We will see that when $\alpha < 0$, an explicit invariant measure for the environment viewed from the point of view of the particle exists, and is absolutely continuous with respect to `IGW`.

2.1 The environment process

For a given tree $\mathcal{T}$ and $x \in \mathcal{T}$, let $\tau_x \mathcal{T}$ denote the shift that moves the root to x. (A special role will be played by τ_x for $x \in D_1(o)$, and by τ_x with $x = \overleftarrow{o}$. We use $\tau^{-1}\mathcal{T} = \tau_{\overleftarrow{o}}\mathcal{T}$ in the latter case.) The environment process $\{\mathcal{T}_t\}_{t\geq 0}$ is defined by $\mathcal{T}_t = \tau_{X_t}\mathcal{T}$. It is straightforward to check that the environment process is a Markov process. In fact, introducing the the operators

$$Df(\mathcal{T}) = f(\tau^{-1}\mathcal{T}) - f(\mathcal{T}),$$

we have that the adjoint operator (with respect to `IGW`) is

$$D^* f(\mathcal{T}) = \frac{1}{m} \sum_{x \in D_1(o)} f(\tau_x \mathcal{T}) - f(\mathcal{T})$$

since

$$\langle g Df \rangle_0 = \langle f D^* g \rangle_0 .$$

Notice that $D^* 1 = d_0/m - 1$.

Define $W(v,n) = Z_n(v)/m^n$. Then $W(v,n)$ is a positive martingale that converge to a random variable denoted W_v. Using the recursions

$$mW_v = \sum_{x \in D_1(v)} W_x, \qquad mW(n,v) = \sum_{x \in D_1(v)} W(n-1,x),$$

we see that $\langle W_v \rangle_0 = 1$ for $v \notin \mathtt{Ray}$. To simplify notation, we write $W_{-j} = W_{v_j}$ with $v_j \in \mathtt{Ray}$ denoting the j-th ancestor of o. Since $W_o(\tau_x \mathcal{T}) = W_x(\mathcal{T})$, we have that $D^* W_o = 0$.

The generator of the environment process is

$$\begin{aligned} L_\alpha f(\mathcal{T}) &= \sum_{x \in D_1(o)} [f(\tau_x \mathcal{T}) - f(\mathcal{T})] + e^{-\alpha} m \, [f(\tau_{-1}\mathcal{T}) - f(\mathcal{T})] \\ &= -mD^* Df(\mathcal{T}) + (e^{-\alpha} - 1) m Df(\mathcal{T}) \end{aligned} \tag{2.1}$$

The adjoint operator (with respect to `IGW`) is $L_\alpha^* = -mD^*D + (e^{-\alpha} - 1)mD^*$. For any $\alpha \in \mathbb{R}$, let μ_α denote any stationary probability measure for L_α, that is μ_α satisfies, for any bounded measurable f,

$$\langle L_\alpha f \rangle_\alpha = 0,$$

where $\langle g \rangle_\alpha = \int g d\mu_\alpha$.

Note that `IGW` is stationary and reversible for L_0. Further, it is ergodic for the environment process. This is elementary to prove, since for any bounded function $f(\mathcal{T})$ such that $L_0 f = 0$, we have that $< |Df|^2 >_0 = 0$, i.e. f is translation invariant for a.e. $\mathcal{T}$ with respect to `IGW`, i.e. constant a.e. . Thus, necessarily, $\mu_0 =$ `IGW`, justifying our notation $\langle \cdot \rangle_0 = \langle \cdot \rangle_{\mathtt{IGW}}$.

In our setup, due to the existence of regeneration times for $\alpha \neq 0$ with bounded expectation, a general ergodic argument ensures the existence of a stationary measure μ_α, which however may fail in general to be absolutely continuous with respect to `IGW`, see [16]. Further, because `IGW` is ergodic and the random walk is elliptic, there is at most one μ_α which is absolutely continuous with respect to `IGW`, since under any such μ_α, the process $\mathcal{T}_t^\alpha$ must be ergodic, see e.g. [18, Corollary 2.1.25] for a similar argument. As we now show, when $\alpha < 0$, the measure μ_α can be constructed explicitly.

Lemma 2.1 *For $\alpha < 0$, the stationary measure μ_α has a density with respect to $\mu_0 =$ `IGW` denoted by $d\mu_\alpha / d\mu_0 = \psi_\alpha$. Further,*

$$\psi_\alpha(\mathcal{T}) = C_\alpha^{-1} Z_\alpha \quad \text{where} \quad Z_\alpha = \sum_{j=0}^{\infty} e^{j\alpha} W_{-j}(\mathcal{T}), \tag{2.2}$$

(2.3) $$C_\alpha = \frac{(1-B)e^\alpha m^{-1}}{1-e^\alpha m^{-1}} + \frac{Be^\alpha}{1-e^\alpha} + 1$$

and

(2.4) $$B = \frac{\sum_k k^2 p_k - m}{m(m-1)}\,.$$

Proof of Lemma 2.1: We show first that C_α provides the correct normalization. In fact, from the relation

(2.5) $$W_{-j} = m^{-1} W_{-j+1} + m^{-1} \sum_{s \in D_1(v_{-j}), s \notin \texttt{Ray}} W_s =: m^{-1}(W_{-j+1} + L_j)$$

and since $\langle W_s \rangle_0 = 1$ if $s \notin \texttt{Ray}$, we obtain

(2.6) $$\langle W_{-j} \rangle_0 = m^{-1} \langle W_{-j+1} \rangle_0 + m^{-1} B(m-1), \quad j \geq 1\,.$$

Since $\langle W_o \rangle = 1$, we deduce that

(2.7) $$\langle W_{-j} \rangle_0 = (1-B)m^{-j} + B, \quad j \geq 0\,.$$

Thus,

$$\begin{aligned}
\langle \sum_{j=0}^\infty e^{j\alpha} W_{-j} \rangle_0 &= 1 + B\sum_{j=1}^\infty e^{j\alpha} + (1-B)\sum_{j=1}^\infty e^{j\alpha} m^{-j} \\
&= 1 + \frac{Be^\alpha}{1-e^\alpha} + \frac{(1-B)e^\alpha m^{-1}}{1-e^\alpha m^{-1}} \\
&= \frac{B}{1-e^\alpha} + \frac{1-B}{1-e^\alpha m^{-1}} = C_\alpha\,,
\end{aligned}$$

as needed.

We next verify that $L_\alpha^* \psi_\alpha = 0$. Since $W_{-j}(\tau^{-1}\mathcal{T}) = W_{-j-1}(\mathcal{T})$, we have

$$\begin{aligned}
D\psi_\alpha &= C_\alpha^{-1} \sum_{j=0}^\infty e^{j\alpha} \left(W_{-j-1}(\mathcal{T}) - W_{-j}(\mathcal{T})\right) \\
&= C_\alpha^{-1} \left(\sum_{j=1}^\infty e^{(j-1)\alpha} W_{-j}(\mathcal{T}) - \sum_{j=0}^\infty e^{j\alpha} W_{-j}(\mathcal{T}) \right) \\
&= C_\alpha^{-1} \sum_{j=0}^\infty (e^{(j-1)\alpha} - e^{j\alpha}) W_{-j}(\mathcal{T}) - C_\alpha^{-1} e^{-\alpha} W_o \\
&= (e^{-\alpha}-1) C_\alpha^{-1} \sum_{j=0}^\infty e^{j\alpha} W_{-j}(\mathcal{T}) - C_\alpha^{-1} e^{-\alpha} W_o \\
&= (e^{-\alpha}-1)\psi_\alpha - C_\alpha^{-1} e^{-\alpha} W_o\,.
\end{aligned}$$

Since $D^* W_o = 0$, we have

$$D^*\left(D\psi_\alpha - (e^{-\alpha}-1)\psi_\alpha\right) = 0\,,$$

i.e.

$$L_\alpha^* \psi_\alpha = 0.$$

□

Before proving Theorem 1.2 for $\alpha < 0$, we need to evaluate the limit of Z_α.

Lemma 2.2 *With notation as in Lemma 2.1, we have*

$$\lim_{\alpha\nearrow 0} \alpha Z_\alpha = B\,, \qquad \mathtt{IGW}-a.s.$$

Proof of Lemma 2.2: Note that the terms L_s appearing in the right side of (2.5) are i.i.d.. Substituting and iterating, we get

$$W_{-k} = \frac{W_o}{m^k} + \frac{L_1}{m^k} + \frac{L_2}{m^{k-1}} + \cdots + \frac{L_k}{m}\,.$$

Therefore,

$$\left(1-\frac{e^\alpha}{m}\right) Z_\alpha = W_o + \frac{1}{m}\sum_{j=1}^\infty e^{\alpha j} L_j =: W_o + M_\alpha\,. \tag{2.8}$$

Note that M_α is a weighted sum of i.i.d. random variables. Further, because $\langle |D_1(v_{-j})|\rangle_0 = \sum k^2 p_k/m$ and $\langle W_s\rangle_0 = 1$, we have that $\lim_{\alpha\nearrow 0}\alpha\langle M_\alpha\rangle_0 = (\sum k^2 p_k - m)/m^2 := \bar{C}$, and that $\mathrm{Var}_{\mathtt{IGW}}(M_\alpha) = O(\alpha)$. It then follows (by an interpolation argument) that that

$$\lim_{\alpha\nearrow 0}\alpha M_\alpha = \bar{C}\,, \quad \mathtt{IGW}-a.s.$$

Substituting in (2.8), this yields the lemma. $\square$

We can now provide the proof of Theorem 1.2 in case $\alpha\nearrow 0$.

Proof of Theorem 1.2 when $\alpha\nearrow 0$: We begin with the computation of v_α. Because μ_α is ergodic and absolutely continuous with respect to IGW, we have that v_α equals the average drift (under μ_α) at o, that is

$$\begin{aligned}
v_\alpha &= m\langle\frac{d_0}{m} - e^{-\alpha}\rangle_\alpha = m\langle D^* 1\rangle_\alpha - m(e^{-\alpha}-1) = m\langle D^* 1\psi_\alpha\rangle_0 - m(e^{-\alpha}-1)\\
&= m\langle D\psi_\alpha\rangle_0 - m(e^{-\alpha}-1)\\
&= m\langle (e^{-\alpha}-1)\psi_\alpha - C_\alpha^{-1}e^{-\alpha}W_o\rangle_0 - m(e^{-\alpha}-1)\\
&= -mC_\alpha^{-1}e^{-\alpha}
\end{aligned}$$

Thus,

$$\lim_{\alpha\nearrow 0}\frac{v_\alpha}{|\alpha|} = -\frac{m^2(m-1)}{\sum k^2 p_k - m}\,. \tag{2.9}$$

It remains to compute the diffusivity $\mathcal{D}^0$ when $\alpha = 0$. Toward this end, one simply repeats the computation in [17, Corollary 1]. One obtains that the diffusivity is

$$\mathcal{D}^0 = \frac{\langle mW_o^2 + \sum_{s\in D_1(o)} W_s^2\rangle_0}{\langle W_o^2\rangle_0^2}\,. \tag{2.10}$$

From the definitions we have that $\langle W_o^2\rangle_0 = (\sum k^2 p_k - m)/m(m-1)$ (see [17, (2)]), and thus

$$\mathcal{D}^0 = \frac{2m^2(m-1)}{\sum k^2 p_k - m}\,. \tag{2.11}$$

Together with (2.9), this completes the proof of Theorem 1.2 when $\alpha\nearrow 0$. $\square$

Remark 2.3 *Note that the construction above fails for $\alpha > 0$, because then Z_α is not defined. The case $\alpha = \infty$ is however special. In that case, the generator is*

$$L_\infty f(\mathcal{T}) = \sum_{x \in D_1(o)} [f(\tau_x \mathcal{T}) - f(\mathcal{T})] \tag{2.12}$$

In particular, one can verify that the measure defined by $d\mu_\infty / d\mu_{GW} = 1/(Cd_o)$ with $C = \sum_k k^{-1} p_k$ is a stationary measure, and that $v_\infty = \langle d_o \rangle_\infty = C$. It follows that the natural invariant measure is not absolutely continuous with respect to IGW.

Remark 2.4 *For $\alpha < 0$ one can construct other invariant measures, that of course are singular with respect to* IGW. *A particular family of such measures is absolutely continuous with respect to the ordinary Galton–Watson measure* GW *(defined as* IGW *but with the standard Galton–Watson measure also for vertices on* Ray*). Indeed, one can verify that the positive function*

$$\psi(\mathcal{T}) = C \sum_{j=1}^{\infty} \prod_{i=1}^{j-1} \frac{d_{-i}}{me^{-\alpha}} = C \sum_{j=1}^{\infty} (me^{-\alpha})^{-j+1} \prod_{i=1}^{j-1} d_{-i} \,, \tag{2.13}$$

with $C = 1 - e^\alpha$, satisfies $\int \psi d\mathtt{GW} = 1$ and $\psi d\mathtt{GW}$ is an invariant measure for L_α. One can also check that the Einstein relation (1.2) *is not satisfied under this measure, emphasizing the role that* IGW *plays in our setup.*

3 Drift towards descendants: proof of Theorem 1.2 for $\alpha \searrow 0$

In the case $\alpha > 0$ we cannot find an explicit expression for the stationary measure so we have to proceed in a different way. We first prove another form of the Einstein relation in terms of the *escape probabilities* (probability of never returning to the origin).

Because we consider the case $\alpha > 0$, there is no difference between considering the walk under the Galton–Watson tree or under IGW – the limiting velocity is the same, i.e. $v_\alpha = \bar{v}_\alpha$. Thus, we only consider the walk $\{Y_t^\alpha\}_{t \ge 0}$ below.

Our approach is to provide an alternative formula for the speed v_α, see Theorem 3.8 below, which is valid for all $\alpha > 0$ small enough. In doing so, we will take advantage of certain recursions, and of the *spine random walk* associated with the walk on the Galton–Watson tree, see Lemma 3.4.

We recall our standing assumptions: $p_0 = 0$, $m > 1$, and $\sum b^k p_k < \infty$ for some $b > 1$. We will throughout drop the superscript α from the notation when it is clear from the context, writing e.g. Y_t for Y_t^α. To introduce our recursions, define $T(x) := \inf\{t \ge 0 : Y_t = x\}$ and $\tau_n := \inf\{t \ge 0 : |Y_t| = n\}$. For a given tree ω, we write $P_{x,\omega}$ for the law of Y_t with $Y_0 = x$. For $0 < |x| \le n$, define

$$\begin{aligned} \beta_n(x) &:= P_{x,\omega}\Big(T(\overleftarrow{x}) > \tau_n\Big), \quad \beta(x) := P_{x,\omega}\Big(T(\overleftarrow{x}) = \infty\Big), \\ \gamma_n(x) &:= E_{x,\omega}\left(\tau_n \wedge T(\overleftarrow{x})\right). \end{aligned}$$

We study the recursions for β_n and γ_n. By the Markov property of $P_{x,\omega}$, for $|x| < n$,

$$\begin{aligned} \gamma_n(x) &= \frac{1}{d_x + \lambda} + \sum_{i=1}^{d_x} \frac{1}{\lambda + d_x} E_{x_i,\omega}\left(\tau_n \wedge T(\overleftarrow{x})\right) \\ &= \frac{1}{d_x + \lambda} + \sum_{i=1}^{d_x} \frac{1}{\lambda + d_x} \Big(E_{x_i,\omega}\left(\tau_n \wedge T(x)\right) + P_{x_i,\omega}(T(x) < \tau_n)\, E_{x,\omega}\left(\tau_n \wedge T(\overleftarrow{x})\right)\Big), \end{aligned}$$

which implies that

$$\gamma_n(x) = \frac{1}{d_x + \lambda} + \sum_{i=1}^{d_x} \frac{1}{\lambda + d_x} \left(\gamma_n(x_i) + (1 - \beta_n(x_i))\gamma_n(x)\right).$$

Hence for any $|x| < n$,

$$\gamma_n(x) = \frac{1 + \sum_{i=1}^{d_x} \gamma_n(x_i)}{\lambda + \sum_{i=1}^{d_x} \beta_n(x_i)},$$

with boundary condition $\gamma_n(x) = 0$ for any $|x| = n$. Similarly, we have

$$\beta_n(x) = \frac{\sum_{i=1}^{d_x} \beta_n(x_i)}{\lambda + \sum_{i=1}^{d_x} \beta_n(x_i)},$$

with $\beta_n(x) = 1$ if $|x| = n$.

Proposition 3.1 *As $\alpha \searrow 0$, $\alpha^{-1}\beta(0)$ converges in law and in expectation to a random variable Y such that*

$$\mathbb{E}(Y) = \frac{m(m-1)}{\mathbb{E}(d_0^2 - d_0)} = \frac{\mathcal{D}^0}{2m} \tag{3.1}$$

This is a form of Einstein relation, as linear response for the escape probability. The law of Y can be identified, see Remark 3.2 below.
Proof: We clearly have that with $B(x) := \frac{1}{\lambda}\sum_{i=1}^{d_x} \beta(x_i)$, it holds that

$$\beta(x) = \frac{B(x)}{1 + B(x)}, \qquad \forall x \neq o,$$

and

$$B(x) = \frac{1}{\lambda} \sum_{i=1}^{d_x} \frac{B(x_i)}{1 + B(x_i)}, \qquad \forall x \in \mathcal{T}. \tag{3.2}$$

Notice that all $B(x)$ are distributed as some random variable, say B, and conditionally on d_x and on the tree up to generation $|x|$, the variables $B(x_i), 1 \le i \le d_x$ are i.i.d. and distributed as B. It follows that

$$\mathbb{E}(B) = e^{\alpha}\mathbb{E}\frac{B}{1+B}, \tag{3.3}$$

and

$$\mathbb{E}(B^2) = \frac{1}{\lambda^2}\left(m\mathbb{E}\left((\frac{B}{1+B})^2\right) + \mathbb{E}(d_o(d_o-1))\left(\mathbb{E}(\frac{B}{1+B})\right)^2\right). \tag{3.4}$$

For any nonnegative r.v. $Z \in L^2$, if we use the notation $\{Z\} := \frac{Z}{\mathbb{E}(Z)}$, then the following elementary inequality holds:

$$\mathbb{E}\left(\{\frac{Z}{1+Z}\}^2\right) \le \mathbb{E}\left(\{Z\}^2\right).$$

By (3.3) and (3.4), we get that

$$\mathbb{E}\left(\{B\}^2\right) = \frac{1}{m}\mathbb{E}\left(\{\frac{B}{1+B}\}^2\right) + \frac{\mathbb{E}(d_o(d_o-1))}{m^2} \le \frac{1}{m}\mathbb{E}\left(\{B\}^2\right) + \frac{\mathbb{E}(d_o(d_o-1))}{m^2},$$

which yields that the second moment of B is uniformly bounded by the square of $\mathbb{E}(B)$: for any $0 < \alpha$,

$$\mathbb{E}(B^2) \leq \frac{\mathbb{E}(d_o(d_o-1))}{m-1} (\mathbb{E}(B))^2.$$

By (3.3), $(1-e^{-\alpha})\mathbb{E}(B) = \mathbb{E}(\frac{B^2}{1+B}) \leq \mathbb{E}(B^2) \leq \frac{\mathbb{E}(d_o(d_o-1))}{m-1} (\mathbb{E}(B))^2$. Hence

$$\mathbb{E}(B) \geq \frac{m(m-1)}{\mathbb{E}(d_o(d_o-1))} (1-e^{-\alpha}).$$

On the other hand, by Jensen's inequality, $\mathbb{E}(B) = e^{\alpha}\mathbb{E}\frac{B}{1+B} \leq e^{\alpha}\frac{\mathbb{E}(B)}{1+\mathbb{E}(B)}$, which implies that

$$\mathbb{E}(B) \leq (e^{\alpha}-1).$$

Therefore, B/α is tight as $\alpha \searrow 0$. In particular, for some sub-sequence $\alpha \searrow 0$, $B(x)/\alpha$ converges in law to some $Y(x)$. Since B/α is bounded in L^2 uniformly in $\alpha > 0$ in a neighborhood of 0, we deduce from (3.2) that

$$Y \overset{\mathrm{d}}{=} \frac{1}{m}\sum_{i=1}^{N} Y_i, \tag{3.5}$$

where N is distributed like d_o and, conditionally on N, (Y_i) are i.i.d and distributed as Y; moreover $\mathbb{E}(Y) = \lim_{\alpha\searrow 0}\mathbb{E}(\frac{B}{\alpha}) > 0$ (the limit along the same sub-sequence).

Dividing

$$(1-e^{-\alpha})\mathbb{E}(B) = \mathbb{E}(\frac{B^2}{1+B}) \leq \mathbb{E}(B^2)$$

by α^2, we get that $\mathbb{E}(Y) = \mathbb{E}(Y^2)$. The same operation in (3.4) gives

$$E(Y)^2 = \frac{m(m-1)}{\mathbb{E}(d_0(d_0-1))}E(Y^2). \tag{3.6}$$

Putting these together we obtain the expression wanted for $E(Y)$. □

Remark 3.2 *It follows from [1, Theorem 16] that the law of Y satisfying* (3.5) *is determined up to a multiplicative constant, and therefore Y equals in distribution aW_o for some constant a. The equality $EY = EY^2$ then implies that Y equals in distribution $W_o/E(W_o^2)$.*

We return to the proof of the Einstein relation concerning velocities. Recall that a *level regeneration time* is a time for which the random walk hits a fresh level and never backtracks, see e.g. [2] for the definition and basic properties. (Level regeneration times are related to, but different from, the regeneration times introduced in [16].) In particular, see [2, Section 4], the inter-regeneration times form an i.i.d. sequence, with all moments bounded. Since $\gamma_n(x)$ is smaller than the n-th level regeneration time (started at x), it follows that the sequence $\gamma_n(x)/n$ is uniformly integrable (under the measure $\mathtt{GW} \times P_{x,\omega}$), and therefore, the convergence in (3.8) holds also in expectation:

$$\lim_{n\to\infty}\frac{\mathbb{E}[\gamma_n(o)]}{n} = \frac{\mathbb{E}(\beta(o))}{v_\alpha}. \tag{3.7}$$

Since $\gamma_n(x) = E_{x,\omega}\left(\tau_n 1_{(\tau_n < T(\overleftarrow{x}))}\right) + O(1)$ and $\frac{\tau_n}{n} \to \frac{1}{v_\alpha}$, $P_{x,\omega}$ a.s. and in L^1 (the latter follows at once from the integrability of regeneration times mentioned above, as τ_n is bounded above by the nth regeneration time), we get that for x fixed,

$$\frac{\gamma_n(x)}{n} \to_{n\to\infty} \frac{1}{v_\alpha}P_{x,\omega}\left(T(\overleftarrow{x}) = \infty\right) = \frac{\beta(x)}{v_\alpha}, \qquad P_\omega a.s. \tag{3.8}$$

So all we need to prove in order to have the Einstein relation for velocities, is that

$$\lim_{\alpha\to 0}\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left(\gamma_n(o)\right)=\frac{1}{m}. \tag{3.9}$$

To this end, define

$$B_n(x):=\frac{1}{\lambda}\sum_{i=1}^{d_x}\beta_n(x_i),\qquad \Gamma_n(x):=\sum_{i=1}^{d_x}\gamma_n(x_i),\qquad |x|<n.$$

Note that showing (3.9) is equivalent to proving that

$$\lim_{\alpha\to 0}\lim_{n\to\infty}\frac{1}{n}\mathbb{E}\left(\Gamma_n(o)\right)=1. \tag{3.10}$$

For $|x|<n-1$,

$$B_n(x)=\frac{1}{\lambda}\sum_{i=1}^{d_x}\frac{B_n(x_i)}{1+B_n(x_i)},\quad \Gamma_n(x)=\frac{1}{\lambda}\sum_{i=1}^{d_x}\frac{1+\Gamma_n(x_i)}{1+B_n(x_i)}. \tag{3.11}$$

Notice that we could define $\Gamma(x):=\lim_{n\to\infty}\frac{\Gamma_n(x)}{n}$, such that

$$\Gamma(x):=\frac{1}{\lambda}\sum_{i=1}^{d_x}\frac{\Gamma(x_i)}{1+B(x_i)} \tag{3.12}$$

As $\alpha\to 0$ we can show that $\Gamma(x)\to aY(x)$ for some constant $a>0$. The problem is that in the limit as $n\to\infty$ we loose information on the value of a (that should be $\mathbb{E}(d_0(d_0-1))/[m(m-1)]$).

In order to determine this constant we have to make a step back and iterate the equations (3.11) and, noticing that $\Gamma_n(x)=0$ for all $|x|=n-1$, we get that

$$\Gamma_n(o)=\sum_{r=1}^{n-1}\frac{1}{\lambda^r}\sum_{|u|=r}\frac{1}{1+B_n(u_1)}\cdots\frac{1}{1+B_n(u_{r-1})}\frac{1}{1+B_n(u_r)}:=\sum_{r=1}^{n-1}\Phi_n(r), \tag{3.13}$$

where $\{u_0,....,u_r\}$ is the shortest path relating the root o to u $[u_0=o,|u_1|=1,...,|u_r|=r]$. Note that $B_n(u_1),...,B_n(u_r)$ are correlated.

Observe that $\Phi_n(r)\le e^{\alpha r}W(o,r)$, consequently $\mathbb{E}(\Phi_n(r))\le e^{\alpha r}$. Since $\alpha>0$ it is hard to control this limit. The aim is to analyze the asymptotic behavior of $\mathbb{E}(\Phi_n(r))$ as $n\to\infty$ and $r\le n$, which will be done in the following two subsections: in the next first subsection we will give a useful representation of $\mathbb{E}(\Phi_n(r))$ based on a spine random walk, whereas in the second subsection we make use of an argument from renewal theory and establish the limit of $\mathbb{E}(\Phi_n(r))$ when $r,n\to\infty$ in an appropriate way.

3.1 Spine random walk representation of $\mathbb{E}(\Phi_n(r))$

Let Ω denote the space of rooted trees with no leaves. Denote by $\widetilde{\Omega}_T$ the space of trees with a marked infinite ray $\texttt{Ray}=(u_n^*)_{n\ge 0}$, with $u_0^*=o$ [$\widetilde{\Omega}_T$ is topologically equal to Ω_T]. Unlike the setup used in Section 2, where e.g. u_1^* was considered a parent of o, we now redefine the notion of descendant in $\widetilde{\Omega}_T$. Namely, for $x\in\mathcal{T}$, $x\neq o$, the parent of x, denoted $\overleftarrow{x}$, is the unique vertex on the geodesic connecting x and o with $|\overleftarrow{x}|=|x|-1$. To avoid confusion, we introduce the normalized progeny of v at level n as $M_n(v)=|\{w:|w-v|=n,|w|=|v|+n\}|/m^n$ We also write $M_n=M_n(o)$.

According to [13], on the space $\widetilde{\Omega}_T$ we may construct a probability $\mathbb{Q}$ such that the marginal of $\mathbb{Q}$ on the space of trees Ω satisfies

$$\frac{d\mathbb{Q}}{d\mathbb{P}}|_{\mathcal{F}_n} := M_n, \qquad n \geq 1,$$

where $\mathcal{F}_n$ is the σ-field generated by the first n generations of ω and $\mathbb{P}$ denotes the Galton–Watson law. Since $p_0 = 0$, $M_\infty > 0$ a.s., moreover,

$$\mathbb{Q}\Big(u_n^* = u \big| \mathcal{F}_\infty\Big) = \frac{1}{M_n m^n}, \qquad \forall\, |u| = n.$$

Under $\mathbb{Q}$, $d_{u_n^*}$ has the size-biased distribution associated with $\{p_k\}$, that is $\mathbb{Q}(d_{u_n^*} = k) = kp_k/m$, u_{n+1}^* is uniformly chosen among the children of u_n^* and, for $v \neq u_{n+1}^*$ with $\overleftarrow{v} = u_n^*$, the sub-trees $\mathcal{T}(v)$ rooted at v, are i.i.d. and have a Galton–Watson law.

Let $(d_i^*, i \geq 1)$ be a sequence of i.i.d. random variables of common law the size-biased of d, and $(\beta_n^{(i,j)}, M_n^{(i,j)}, n \geq 1)_{i,j\geq 0}$ be i.i.d copies of $(\beta_n, M_n(o), n \geq 0)$ $[\beta_0 = 1]$, independent of $(d_i^*)_{i\geq 1}$. We denote by $\beta^{(i,j)}$ the limit of $\beta_n^{(i,j)}$ as $n \to \infty$. Let us define another sequence $(M_j^{*,n})_{1\leq j\leq n}$ by

$$M_j^{*,n} := \frac{1}{m} M_{j+1}^{*,n} + \frac{1}{m} \sum_{k=1}^{d_j^*-1} M_{n-(j+1)}^{(j,k)}, \qquad j < n, \tag{3.14}$$

with $M_n^{*,n} := 1$. We write simply M_j^* for $M_j^{*,\infty}$ for all $j \geq 0$. Notice that under $\mathbb{Q}$, $(M_n(u_j^*), 0 \leq j \leq n)_{n\geq 0}$ has the same law as $(M_j^{*,n}, 0 \leq j \leq n)_{n\geq 0}$.

Consider a random walk $(S_., P)$ on $\mathbb{Z}$ with step distribution

$$P(S_i - S_{i-1} = 1) = \frac{\lambda}{\lambda + m^2}, \qquad P(S_i - S_{i-1} = -1) = \frac{m^2}{\lambda + m^2}, \qquad i \geq 1.$$

The random walk S is assumed to be independent of $(d_\cdot^*, \beta_\cdot^{(i,j)}, M_\cdot^{(i,j)}, i, j \geq 1)$. Define

$$Z_n(r) := E_{0,\omega}\Big(\mathbf{1}_{(\tau_S(-r) < \tau_S(n-r))} \prod_{i=0}^{\tau_S(-r)-1} f_{n-r}(S_i)\Big), \qquad 1 \leq r \leq n,$$

where the expectation $E_{0,\omega}$ only takes with respect to the random walk S with $S_0 = 0$, and

$$f_n(x) := \frac{m^2 + \lambda}{m(1 + \lambda + \lambda a_x^{(n)})} = \frac{m^2 + \lambda}{m(1 + \lambda + \sum_{k=1}^{d_x^*-1} \beta_{n-x-1}^{(x,k)})} \tag{3.15}$$

for $0 \leq x < n$, and $a_j^{(n)} := \frac{1}{\lambda} \sum_{k=1}^{d_j^*-1} \beta_{n-j-1}^{(j,k)}$ for any $j < n$. Let $(d_y^*, \beta_\cdot^{(y,k)})_{y\in\mathbb{Z}, k\geq 1}$ be a family of i.i.d. copies of $(d^*, \beta_\cdot)$ [recalling that under $\mathbb{Q}$, d and $(\beta_\cdot)$ are independent]. We extend the definitions of $f_n(x)$ and $a_x^{(n)}$ in the obvious way such that (3.15) holds for all $x \in \mathbb{Z} \cap (-\infty, n-1]$. The main result of this subsection is the following representation for $\mathbb{E}(\Phi_n(r))$:

Proposition 3.3 *For any $1 \leq r \leq n$,*

$$\mathbb{E}(\Phi_n(r)) = \mathbb{Q}\left[\frac{Z_n(r)}{M_0^{*,n-r}}\right] \tag{3.16}$$

Before entering into the proof of Proposition 3.3, we mention that the random walk $(S_., P)$ may find its root in the following lemma whose proof can be found in [6, Appendix]:

Lemma 3.4 (Spine random walk) *Let $n > k \geq 2$. Let $b_{j+1} > 0$ and $a_j \geq 0$ for all $0 \leq j < n$. Define $(z_j)_{0\leq j\leq n}$ by $z_n = 0$ and*

$$z_j := \frac{1}{1 + a_j + b_{j+1}(1 - z_{j+1})}, \qquad 0 \leq j \leq n-1.$$

Let (S_m) be a Markov chain on $\{0, 1, ..., n\}$ with probability transition $\widetilde{P}(S_m = j+1 | S_{m-1} = j) = \frac{b_{j+1}}{1+b_{j+1}}$ and $\widetilde{P}(S_m = j-1 | S_{m-1} = j) = \frac{1}{1+b_{j+1}}$, and denote by $\widetilde{P}_r$ the law of the chain (S_m) with $S_0 = r$. Then, for any $1 \leq r < n$,

$$\prod_{j=1}^{r} z_j = \widetilde{E}_r\Big(\mathbf{1}_{(\tau_S(0) < \tau_S(n))} \prod_{j=1}^{n-1} \Big(\frac{1 + b_{j+1}}{1 + b_{j+1} + a_j}\Big)^{L^j_{\tau_S(0)}}\Big),$$

with $\tau_S(x) := \inf\{j \geq 1 : S_j = x\}$ the first hitting time of S at x and $L^x_m := \sum_{i=0}^{m-1} 1_{(S_i = x)}$ is the local time at x.

Proof of Proposition 3.3: Observe that $\beta_n(x) = \frac{B_n(x)}{1+B_n(x)}$ and

$$\Phi_n(r) = \frac{1}{\lambda^r} \sum_{|u|=r} (1 - \beta_n(u_1)) \cdots (1 - \beta_n(u_r)).$$

By the change of measure, we have for any $F \geq 0$,

$$\mathbb{E}\left[\sum_{|u|=n} F(\beta_n(u_1), d_{u_1}, ..., \beta_n(u_n), d_{u_n})\right] = m^n \mathbb{Q}\left[F(\beta_n(u_1^*), d_{u_1^*}, ..., \beta_n(u_n^*), d_{u_n^*})\right].$$

It follows that for $r < n$,

$$\begin{aligned} & m^r\, \mathbb{Q}\Big((1 - \beta_n(u_1^*)) \cdots (1 - \beta_n(u_r^*)) \frac{1}{M_n(u_r^*)}\Big) \\ = \;& m^{-(n-r)} \mathbb{E}\left[\sum_{|v|=n} (1 - \beta_n(v_1)) \cdots (1 - \beta_n(v_r)) \frac{1}{M_n(v_r)}\right] \\ = \;& \mathbb{E}\left[\sum_{|u|=r} (1 - \beta_n(u_1)) \cdots (1 - \beta_n(u_r))\right], \end{aligned}$$

where the term $m^{-(n-r)} \frac{1}{M_n(v_r)}$ disappears when one takes the sum over $|v| = n$ by keeping $v_r = u$. It follows that

$$\mathbb{E}(\Phi_n(r)) = \frac{m^r}{\lambda^r} \mathbb{Q}\Big((1 - \beta_n(u_1^*)) \cdots (1 - \beta_n(u_r^*)) \frac{1}{M_n(u_r^*)}\Big), \tag{3.17}$$

and exactly the same as (3.13), by iterating the equations on B_n, we get that for any $r \leq n-1$,

$$B_n(o) = \frac{1}{\lambda^r} \sum_{|u|=r} \frac{1}{1 + B_n(u_1)} \cdots \frac{1}{1 + B_n(u_{r-1})} \frac{B_n(u_r)}{1 + B_n(u_r)}.$$

Hence,

$$\mathbb{E}(B_n(o)) = \frac{m^r}{\lambda^r} \mathbb{Q}\Big((1 - \beta_n(u_1^*)) \cdots (1 - \beta_n(u_{r-1}^*)) \beta_n(u_r^*) \frac{1}{M_n(u_r^*)}\Big). \tag{3.18}$$

Note that

$$\beta_n(u_j^*) = \frac{\beta_n(u_{j+1}^*) + \sum_{v\neq u_{j+1}^*, \overleftarrow{v}=u_j^*} \beta_n(v)}{\lambda + \beta_n(u_{j+1}^*) + \sum_{v\neq u_{j+1}^*, \overleftarrow{v}=u_j^*} \beta_n(v)},$$

and

$$M_n(u_j^*) = \frac{1}{m} \sum_{v\neq u_{j+1}^*, \overleftarrow{v}=u_j^*} M_n(v) + \frac{1}{m} M_n(u_{j+1}^*).$$

Under $\mathbb{Q}$, for such $|v| = j+1$, $(\beta_n(v), M_n(v))$ are i.i.d. and distributed as $(\beta_{n-j-1}, M_{n-j-1}(o))$ (under $\mathbb{P}$). Define a sequence $(\xi_i^{(n)})_{1\leq i\leq n}$ by

$$\xi_j^{(n)} := \frac{\xi_{j+1}^{(n)} + \sum_{k=1}^{d_j^*-1} \beta_{n-j-1}^{(j,k)}}{\lambda + \xi_{j+1}^{(n)} + \sum_{k=1}^{d_j^*-1} \beta_{n-j-1}^{(j,k)}}, \quad j < n,$$

and $\xi_n^{(n)} := 1$. Then $(\beta_n(u_j^*), M_n(u_j^*))_{j\leq n}$ is distributed as $(\xi_j^{(n)}, M_j^{*,n})_{j\leq n}$, which implies that

$$\mathbb{E}(\Phi_n(r)) \quad = \quad \frac{m^r}{\lambda^r} \mathbb{Q}\Big((1-\xi_1^{(n)})\cdots(1-\xi_r^{(n)}) \frac{1}{M_r^{*,n}}\Big). \tag{3.19}$$

We can represent $1-\xi_j^{(n)}$ as the probability for a one-dimensional random walk in a random environment (RWRE) with cemetery point, starting from j, to hit $j-1$ before n. Applying Lemma 3.4 to $a_j^{(n)} := \frac{1}{\lambda}\sum_{k=1}^{d_j^*-1} \beta_{n-j-1}^{(j,k)}$ and $b_{j+1} := \frac{1}{\lambda}$, we see that

$$\begin{aligned}
\prod_{j=1}^{r}(1-\xi_j^{(n)}) \quad &= \quad \widetilde{E}_{r,\omega}\Big(\mathbf{1}_{(\tau_S(0)<\tau_S(n))} \prod_{j=1}^{n-1} \Big(\frac{1+\lambda}{1+\lambda+\lambda a_j^{(n)}}\Big)^{L^j_{\tau_S(0)}}\Big) \\
&= \quad \widetilde{E}_{r,\omega}\Big(\mathbf{1}_{(\tau_S(0)<\tau_S(n))} \prod_{i=0}^{\tau_S(0)-1} \frac{1+\lambda}{1+\lambda+\lambda a_{S_i}^{(n)}}\Big),
\end{aligned}$$

where $(S_i)_{i\geq 0}$ is a random walk on $\mathbb{Z}$ with step distribution $\widetilde{P}(S_i - S_{i-1} = 1) = \frac{1}{1+\lambda}$ and $\widetilde{P}(S_i - S_{i-1} = -1) = \frac{\lambda}{1+\lambda}$ for $i \geq 1$, and the expectation $\widetilde{E}_{r,\omega}$ is taken with respect to (S_m) with $S_0 = r$.

Define the probability P with

$$\frac{dP}{d\widetilde{P}}\Big|_{\sigma\{S_0,\ldots,S_n\}} = \Big(\frac{\lambda}{m}\Big)^{S_n-S_0} \Big(\frac{m(1+\lambda)}{m^2+\lambda}\Big)^n, \qquad n \geq 0.$$

Then,

$$\frac{m^r}{\lambda^r}\prod_{j=1}^{r}(1-\xi_j^{(n)}) = E_{r,\omega}\Big(\mathbf{1}_{(\tau_S(0)<\tau_S(n))} \prod_{i=0}^{\tau_S(0)-1} f_n(S_i)\Big) := \widetilde{Z}_n(r).$$

With the notation of $\widetilde{Z}_n(r)$, we get that

$$\mathbb{E}\left(\Phi_n(r)\right) = \mathbb{Q}\left[\frac{\widetilde{Z}_n(r)}{M_r^{*,n}}\right]. \tag{3.20}$$

Observe that for any $r < n$, under $\mathbb{Q}$, $(f_n(x+r), d_{r+y}^*, M_{r+\cdot}^{*,r+k})_{x\leq n-r, y\geq 0, k\geq 0}$ has the law as $(f_{n-r}(x), d_y^*, M_{\cdot}^{*,k})_{x\leq n-r,y\geq 0,k\geq 0}$. This invariance by linear translation and (3.20) yield Proposition 3.3. $\square$

We end this subsection by the following remark:

Remark 3.5 *With the the same notations as in Proposition 3.3, we have*

$$\mathbb{E}(B_n(o)) \leq \frac{m}{\lambda}\mathbb{Q}\left[\frac{Z_n(r-1)}{M_1^{*,n-(r-1)}}\right], \tag{3.21}$$

$$\mathbb{E}(B_n(o)) \geq \frac{m}{\lambda}\mathbb{Q}\left[\frac{Z_n(r-1)}{M_1^{*,n-(r-1)}}\,\frac{\sum_{k=1}^{d_1^*-1}\beta_{n-r-1}^{(1,k)}}{\lambda+\sum_{k=1}^{d_1^*-1}\beta_{n-r-1}^{(1,k)}}\right]. \tag{3.22}$$

Proof of Remark 3.5: In the same way which leads to (3.19), we get from (3.18) that

$$\begin{aligned}
\mathbb{E}(B_n(o)) &= \frac{m^r}{\lambda^r}\,\mathbb{Q}\Big((1-\xi_1^{(n)})\cdots(1-\xi_{r-1}^{(n)})\xi_r^{(n)}\,\frac{1}{M_n(u_r^*)}\Big)\\
&= \frac{m^r}{\lambda^r}\,\mathbb{Q}\Big(\Big[\prod_{i=1}^{r-1}(1-\xi_i^{(n)})-\prod_{i=1}^{r}(1-\xi_i^{(n)})\Big]\,\frac{1}{M_r^{*,n}}\Big)\\
&= \frac{m}{\lambda}\,\mathbb{Q}\Big(\frac{\widetilde{Z}_n(r-1)}{M_r^{*,n}}\Big)-\mathbb{Q}\Big(\frac{\widetilde{Z}_n(r)}{M_r^{*,n}}\Big). \qquad (3.23)
\end{aligned}$$

On the other hand,

$$\xi_r^{(n)}=\frac{\xi_{r+1}^{(n)}+\sum_{k=1}^{d_r^*-1}\beta_{n-r-1}^{(j,k)}}{\lambda+\xi_{r+1}^{(n)}+\sum_{k=1}^{d_r^*-1}\beta_{n-r-1}^{(r,k)}}\geq\frac{\sum_{k=1}^{d_r^*-1}\beta_{n-r-1}^{(r,k)}}{\lambda+\sum_{k=1}^{d_r^*-1}\beta_{n-r-1}^{(r,k)}},$$

hence

$$\begin{aligned}
\mathbb{E}(B_n(o)) &\geq \frac{m^r}{\lambda^r}\,\mathbb{Q}\Big((1-\xi_1^{(n)})\cdots(1-\xi_{r-1}^{(n)})\frac{\sum_{k=1}^{d_r^*-1}\beta_{n-r-1}^{(r,k)}}{\lambda+\sum_{k=1}^{d_r^*-1}\beta_{n-r-1}^{(r,k)}}\,\frac{1}{M_r^{*,n}}\Big)\\
&= \frac{m}{\lambda}\,\mathbb{Q}\Big(\widetilde{Z}_n(r-1)\,\frac{\sum_{k=1}^{d_r^*-1}\beta_{n-r-1}^{(r,k)}}{\lambda+\sum_{k=1}^{d_r^*-1}\beta_{n-r-1}^{(r,k)}}\,\frac{1}{M_r^{*,n}}\Big), \qquad (3.24)
\end{aligned}$$

yielding the assertions in Remark 3.5. □

3.2 An argument based on renewal theory

The main result is Lemma 3.7 which evaluates the limit of $\mathbb{E}(\Phi_n(r))$ and in turn gives the velocity representation in Theorem 3.8. The analysis is based on the use of a renewal structure in the representation of Proposition 3.3. Under P, (S_i) drifts to $-\infty$. Denote by $(R_0 := 0) < R_1 < R_2 < \ldots$ the regeneration times for (S_i), that is $R_i = \min\{n > R_{i-1} : \{S_i\}_{j=0}^n \cap \{S_j\}_{j>n} = \emptyset\}$. The sequence $\{S_{j+R_i} - S_{R_i}, 0 \leq j \leq R_{i+1}-R_i\}_{i\geq 1}$ is clearly i.i.d and has as common distribution that of $\{S_j, 0 \leq j \leq R_1\}$ conditioned on $\{\tau_S(1) = \infty\}$. Further, because

$$E(S_{i+1}-S_i)=\frac{\lambda-m^2}{\lambda+m^2}\leq-\frac{m-1}{m+1},$$

it is straightforward to check that there exists a constant $\kappa > 0$, independent of α, so that

$$E(e^{\kappa R_1})<\infty\,,\quad E(e^{\kappa(R_2-R_1)})<\infty\,. \tag{3.25}$$

Define

$$\zeta_j := \prod_{i=R_{j-1}}^{R_j-1} f_\infty(S_i), \qquad j\geq 1,$$

where $f_\infty(x)$, and $a_x^{(\infty)}$, are defined by

$$f_\infty(x) := \frac{m^2+\lambda}{m(1+\lambda+\lambda a_x^{(\infty)})} := \frac{m^2+\lambda}{m(1+\lambda+\sum_{k=1}^{d_x^*-1}\beta^{(x,k)})}, \qquad x\in\mathbb{Z}. \tag{3.26}$$

Remark that under $\mathbb{Q}\otimes P$, $(\zeta_j, j\ge 2)$ are i.i.d. and independent of ζ_1. Define further

$$h(y) := \mathbb{Q}\otimes P\left[\prod_{i=0}^{\tau_S(-y)-1} f_\infty(S_i)1_{(\tau_S(-y)\le R_1)}\,\Big|\,\tau_S(1)=\infty\right], \qquad y\ge 1. \tag{3.27}$$

We extend the definition of h to $\mathbb{Z}$ by letting $h(y):=0$ if $y\le 0$.

Lemma 3.6 *Assume that*

$$\sum_{y=1}^{\infty} h(y) < \infty, \tag{3.28}$$

$$\mathbb{Q}\otimes P\left[\frac{\zeta_1}{M_1^*}+\zeta_1+|S_{R_2}-S_{R_1}|\,\zeta_2\right]<\infty. \tag{3.29}$$

Then,

$$\mathbb{Q}\otimes P\,[\zeta_2]=1.$$

We will see below, see Lemma 3.9, that (3.28) and (3.29) both hold for all $\alpha>0$ small enough.

Proof: Almost surely, $\beta_n(x)\downarrow\beta(x)$. Then for a fixed r, almost surely,

$$Z_n(r)\to Z_\infty(r) := E_{0,\omega}\Big(\prod_{i=0}^{\tau_S(-r)-1} f_\infty(S_i)\Big),$$

and $M_1^{*,n}\to M_1^*$. Applying Fatou's lemma in the expectation under $\mathbb{Q}$ in (3.22), we get that for any r,

$$\begin{aligned}\frac{\lambda}{m}\mathbb{E}(B) &\ge \mathbb{Q}\left[\frac{Z_\infty(r-1)}{M_1^*}\,\frac{\sum_{k=1}^{d_1^*-1}\beta^{(1,k)}}{\lambda+\sum_{k=1}^{d_1^*-1}\beta^{(1,k)}}\right]\\ &= \mathbb{Q}\otimes P\left[\prod_{i=0}^{\tau_S(1-r)-1} f_\infty(S_i)\,\frac{1}{M_1^*}\,\frac{\sum_{k=1}^{d_1^*-1}\beta^{(1,k)}}{\lambda+\sum_{k=1}^{d_1^*-1}\beta^{(1,k)}}\right].\end{aligned}$$

We can not directly let $r\to\infty$ inside the above expectation, so we decompose this expectation by the regeneration times $0<R_1<R_2<\ldots$. Write

$$\zeta_1' := \frac{\zeta_1}{M_1^*}\,\frac{\sum_{k=1}^{d_1^*-1}\beta^{(1,k)}}{\lambda+\sum_{k=1}^{d_1^*-1}\beta^{(1,k)}}.$$

Then

$$\begin{aligned}\frac{\lambda}{m}\mathbb{E}(B) &\ge \sum_{k=2}^{\infty}\mathbb{Q}\otimes P\left[1_{(R_k<\tau_S(1-r)\le R_{k+1})}\,\zeta_1'\prod_{i=R_1}^{R_k-1} f_\infty(S_i)\prod_{i=R_k}^{\tau_S(1-r)-1} f_\infty(S_i)\right]\\ &= \sum_{k=2}^{\infty}\mathbb{Q}\otimes P\left[1_{(R_k<\tau_S(1-r))}\,\zeta_1'\prod_{i=R_1}^{R_k-1} f_\infty(S_i)\,h(r-1+S_{R_k})\right],\end{aligned}$$

by using the Markov property of S at R_k. Observe that $(\zeta_j, S_{R_j} - S_{R_{j-1}})_{j\geq 2}$ are i.i.d. under the annealed measure $\mathbb{Q}\otimes P$, and are independent of (ζ_1', S_{R_1}). By replacing $r-1$ by r, we get that for any r,

$$\sum_{k=2}^{\infty} \mathbb{Q}\otimes P\left[1_{(S_{R_k}>-r)}\,\zeta_1' \prod_{j=2}^{k}\zeta_j\, h(r+S_{R_k})\right] \leq \frac{\lambda}{m}\mathbb{E}(B). \tag{3.30}$$

Now, we claim that

$$\mathbb{Q}\otimes P\,[\zeta_2] \leq 1. \tag{3.31}$$

To prove (3.31), we assume that $a := \mathbb{Q}\otimes P\,[\zeta_2] > 1$ and show that it leads to a contradiction with (3.30). Toward this end, define a distribution U on $\mathbb{Z}_+$ by

$$U(x) := \frac{\mathbb{Q}\otimes P\left[1_{(S_{R_2}-S_{R_1}=-x)}\zeta_2\right]}{\mathbb{Q}\otimes P\,[\zeta_2]}, \qquad x\geq 0.$$

Then (3.30) becomes

$$\begin{aligned}\frac{\lambda}{m}\mathbb{E}(B) &\geq \sum_{k=2}^{\infty} a^{k-1}\,\mathbb{Q}\otimes P\left[1_{(S_{R_1}>-r)}\,\zeta_1' \sum_{x=0}^{r+S_{R_1}} h(r+S_{R_1}-x)U^{\otimes(k-1)}(x)\right]\\ &\geq a^{l-1}\sum_{k=l}^{\infty} \mathbb{Q}\otimes P\left[1_{(S_{R_1}>-r)}\,\zeta_1' \sum_{x=0}^{r+S_{R_1}} h(r+S_{R_1}-x)U^{\otimes(k-1)}(x)\right],\end{aligned}$$

for any $l\geq 2$.

Since $\sum_{k=1}^{l-1}\mathbb{Q}\otimes P\left[1_{(S_{R_1}>-r)}\,\zeta_1' \sum_{x=0}^{r+S_{R_1}} h(r+S_{R_1}-x)U^{\otimes(k-1)}(x)\right] \to 0$ as $r\to\infty$ [by the dominated convergence under (3.28) and the integrability of $\zeta_1' \leq \frac{\zeta_1}{M_1^*}$ under (3.29)], we get that for any fixed ℓ,

$$\begin{aligned}a^{1-l}\,\frac{\lambda}{m}\mathbb{E}(B) &\geq \sum_{k=1}^{\infty} \mathbb{Q}\otimes P\left[1_{(S_{R_1}>-r)}\,\zeta_1' \sum_{x=0}^{r+S_{R_1}} h(r+S_{R_1}-x)U^{\otimes(k-1)}(x)\right] + o(1)\\ &= \mathbb{Q}\otimes P\,[\zeta_1']\,\frac{\sum_{x=0}^{\infty} h(x)}{\sum_{x=0}^{\infty} xU(x)} + o(1), \qquad r\to\infty,\end{aligned}$$

by applying the renewal theorem [7, pg. 362], using (3.29). Thus we get some constant $C>0$ such that $\frac{\lambda}{m}\mathbb{E}(B) \geq a^{l-1}C$ for any $\ell\geq 2$, which is impossible if $a>1$. Hence we proved (3.31).

It remains to show

$$\mathbb{Q}\otimes P\,[\zeta_2] \geq 1. \tag{3.32}$$

The proof of this part is similar, we shall use (3.21) instead of (3.22). Set

$$\bar{f}_S(r) := \prod_{i=0}^{\tau_S(-r)-1} f_\infty(S_i)\,.$$

Since $f_\infty(x) \geq f_\ell(x)$ for any ℓ, we get that

$$\frac{\lambda}{m}\mathbb{E}(B_n(o)) \leq \mathbb{Q}\otimes P\left[\mathbf{1}_{(\tau_S(1-r)<\tau_S(n-(r-1)))}\bar{f}_S(r-1)\,\frac{1}{M_1^{*,n-(r-1)}}\right].$$

Taking $r = n$ gives that

$$\frac{\lambda}{m}\mathbb{E}(B_n(o)) \le \mathbb{Q}\otimes P\left[\mathbf{1}_{(\tau_S(1-n)<\tau_S(1))}\bar{f}_S(n-1)\right] .$$

Since $\mathbb{E}(B) \le \mathbb{E}(B_n(o))$, we obtain that for any n,

$$\begin{aligned}\frac{\lambda}{m}\mathbb{E}(B) &\le \mathbb{Q}\otimes P\left[\mathbf{1}_{(\tau_S(-n)<\tau_S(1))}\bar{f}_S(n)\right]\\ &\le \mathbb{Q}\otimes P\left[\mathbf{1}_{(\tau_S(-n)\le R_1,\,\tau_S(-n)<\tau_S(1)))}\bar{f}_S(n)\right] + \sum_{k=1}^{\infty}\mathbb{Q}\otimes P\left[\mathbf{1}_{(R_k<\tau_S(-n)\le R_{k+1})}\bar{f}_S(n)\right].\end{aligned}$$

By the Markov property at $\tau_S(-n)$, the first term equals

$$\frac{\mathbb{Q}\otimes P\left[\mathbf{1}_{(\tau_S(-n)\le R_1)}\bar{f}_S(n)\mathbf{1}_{(\tau_S(1)=\infty)}\right]}{P_{-n}(\tau_S(1)=\infty)} = \frac{P(\tau_S(1)=\infty)}{P_{-n}(\tau_S(1)=\infty)}h(n) \to 0\,,$$

since $P_{-n}(\tau_S(1) = \infty) \ge c$ for some constant $c > 0$ and $h(n) \to 0$. Then, recalling that h vanishes at $\mathbb{Z}_-$, we get

$$\frac{\lambda}{m}\mathbb{E}(B) \le o(1) + \sum_{k=1}^{\infty}\mathbb{Q}\otimes P\left[\zeta_1\prod_{j=2}^{k}\zeta_j\, h(n+S_{R_k})\right] ,$$

with ζ_j and h defined as before. If $a := \mathbb{Q}\otimes P\,[\zeta_2] < 1$, then with the distribution $U(\cdot)$ introduced before,

$$\begin{aligned}\sum_{k=1}^{\infty}\mathbb{Q}\otimes P\left[\zeta_1\prod_{j=2}^{k}\zeta_j\, h(n+S_{R_k})\right] &= \sum_{k=1}^{\infty}a^{k-1}\,\mathbb{Q}\otimes P\left[\zeta_1\sum_{x=0}^{n+S_{R_1}}h(n+S_{R_1}-x)U^{\otimes(k-1)}(x)\right]\\ &:= \sum_{k=1}^{\infty}a^{k-1}\,b_k^{(n)}.\end{aligned}$$

Note that $\max_n b_k^{(n)} \le \mathbb{Q}\otimes\widetilde{P}\,[\,\zeta_1]\,\sum_{x=0}^{\infty}h(x)\,U^{\otimes(k-1)}(x)$, and that, due to (3.28), $\lim_{n\to\infty} b_k^{(n)} = 0$. The dominated convergence theorem then implies that $\sum_{k=1}^{\infty}a^{k-1}\,b_k^{(n)} \to 0$ which in turn yields $\frac{\lambda}{m}\mathbb{E}(B) \le o(1)$, a contradiction. Thus $\mathbb{Q}\otimes P\,[\zeta_2] \ge 1$. This completes the proof of the lemma. □

Lemma 3.7 *Assume* (3.28), (3.29) *and that for some* $p > 1$,

(3.33) $$\mathbb{Q}\otimes P((\zeta_2)^p) < \infty.$$

Furthermore, we assume that

(3.34) $$\lim_{r\to\infty}\mathbb{Q}\otimes P\left[\mathbf{1}_{(\tau_S(-r)\le R_1)}\prod_{i=0}^{\tau_S(-r)-1}f_\infty(S_i)\,\frac{1}{M_0^*}\right] = 0,$$

where as before, R_1 is the first regeneration time for S under P and M_0^* is define in (3.14). Then, for any $\varepsilon > 0$,

(3.35) $$\lim_{n\to\infty}\max_{\varepsilon n\le r\le(1-\varepsilon)n}\left|\mathbb{E}(\Phi_n(r)) - \frac{\mathbb{Q}\otimes P\left(\frac{\zeta_1}{M_0^*}\right)\sum_{y\ge1}h(y)}{\mathbb{Q}\otimes P\left(\zeta_2|S_{R_2}-S_{R_1}|\right)}\right| = 0.$$

Moreover,

$$\sup_{r\ge 1,\, n\ge r} \mathbb{E}(\Phi_n(r)) < \infty. \tag{3.36}$$

Proof: We are going to prove the following stronger assertions:

$$\limsup_{r\to\infty} \sup_{n\ge r} \mathbb{E}(\Phi_n(r)) \le \frac{\mathbb{Q}\otimes P\left(\frac{\zeta_1}{M_0^*}\right) \sum_{y\ge 1} h(y)}{\mathbb{Q}\otimes P\left(\zeta_2 |S_{R_2} - S_{R_1}|\right)}, \tag{3.37}$$

$$\liminf_{n\to\infty} \max_{\varepsilon n\le r\le (1-\varepsilon)n} \mathbb{E}(\Phi_n(r)) \ge \frac{\mathbb{Q}\otimes P\left(\frac{\zeta_1}{M_0^*}\right) \sum_{y\ge 1} h(y)}{\mathbb{Q}\otimes P\left(\zeta_2 |S_{R_2} - S_{R_1}|\right)}. \tag{3.38}$$

Clearly, (3.36) is an immediate consequence of (3.37) and (3.35) follows from (3.37) and (3.38).

Proof of (3.37) : Let us introduce the notation: for $\ell \ge 1$,

$$\zeta_j(\ell) := \prod_{i=R_{j-1}}^{R_j - 1} f_\ell(S_i), \qquad j\ge 1.$$

By (3.16),

$$\mathbb{E}(\Phi_n(r)) = \mathbb{Q}\otimes P\left[\mathbf{1}_{(\tau_S(-r)<\tau_S(n-r))} \prod_{i=0}^{\tau_S(-r)-1} f_{n-r}(S_i)\, \frac{1}{M_0^*}\right].$$

Noticing that $\mathbf{1}_{(\tau_S(-r)<\tau_S(n-r))} = 1_{(\tau_S(-r)\le R_1\wedge \tau_S(n-r))} + \sum_{k=1}^\infty 1_{(R_1<\tau_S(n-r),\, R_k<\tau_S(-r)\le R_{k+1})}$, we get

$$\mathbb{E}(\Phi_n(r)) = I_{(3.39)}(0) + \sum_{k=1}^\infty I_{(3.39)}(k), \tag{3.39}$$

where

$$\begin{aligned} I_{(3.39)}(0) &:= \mathbb{Q}\otimes P\left[\mathbf{1}_{(\tau_S(-r)<R_1\wedge\tau_S(n-r))} \prod_{i=0}^{\tau_S(-r)-1} f_{n-r}(S_i)\, \frac{1}{M_0^*}\right], \\ I_{(3.39)}(k) &:= \mathbb{Q}\otimes P\left[1_{(R_1<\tau_S(n-r),\, R_k<\tau_S(-r)\le R_{k+1})} \frac{\zeta_1(n-r)}{M_0^*} \prod_{j=2}^{k} \zeta_j(n-r) \prod_{i=R_k}^{\tau_S(-r)-1} f_{n-r}(S_i)\right] \\ &= \mathbb{Q}\otimes P\left[1_{(R_1<\tau_S(n-r),\, R_k<\tau_S(-r))} \frac{\zeta_1(n-r)}{M_0^*} \prod_{j=2}^{k} \zeta_j(n-r)\, h_{n-r}(r+S_{R_k})\right], \end{aligned}$$

by the Markov property at R_k (with convention $\prod_\emptyset \equiv 1$) and with

$$h_\ell(y) := \mathbb{Q}\otimes P\left[\prod_{i=0}^{\tau_S(-y)-1} f_\ell(S_i) 1_{(\tau_S(-y)\le R_1)} \,\Big|\, \tau_S(1)=\infty\right], \qquad y\ge 1,\, \ell\ge 1.$$

We also define $h_L(y) := 0$ for all $y\le 0$. Since $f_\ell(x)\le f_\infty(x)$, $h_\ell(x)\le h(x)$, $\zeta_j(n-r)\le \zeta_j$ for any $j\ge 1$, we have

$$I_{(3.39)}(0) \le \mathbb{Q}\otimes P\left[\mathbf{1}_{(\tau_S(-r)<R_1)} \prod_{i=0}^{\tau_S(-r)-1} f_\infty(S_i)\, \frac{1}{M_0^*}\right] \to 0, \qquad \text{as } r\to\infty,$$

by (3.34). Further,

$$\sum_{k=1}^{\infty} I_{(3.39)}(k) \leq \sum_{k=1}^{\infty} \mathbb{Q}\otimes P\left[1_{(S_{R_1}>-r)}\frac{\zeta_1}{M_0^*}\prod_{j=2}^{k}\zeta_j\, h(r+S_{R_k})\right]$$

(3.40)
$$= \sum_{k=1}^{\infty} \mathbb{Q}\otimes P\left[1_{(S_{R_1}>-r)}\frac{\zeta_1}{M_0^*}\sum_{x=0}^{r+S_{R_1}} h(r+S_{R_1}-x)\, U^{\otimes(k-1)}(x)\right],$$

where

$$U(x) := \mathbb{Q}\otimes P\left[1_{(S_{R_2}-S_{R_1}=-x)}\,\zeta_2\right], \qquad x \geq 0,$$

is a distribution by Lemma 3.7.

Note that $M_0^* \geq \frac{M_1^*}{m}$, $\mathbb{Q}\otimes P\left(\frac{\zeta_1}{M_0^*}\right) \leq m\,\mathbb{Q}\otimes P\left(\frac{\zeta_1}{M_1^*}\right) < \infty$ by (3.29). Recall that $h(y)=0$ for all $y \leq 0$. By the renewal theorem and (3.28), as $r\to\infty$, the sum in (3.40) converges to

$$\frac{\mathbb{Q}\otimes P\left(\frac{\zeta_1}{M_0^*}\right)\sum_{y\geq 1} h(y)}{\mathbb{Q}\otimes P\left(\zeta_2|S_{R_2}-S_{R_1}|\right)}.$$

The estimate (3.37) follows.

Proof of (3.38) : The idea is to replace $\zeta_2(n-r)$ by $\zeta_2 \equiv \zeta_2(\infty)$ in (3.39). Let $\ell = n-r$ and recall that $f_\ell(x) := \frac{m^2+\lambda}{m(1+\lambda+\lambda a_x^{(\ell)})} = \frac{m^2+\lambda}{m(1+\lambda+\sum_{k=1}^{d_x^*}\beta_{\ell-x-1}^{(x,k)})}$, for $x<\ell$. Then

$$0 \leq f_\infty(x) - f_\ell(x) = \frac{(m^2+\lambda)\lambda(a_x^{(\ell)}-a_x^{(\infty)})}{m(1+\lambda+\lambda a_x^{(\ell)})\,(1+\lambda+\lambda a_x^{(\infty)})} = f_\infty(x)\,\frac{\lambda(a_x^{(\ell)}-a_x^{(\infty)})}{1+\lambda+\lambda a_x^{(\ell)}}.$$

It follows that for any $j\geq 2$,

$$\zeta_j(\ell) = \zeta_j \prod_{i=R_{j-1}}^{R_j-1}\left[1-\frac{\lambda(a_{S_i}^{(\ell)}-a_{S_i}^{(\infty)})}{1+\lambda+\lambda a_{S_i}^{(\ell)}}\right] := \zeta_j \,\times\, \Lambda_j(\ell).$$

Fix a large integer L. Using (3.39) and the fact that $h_\cdot(y)$ is nondecreasing for any y, we deduce that for all $n-r\geq L$ and any large constant $C>0$ [the constant C will be chosen later on],

$$\mathbb{E}(\Phi_n(r)) \geq \sum_{k=1}^{Cn}\mathbb{Q}\otimes P\left[1_{(R_1<\tau_S(L),\, R_k<\tau_S(-r))}\frac{\zeta_1(L)}{M_0^*}\prod_{j=2}^{k}\zeta_j\prod_{j=2}^{k}\Lambda_j(n-r)h_L(r+S_{R_k})\right].$$

The first step is to replace $\Lambda_j(n-r)$ by 1, then we have to check that the error term is uniformly small:

(3.41)
$$I_{(3.41)} := \sum_{k=1}^{Cn}\mathbb{Q}\otimes P\left[1_{(R_1<\tau_S(L),\, R_k<\tau_S(-r))}\frac{\zeta_1(L)}{M_0^*}\prod_{j=2}^{k}\zeta_j\left[1-\prod_{j=2}^{k}\Lambda_j(n-r)\right]h_L(r+S_{R_k})\right] \to 0,$$

as $r\to\infty$ and $\varepsilon n \leq r \leq (1-\varepsilon)n$. Let us postpone for the moment the proof of (3.41). Going back to $\mathbb{E}(\Phi_n(r))$, we obtain that

$$\begin{aligned}
& \mathbb{E}(\Phi_n(r)) \\
\geq\ & \sum_{k=1}^{Cn} \mathbb{Q}\otimes P\left[1_{(R_1<\tau_S(L),\, R_k<\tau_S(-r))}\frac{\zeta_1(L)}{M_0^*}\prod_{j=2}^{k}\zeta_j\, h_L(r+S_{R_k})\right] - I_{(3.41)} \\
\geq\ & \sum_{k=1}^{\infty} \mathbb{Q}\otimes P\left[1_{(R_1<\tau_S(L),\, R_k<\tau_S(-r))}\frac{\zeta_1(L)}{M_0^*}\prod_{j=2}^{k}\zeta_j\, h_L(r+S_{R_k})\right] - I_{(3.41)} - I_{(3.42)}, \qquad (3.42)
\end{aligned}$$

with

$$I_{(3.42)} := \sum_{k=Cn+1}^{\infty} \mathbb{Q}\otimes P\left[1_{(R_1<\tau_S(L),\, R_k<\tau_S(-r))}\frac{\zeta_1(L)}{M_0^*}\prod_{j=2}^{k}\zeta_j\, h_L(r+S_{R_k})\right].$$

If we can prove that for a well-chosen C, $I_{(3.42)}$ goes to zero uniformly for $r\to\infty$ and $\varepsilon n \leq r \leq (1-\varepsilon)n$, then by applying the renewal theorem (L fixed, $r\to\infty$) to the sum in (3.42), we get that

$$\liminf_{n\to\infty}\min_{n-r\geq\varepsilon n}\mathbb{E}(\Phi_n(r)) \geq \frac{\mathbb{Q}\otimes P\left(\frac{\zeta_1(L)}{M_0^*}1_{(R_1<\tau_S(L))}\right)\sum_{y\geq 1}h_L(y)}{\mathbb{Q}\otimes P\left(\zeta_2|S_{R_2}-S_{R_1}|\right)}.$$

Letting $L\to\infty$ gives the lower bound (3.38).

It remains to show that $I_{(3.42)}$ and $I_{(3.41)}$ go to zero uniformly for $r\to\infty$ and $\varepsilon n\leq r\leq (1-\varepsilon)n$. We first deal with $I_{(3.42)}$. Let $h^* := \max_{x\geq 0} h(x)$. We have

$$\begin{aligned}
I_{(3.42)} &\leq h^* \sum_{k=Cn+1}^{\infty}\mathbb{Q}\otimes P\left[1_{(R_k<\tau_S(-r))}\frac{\zeta_1(L)}{M_0^*}\prod_{j=2}^{k}\zeta_j\right] \\
&\leq h^* \sum_{k=Cn+1}^{\infty}\mathbb{Q}\otimes P\left[\frac{\zeta_1(L)}{M_0^*}1_{(S_{R_k}-S_{R_1}>-r)}\prod_{j=2}^{k}\zeta_j\right] \\
&= h^* \sum_{k=Cn+1}^{\infty}\mathbb{Q}\otimes P\left[\frac{\zeta_1(L)}{M_0^*}\right]\mathbb{Q}\otimes P\left[1_{(S_{R_k}-S_{R_1}>-r)}\prod_{j=2}^{k}\zeta_j\right],
\end{aligned}$$

by the independence between $(\zeta_1(L), M_0^*)$ and $(S_{R_k}-S_{R_1}, \zeta_j, j\geq 2)$. Recalling that $\mathbb{Q}\otimes P(\zeta_2)=1$. Let $\widehat{P}$ be a new probability measure defined by $\frac{d\widehat{P}}{d\mathbb{Q}\otimes P}=\zeta_2$, then under $\widehat{P}$, $S_{R_1}-S_{R_k}$ is the sum of $k-1$ positive i.i.d. variables with mean $\mathbb{Q}\otimes P(\zeta_2(S_{R_1}-S_{R_2})) := a\in(0,\infty)$ by (3.29). Taking $C := \frac{2}{a}$. Then by Cramer's bound, there exists some $c_0>0$ such that

$$\mathbb{Q}\otimes P\left[1_{(S_{R_k}-S_{R_1}>-r)}\prod_{j=2}^{k}\zeta_j\right] = \widehat{P}\Big(S_{R_1}-S_{R_k}<r\Big)\leq e^{-c_0k},$$

for any $k>Cn$ and $r\leq n$. It follows

$$I_{(3.42)}\leq h^*\,\mathbb{Q}\otimes P\left[\frac{\zeta_1(L)}{M_0^*}\right]\sum_{k=Cn+1}^{\infty}e^{-c_0k}\to 0,$$

uniformly as $r\leq n$ and $r\to\infty$.

It remains to check (3.41) [with $C := \frac{2}{a}$ chosen before]. We first observe that $h_l(x) \le h(x) \le h^*$ and that

$$\begin{aligned} I_{(3.41)} &\le h^* \sum_{k=1}^{Cn} \mathbb{Q} \otimes P \left[\frac{\zeta_1}{M_0^*} \prod_{j=2}^{k} \zeta_j \Big(1 - \prod_{j=2}^{k} \Lambda_j(n-r) \Big) \right] \\ &= h^* \sum_{k=1}^{Cn} \widehat{E} \left[\frac{\zeta_1}{M_0^*} \Big(1 - \prod_{j=2}^{k} \Lambda_j(n-r) \Big) \right] \\ &= h^* \sum_{k=1}^{Cn} \widehat{E} \left[\frac{\zeta_1}{M_0^*} \right] \left[1 - (\widehat{E}(\Lambda_2(n-r)))^k \right], \end{aligned}$$

where the annealed expectation $\widehat{E}$ has the density ζ_2 with respect to $\mathbb{Q} \otimes P$ and under $\widehat{E}$, Λ_j are i.i.d and independent of ζ_1.

To proceed, we employ the following estimate, which will be proved below: there exists a constant c_1 (that may depend on α) so that $\forall \ell \ge \ell_0$,

$$1 - \widehat{E}(\Lambda_2(\ell)) \le e^{-c_1 \ell}. \tag{3.43}$$

Since $n - r \ge \varepsilon n$, (3.43) yields that $I_{(3.41)} \to 0$ as stated in (3.41).

It remains to check (3.43). $\beta_\ell(o) - \beta(o)$ corresponds to the probability that an excursion of the tree-valued walk is higher than ℓ; the latter is dominated by the probability that a level regeneration distance is larger than ℓ, which decays exponentially by [2, Lemma 4.2(i)]. It follows that

$$\mathbb{P}(\beta_\ell(o) - \beta(o) > e^{-c_2 \ell}) \le e^{-c_2 \ell}, \qquad \forall \ell \ge \ell_0, \tag{3.44}$$

where c_2 may depend on α. Then $\mathbb{E}((\beta_\ell(o) - \beta(o)) \le 2e^{-c_2 \ell}$. Notice that for $R_1 \le i < R_2$, $S_i < 0$ hence

$$\sum_{i=R_1}^{R_2-1} (a_{S_i}^{(\ell)} - a_{S_i}^{(\infty)}) = \sum_{x \le 0} (a_x^{(\ell)} - a_x^{(\infty)})(L_{R_2}^x - L_{R_1}^x) \le \frac{1}{\lambda} \sum_{x \le 0} \sum_{k=1}^{d_x^* - 1} (\beta_\ell^{(x,k)} - \beta^{(x,k)})(L_{R_2}^x - L_{R_1}^x),$$

implying that

$$\mathbb{Q} \otimes P \left[\sum_{i=R_1}^{R_2-1} (a_{S_i}^{(\ell)} - a_{S_i}^{(\infty)}) \right] \le \frac{2}{\lambda} \mathbb{Q}(d^* - 1)\, E(R_2 - R_1) e^{-c_2 \ell}.$$

By Hölder's inequality with $\frac{1}{p} + \frac{1}{q} = 1$,

$$\begin{aligned} \widehat{E} \left[1_{(\sum_{i=R_1}^{R_2-1} (a_{S_i}^{(\ell)} - a_{S_i}^{(\infty)}) > e^{-c_3 \ell})} \right] &= \mathbb{Q} \otimes P \left[\zeta_2 1_{(\sum_{i=R_1}^{R_2-1} (a_{S_i}^{(\ell)} - a_{S_i}^{(\infty)}) > e^{-c_3 \ell})} \right] \\ &\le (\mathbb{Q} \otimes P((\zeta_2)^p))^{1/p} \left(\mathbb{Q} \otimes P(\sum_{i=R_1}^{R_2-1} (a_{S_i}^{(\ell)} - a_{S_i}^{(\infty)}) > e^{-c_3 \ell}) \right)^{1/q} \\ &\le e^{-c_3 \ell}, \end{aligned}$$

for some constant $c_3 = c_3(\alpha, p, q, c_2) > 0$ and for all large ℓ. Now, using the elementary inequality: for any $j \ge 1$ and $x_1, ..., x_j \in [0,1]$, $1 - \prod_{i=1}^{j}(1 - x_i) \le \sum_{i=1}^{j} x_i$, we get

$$1 - \Lambda_2(\ell) \le \sum_{i=R_1}^{R_2-1} \frac{\lambda(a_{S_i}^{(\ell)} - a_{S_i}^{(\infty)})}{1 + \lambda + \lambda a_{S_i}^{(\ell)}} \le \sum_{i=R_1}^{R_2-1} (a_{S_i}^{(\ell)} - a_{S_i}^{(\infty)}).$$

Therefore

$$1 - \widehat{E}(\Lambda_2(\ell)) \le 2\, e^{-c_3\ell},$$

proving (3.43). The proof of the lemma is now complete. □

Noticing that $\mathbb{E}(\Gamma_n(o)) = m\,\mathbb{E}(\gamma_{n-1}(o))$. By (3.7), (3.13) and Lemma 3.7, we immediately obtain the following representation of the velocity v_α.

Theorem 3.8 (Velocity representation) *Assume* (3.28)*,* (3.29)*,* (3.33) *and* (3.34)*. Recall the functions f_∞ and h, see* (3.27) *and* (3.26)*, and* (3.14) *for M_0^*. Then,*

$$\frac{m\,\mathbb{E}(\beta)}{v_\alpha} = \frac{\mathbb{Q}\otimes P\left[\prod_{i=0}^{R_1-1} f_\infty(S_i)\,\frac{1}{M_0^*}\right]}{\mathbb{Q}\otimes P\left[\prod_{i=R_1}^{R_2-1} f_\infty(S_i)\,|S_{R_2}-S_{R_1}|\right]}\,\sum_{y\ge 1} h(y)\,.$$

Before applying Theorem 3.8, we show that the conditions for the representation of v_α hold when α is small enough. Recall our standing assumption that $p_0 = 0$, and the constant κ, see (3.25).

Lemma 3.9 *There exists an $\alpha_0 = \alpha_0(m,\kappa)$ such that if $0 < \alpha < \alpha_0$ then* (3.28)*,* (3.29)*,* (3.33) *and* (3.34) *hold.*

Proof: Note that $f_\infty(x) \le (m^2+\lambda)/(m+m\lambda)$ and the right side is a bounded differentiable function of α, which equals 1 at $\alpha = 0$. It follows that $f_\infty(x) \le 1 + c\alpha$ for some constant $c = c(m)$.

In what follows, we will make sure to use constants that do not depend on α. Note that, since $\tilde{P}(\tau_S(1) = \infty)$ is bounded away from 0 uniformly in α,

$$\sum_{y=1}^{\infty} h(y) \le C \sum_{n=0}^{\infty} (1+c\alpha)^n \tilde{P}(R_1 \ge n) \le C' \sum_{n=0}^{\infty} (1+c\alpha)^n e^{-\kappa n}\,,$$

where $C' = C'(\kappa, m)$ and we used (3.25). In particular, for $\alpha < \alpha_0(m,\kappa)$, we deduce (3.28).

The proof of (3.29) is similar: since $|S_{R_2} - S_{R_1}| < R_2 - R_1$, the exponential moments (3.25) imply that it is enough to check that $\mathbb{Q}\otimes P[\zeta_1] < \infty$ and $\mathbb{Q}\otimes P[\zeta_2^{1+\delta}] < \infty$ for some $\delta > 0$ independent of α. Using again the estimate $f_\infty(x) \le 1 + c\alpha$ and the independence between M_0^* and R_1, we see that (3.29), (3.33) and (3.34) follow at once from (3.25). □

Proof of Theorem 1.1 (case $\alpha \searrow 0$). By proposition 3.1,

$$\lim_{\alpha\searrow 0} \frac{\mathbb{E}(\beta(x))}{\alpha} = \frac{\mathcal{D}^0}{2m}\,.$$

Then by the velocity representation for v_α (Theorem 3.8), it is enough to prove that

$$\lim_{\alpha\searrow 0} \frac{\mathbb{Q}\otimes \widetilde{E}\left[\prod_{i=0}^{R_1-1} f_\infty(S_i)\,\frac{1}{M_0^*}\right]}{\mathbb{Q}\otimes P\left[\prod_{i=R_1}^{R_2-1} f_\infty(S_i)\,|S_{R_2}-S_{R_1}|\right]}\,\sum_{y\ge 1} h(y) = 1\,. \tag{3.45}$$

Since we are interested in the limit $\alpha \searrow 0$, we may and will assume throughout that $\alpha < \alpha_0(m,\kappa)$ the constant appeared in Lemma 3.9. We write in this proof $A \sim_\alpha B$ if $(A-B)/\alpha \to_{\alpha\searrow 0} 0$.

Note that $f_\infty(x) \le 1 + c\alpha$ for some constant $c > 0$. Mimicking the proof of Lemma 3.9, we therefore get that

$$\mathbb{Q} \otimes P\left[\prod_{i=R_1}^{R_2-1} f_\infty(S_i)\,|S_{R_2} - S_{R_1}|\right] \sim_\alpha E\left[|S_{R_2} - S_{R_1}|\right] = \frac{1}{P(\tau_S(1) = \infty)} \sim_\alpha \frac{m}{m-1}.$$

In the same way, $h(y) \sim_\alpha P(\tau_S(-y) \le R_1 | \tau_S(1) = \infty)$ for $y \ge 1$, hence

$$\sum_{y\ge 1} h(y) \sim_\alpha E(|S_{R_1}||\tau_S(1) = \infty) \sim_\alpha \frac{m}{m-1}.$$

Finally, as $\alpha \to 0$,

$$\mathbb{Q} \otimes P\left[\prod_{i=1}^{R_1-1} f_\infty(S_i)\,\frac{1}{M_0^*}\right] \sim_\alpha \mathbb{Q}\left[\frac{1}{M_0^*}\right] = 1,$$

implying (3.45) and completing the proof of Theorem 1.1. □

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